Growth properties of Nevanlinna matrices for rational moment problems

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We consider rational moment problems on the real line with their associated orthogonal rational functions. There exists a Nevanlinna type parameterization relating to the problem, with associated Nevanlinna matrices of functions having singularities in the closure of the set of poles of the rational functions belonging to the problem. We prove results related to the growth at the singularities of the functions in a Nevanlinna matrix, and in particular provide bounds on the growth analogous to the corresponding result in the classical polynomial case, when the number of singularities is finite.

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Growth properties of Nevanlinna matrices for rational moment problems

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Abstract

We consider rational moment problems on the real line with their associated orthogonal rational functions. There exists a Nevanlinna type parameterization relating to the problem, with associated Nevanlinna matrices of functions having singularities in the closure of the set of poles of the rational functions belonging to the problem. We prove results related to the growth at the singularities of the functions in a Nevanlinna matrix, and in particular provide bounds on the growth analogous to the corresponding result in the classical polynomial case, when the number of singularities is finite.

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1 Introduction

We use the following notations. $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}}$ the one point compactification of $\mathbb{C}$ (the extended complex plane), $\mathbb{R}$ the real line, $\hat{\mathbb{R}}$ the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$, $\mathbb{U}$ the open upper half-plane, $\hat{\mathbb{U}}$ the closure of $\mathbb{U}$ in $\mathbb{C}$.

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A function f is called a *Pick function* if it is holomorphic in $\mathbb{U}$ and maps $\mathbb{U}$ into $\hat{\mathbb{U}}$. A Pick function is either a constant in $\hat{\mathbb{R}}$ or maps $\mathbb{U}$ into $\mathbb{U}$.

Let μ be a finite positive measure on $\mathbb{R}$. The *Stieltjes transform* S_μ of μ is defined as

$$S_\mu(z) = \int\limits_{\mathbb{R}} C(t,z)\,\mathrm{d}\mu(t), \qquad C(t,z) = \frac{1}{t-z}.$$

The *Herglotz-Riesz-Nevanlinna transform* Ω_μ of μ is defined as

$$\Omega_\mu(z) = \int\limits_{\mathbb{R}} D(t,z)\,\mathrm{d}\mu(t), \qquad D(t,z) = \frac{1+tz}{t-z}.$$

Both of these functions are Pick functions. Furthermore

$$\Omega_\mu(z) = (1+z^2)S_\mu(z) + \int\limits_{\mathbb{R}} \mathrm{d}\mu(t).$$

Thus for fixed z there is a one-to-one correspondence between Ω_μ and S_μ as functions of μ.

Let M be a Hermitian, positive definite linear functional on the space $\mathcal{P}$ of polynomials, and define its moments c_n by $c_n = M[z^n]$, $n = 0, 1, 2, \ldots$. A solution of the *Hamburger moment problem* for $\{c_n\}$ (or M) is a measure μ on $\mathbb{R}$ which satisfies $\int_{\mathbb{R}} t^n\,\mathrm{d}\mu(t) = c_n$ for all n. (Such measures exist.) A moment problem is called *determinate* if it has exactly one solution, *indeterminate* if it has more than one solutions.

There is a one-to-one correspondence between all Pick functions f and all solutions μ of an indeterminate problem given by

$$S_\mu(z) = -\frac{A(z)f(z) - C(z)}{B(z)f(z) - D(z)}$$

(*Nevanlinna parameterization* of the solutions). Here A, B, C, D are entire transcendent functions where the growth is restricted as follows: Let F be any of the functions A, B, C, D. Then for every positive ε, there exists a constant $M(\varepsilon)$ such that

$$|F(z)| \le M(\varepsilon)\exp\{\varepsilon|z|\}.$$

(Thus the function is of at most minimal type of order 1.)

For detailed treatments of important aspects of the Hamburger moment problem, see e.g. [1,3–5,11–13,16,22–26].

The *strong Hamburger moment problem* is analogous to the classical problem, with the space of polynomials replaced by the space of Laurent polynomials (linear combinations of z^k, $k = 0, \pm 1, \pm 2, \ldots$.) A similar parameterization of the set of solutions of an indeterminate problem holds, with the appropriate functions A, B, C, D holomorphic in $\mathbb{C} \setminus \{0\}$. When F is any of the functions A, B, C, D, there exist for every positive ε, constants $M_\infty(\varepsilon)$ and $M_0(\varepsilon)$ such that

$$|F(z)| \leq M_\infty(\varepsilon) \exp(\varepsilon|z|) \quad \text{and} \quad |F(z)| \leq M_0(\varepsilon) \exp(\varepsilon/|z|).$$

For detailed treatments on the theory of strong Hamburger moment problems, see e.g., [14,17–21].

In this paper, we treat a *rational moment problem*, where polynomials are replaced by rational functions with prescribed poles in $\hat{\mathbb{R}}$. A Nevanlinna parameterization for solutions of an indeterminate problem in terms of Ω_μ and Pick functions was proved by A. Almendral in [2].

The classical Hamburger moment problem is a special case of the rational problem under consideration. Thus in this case there is an alternative parameterization in terms of Ω_μ.

Our aim in this paper is to establish growth conditions at the singularities of the functions A, B, C, D appearing in the parameterization formula.

In Section 2 we introduce the rational spaces on which the rational moment problems are defined, and sketch the theory of orthogonal rational functions and their use in the theory of rational moment problems, including the Nevanlinna parameterization of the solutions of indeterminate problems. Section 3 is devoted to establishing a Riesz type criterion for such indeterminate problems when the number of singularities is finite. This criterion is crucial for the further development of the growth properties. (For the classical Riesz criterion, see e.g., [1],[22–24].) Finally in Section 4 we prove our result on the restriction on the growth of the functions A, B, C, D at the singularities.

The organization and presentation of the material in Sections 3 and 4 is strongly influenced by Akhiezer's work [1] on the classical moment problem. Other very instructive treatments of the classical problem can be found in the treatises by M. Riesz [22–24] and by Shohat and Tamarkin [25] and Stone [26]. This classical approach has to be modified in a number of ways, but the final results are of basically the same structure.

Remark 1.1 A parameterization result for rational moment problems associated with poles outside the closed unit disk and measures on the unit circle $\mathbb{T}$ was proved in [10]. Here Ω_μ is replaced by the Herglotz-Riesz transform $\int_{\mathbb{T}} \frac{t+z}{t-z} \, \mathrm{d}\mu(t)$ and Pick functions are replaced by Carathéodory functions (holomorphic in the open unit disk and mapping this disk to the closed right half-plane). All the isolated singularities of the relevant functions are poles in this case.

2 Orthogonal rational functions and rational moment problems

Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of arbitrary points (interpolation points or singularities) in $\hat{\mathbb{R}} \setminus \{0\}$, $\alpha_0 = \infty$. We denote by G the set of points α in $\hat{\mathbb{R}} \setminus \{0\}$ for which there is at least one k such that $\alpha_k = \alpha$. For $\alpha \in G$ we denote by Γ_α the subsequence of $\{\alpha_k\}_{k=1}^{\infty}$ consisting of those α_k for which $\alpha_k = \alpha$.

Set

$$\pi_0 = 1, \quad \pi_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{\alpha_k}\right), \quad n = 1, 2, \ldots, \quad b_n(z) = \frac{z^n}{\pi_n(z)}, \quad n = 0, 1, 2, \ldots.$$

The set $\{b_0, b_1, \ldots, b_n\}$ is a basis for the space

$$\mathcal{L}_n = \left\{ \frac{p(z)}{\pi_n(z)} : p \in \mathcal{P}_n \right\}$$

where $\mathcal{P}_n$ denotes the space of polynomials of degree at most n. We set $\mathcal{L}_\infty = \cup_{n=0}^{\infty} \mathcal{L}_n$. We shall also consider the space $\mathcal{R}_\infty = \mathcal{L}_\infty \cdot \mathcal{L}_\infty$ consisting of products of two functions in $\mathcal{L}_\infty$. Note that if Γ_α is infinite for all $\alpha \in G$, then $\mathcal{R}_\infty = \mathcal{L}_\infty$.

Remark 2.1 The space of Laurent polynomials is not formally included in this setting. The exclusion of the origin as interpolation point is for technical reasons. A discussion of basic properties in the general case when also the origin is included among the possible interpolation points can be found in [9].

Let M be a Hermitian, positive definite linear functional on $\mathcal{R}_\infty$. Thus $M[\overline{f}] = \overline{M[f]}$ for $f \in \mathcal{R}_\infty$ and $M[g \cdot \overline{g}] > 0$ for $g \in \mathcal{L}_\infty$, $g \neq 0$. For convenience we assume M normalized such that $M[1] = 1$. The moments $\mu_{m,n}$ of M are defined as

$$\mu_{m,n} = M[b_m \cdot b_n].$$

(Note that $\overline{b}_n = b_n$.) A measure μ on $\mathbb{R}$ is said to solve the *rational moment problem on* $\mathcal{L}_\infty$ if b_m is integrable with respect to μ and

$$\int_{\mathbb{R}} b_m(t) \, \mathrm{d}\mu(t) = \mu_{m,0} \quad \text{for} \quad m = 0, 1, 2, \ldots.$$

Equivalently

$$\int_{\mathbb{R}} g(t) \, \mathrm{d}\mu(t) = M[g] \quad \text{for} \quad g \in \mathcal{L}_\infty.$$

A measure μ on $\mathbb{R}$ is said to solve the *rational moment problem on* $\mathcal{R}_\infty$ if $b_m \cdot b_n$ is integrable with respect to μ and

$$\int\limits_{\mathbb{R}} b_m(t) b_n(t)\, \mathrm{d}\mu(t) = \mu_{m,n} \quad \text{for} \quad m, n = 0, 1, 2, \ldots.$$

Equivalently

$$\int\limits_{\mathbb{R}} f(t)\, \mathrm{d}\mu(t) = M[f] \quad \text{for} \quad f \in \mathcal{R}_\infty.$$

A solvable rational moment problem is said to be *determinate* if it has exactly one solution, *indeterminate* if it has more than one solution. We denote by $\mathcal{M}(\mathcal{L}_\infty)$ the set of solutions of the problem on $\mathcal{L}_\infty$, and by $\mathcal{M}(\mathcal{R}_\infty)$ the set of solutions of the problem on $\mathcal{R}_\infty$.

Let $\{\varphi_n\}_{n=0}^\infty$ be the sequence of functions obtained by orthonormalization (with respect to M) of the sequence $\{b_n\}_{n=0}^\infty$. We fix them uniquely by multiplying with a unimodular constant, so that the coefficient of b_n in the expansion of φ_n with respect to the basis $\{b_n\}$ is positive.

The function φ_n has the form $\varphi_n(z) = \frac{p_n(z)}{\pi_n(z)}$, $p_n \in \mathcal{P}_n$. Note that by our normilization, the coefficients are real, hence $\varphi_n(x)$ is real for $x \in \mathbb{R}$. The functions ψ_n of the second kind are defined by

$$\psi_0(z) = -z, \quad \psi_n(z) = M_t\left[D(t,z)\{\varphi_n(t) - \varphi_n(z)\}\right], \quad n = 1, 2, \ldots.$$

We shall also consider the rational functions σ_n given by (M_t refers to M applied to t-variable)

$$\sigma_n(z) = M_t\left[C(t,z)\{\varphi_n(t) - \varphi_n(z)\}\right], \quad n = 0, 1, 2, \ldots.$$

We observe that both ψ_n and φ_n belong to $\mathcal{L}_n$, and that both functions are real for real z. Furthermore we find that

$$\sigma_n(z) = \frac{1}{1+z^2}[z\varphi_n(z) + \psi_n(z)], \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

The sequences $\{\varphi_n\}$, $\{\psi_n\}$, and $\{\sigma_n\}$ satisfy a three-term recurrence relation of the form

$$\begin{bmatrix} \sigma_n(z) \\ \psi_n(z) \\ \varphi_n(z) \end{bmatrix} = \left\{ E_n \frac{z}{1 - z/\alpha_n} + B_n \frac{1 - z/\alpha_{n-2}}{1 - z/\alpha_n} \right\} \begin{bmatrix} \sigma_{n-1}(z) \\ \psi_{n-1}(z) \\ \varphi_{n-1}(z) \end{bmatrix} + C_n \frac{1 - z/\alpha_{n-2}}{1 - z/\alpha_n} \begin{bmatrix} \sigma_{n-2}(z) \\ \psi_{n-2}(z) \\ \varphi_{n-2}(z) \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} \sigma_0(z) \\ \psi_0(z) \\ \varphi_0(z) \end{bmatrix} = \begin{bmatrix} 0 \\ -z \\ 1 \end{bmatrix}.$$

Here B_n, C_n, E_n are real numbers satisfying $E_n = -C_n E_{n-1}$ for $n = 2, 3, \ldots$. See [8, Sections 11.1, 11.2, 11.9].

Note that σ_1 has the form $\sigma_1(z) = \kappa/(1 - z/\alpha_1)$, where κ is a constant. We define

$$\chi_n(z) = \kappa^{-1}(1 - z/\alpha_1)\sigma_{n+1}(z), \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Note that $\chi_n(x)$ is real for real x.

The sequence $\{\chi_n\}$ satisfies the recurrence relation

$$\chi_n(z) = \left\{ E_{n+1} \frac{z}{1 - z/\alpha_{n+1}} + B_{n+1} \frac{1 - z/\alpha_{n-1}}{1 - z/\alpha_{n+1}} \right\} \chi_{n-1}(z) + C_n \frac{1 - z/\alpha_{n-1}}{1 - z/\alpha_{n+1}} \chi_{n-2}(z)$$

for $n = 2, 3, \ldots$, with $\chi_0 = 1$.

Set

$$\tilde{\pi}_0 = 1, \quad \tilde{\pi}_n(z) = \prod_{k=2}^{n+1} \left(1 - \frac{z}{\alpha_k}\right), \quad n = 1, 2, \ldots \text{ and, } \quad \tilde{b}_n(z) = \frac{z^n}{\tilde{\pi}_n(z)}, \quad \text{for } n = 0, 1, 2, \ldots.$$

Let $\tilde{\mathcal{L}}_n$ denote the space spanned by $\{\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_n\}$, and set $\tilde{\mathcal{L}}_\infty = \cup_{n=0}^\infty \tilde{\mathcal{L}}_n$, $\tilde{\mathcal{R}}_\infty = \tilde{\mathcal{L}}_\infty \cdot \tilde{\mathcal{L}}_\infty$. We then have $\chi_n \in \tilde{\mathcal{L}}_n$.

According to the Favard type theorem for orthogonal rational functions (see [8, Section 11.9]), it follows that there is a positive functional $\tilde{M}$ on $\tilde{\mathcal{R}}_\infty$ such that the sequence $\{\chi_n\}$ is orthonormal with respect to $\tilde{M}$. We can then consider moment problems on $\tilde{\mathcal{L}}_\infty$ and $\tilde{\mathcal{R}}_\infty$ for the functional $\tilde{M}$. We shall call these moment problems *associated moment problems*. Since $\tilde{M}$ is positive, the moment problem on $\tilde{\mathcal{L}}_\infty$ is always solvable.

We shall use the notation

$$\omega_n(z) = 1 + \sum_{k=1}^{n-1} |\varphi_k(z)|^2, \quad \Omega_n(z) = 1 + \sum_{k=1}^{n-1} |\psi_k(z)|^2, \quad \tilde{\omega}_n(z) = 1 + \sum_{k=1}^{n-1} |\chi_k(z)|^2.$$

We also set

$$\omega_{\alpha,n}(z) = \sum_{\substack{k=1\\ \alpha_k \in \Gamma_\alpha}}^{n-1} |\varphi_k(z)|^2.$$

Note that $\omega_n(z) = 1 + \sum_{\alpha \in G} \omega_{\alpha,n}(z)$.

Let x_0 be a point in $\mathbb{R} \setminus [\hat{G} \cup \{0\}]$, where $\hat{G}$ denotes the closure in $\hat{\mathbb{C}}$ of the set G of interpolation points. For technical reasons, x_0 is chosen such that $\psi_n(x_0) \neq 0$ and $q_n(\alpha_k, x_0) \neq 0$ for $k = 1, 2, \ldots, n$, for all n, where $q_n(z, \tau)$ is the numerator polynomial of the rational function $\varphi_n(z) + \tau \frac{1 - z/\alpha_{n-1}}{1 - z/\alpha_n} \varphi_{n-1}(z)$. Such choice is always possible, see [8, Lemma 11.5.4]. In the following x_0 shall be kept fixed, and will not be included in the notation for A_n, B_n, C_n, D_n below. We set $H(z, x_0) = \frac{x_0 - z}{x_0 z}$ and define

$$\begin{aligned}
A_n(z) &= H(z, x_0) \left[1 + \sum_{k=1}^{n-1} \psi_k(x_0) \psi_k(z) \right] \\
B_n(z) &= H(z, x_0) \left[D(z, x_0) - \sum_{k=1}^{n-1} \psi_k(x_0) \varphi_k(z) \right] \\
C_n(z) &= H(z, x_0) \left[D(z, x_0) + \sum_{k=1}^{n-1} \varphi_k(x_0) \psi_k(z) \right] \\
D_n(z) &= H(z, x_0) \left[1 + \sum_{k=1}^{n-1} \varphi_k(x_0) \varphi_k(z) \right]
\end{aligned}$$

(Note that the definitions differ from those used in [2] by a real constant factor E_n.)

We set $\mathbb{C}_G = \hat{\mathbb{C}} \setminus \left[\hat{G} \cup \{-\mathrm{i}, \mathrm{i}\} \right]$. For $z \in \mathbb{C}_G$ and $t \in \hat{\mathbb{R}}$ we define

$$T_n(z, t) = -\frac{A_n(z) t - C_n(z)}{B_n(z) t - D_n(z)}$$

(which means $-A_n(z)/B_n(z)$ when $t = \infty$). The functions A_n, B_n, C_n, D_n can also be expressed in the following way:

$$\begin{aligned}
A_n(z) &= \frac{1}{E_n x_0 z} \left[f_n(x_0, z) \psi_n(x_0) \psi_{n-1}(z) - f_n(z, x_0) \psi_{n-1}(x_0) \psi_n(z) \right] \\
B_n(z) &= \frac{1}{E_n x_0 z} \left[f_n(x_0, z) \psi_n(x_0) \varphi_{n-1}(z) - f_n(z, x_0) \psi_{n-1}(x_0) \varphi_n(z) \right] \\
C_n(z) &= \frac{1}{E_n x_0 z} \left[f_n(x_0, z) \varphi_n(x_0) \psi_{n-1}(z) - f_n(z, x_0) \varphi_{n-1}(x_0) \psi_n(z) \right] \\
D_n(z) &= \frac{1}{E_n x_0 z} \left[f_n(x_0, z) \varphi_n(x_0) \varphi_{n-1}(z) - f_n(z, x_0) \varphi_{n-1}(x_0) \varphi_n(z) \right]
\end{aligned}$$

where $f_n(z,w) = \left(1 - \frac{z}{\alpha_{n-1}}\right)\left(1 - \frac{w}{\alpha_n}\right)$.

It follows by a simple argument from [8, Corollary 11.5.6] that the functions $B_n(z)t - D_n(z)$, $t \in \hat{\mathbb{R}}$, have all their zeros on $\mathbb{R}$. According to [8, Lemma 11.10.6], the function $z \to T_n(z,t)$ for $t \in \hat{\mathbb{R}}$ is a Pick function, hence all the zeros of $A_n(z)t - C_n(z)$ are also real.

The index n (or the function φ_n) is said to be *regular* if $p_n(\alpha_{n-1}) \neq 0$ ($p_n(\infty) \neq 0$ means that p_n has degree exactly like n).

For z fixed, the linear fractional transformation $t \longrightarrow T_n(z,t)$ maps for a regular index n the closed lower half-plane onto a proper closed disk $\Delta_n(z)$ in the open right half-plane. When $m > n$, we have $\Delta_m(z) \subset \Delta_n(z)$. Let Λ denote the sequence of regular indices, and set

$$\Delta_\infty(z) = \bigcap_{n \in \Lambda} \Delta_n(z).$$

The $\Delta_\infty(z)$ is a proper, closed disk or a single point, independent of z in $\mathbb{C}_G$. Furthermore $\Delta_\infty(z)$ is a proper disk if and only if the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ converges locally uniformly in the domain $\mathbb{C}_G$. This is the case if and only if the series $\sum_{k=1}^{\infty} |\psi_k(z)|^2$ converges.

We shall in the following assume that the set Λ *is infinite*. For simplicity of notation we let without loss of generality Λ consist of the natural numbers. We shall use the notation

$$\omega(z) = 1 + \sum_{k=1}^{\infty} |\varphi_k(z)|^2, \quad \Omega(z) = 1 + \sum_{k=1}^{\infty} |\psi_k(z)|^2, \quad \tilde{\omega}(z) = 1 + \sum_{k=1}^{\infty} |\chi_k(z)|^2.$$

The following inclusions hold:

$$\{\Omega_\mu(z) : \mu \in \mathcal{M}(\mathcal{R}_\infty)\} \subset \Delta_\infty(z) \subset \{\Omega_\mu(z) : \mu \in \mathcal{M}(\mathcal{L}_\infty)\}.$$

It follows that if the moment problem on $\mathcal{R}_\infty$ is indeterminate, then the series $\sum_{k=1}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=1}^{\infty} |\psi_k(z)|^2$ converge. Furthermore, if the series $\sum_{k=1}^{\infty} |\varphi_k(z)|^2$ converges, then the moment problem on $\mathcal{L}_\infty$ is indeterminate. Now assume that the moment problem for M on $\mathcal{R}_\infty$ is indeterminate. Then $\sum_{k=1}^{\infty} |\psi_k(z)|^2$ converges, hence also $\sum_{k=1}^{\infty} |\chi_k(z)|^2$ converges. Thus the associated moment problem for $\tilde{M}$ on $\tilde{\mathcal{L}}_\infty$ is indeterminate. *Because of the closely related recursion formulas, it is reasonable to expect that the moment problem for $\tilde{M}$ on $\tilde{\mathcal{R}}_\infty$ is indeterminate when the problem for M on $\mathcal{R}_\infty$ is indeterminate. We have no proof of this, but we shall make this assumption in the proof of (3.9) and Proposition 4.2 for F equal to A or C.* However, when all the sets Γ_α are infinite, then $\mathcal{R}_\infty = \mathcal{L}_\infty$ and the moment problem on $\mathcal{L}_\infty$ and $\mathcal{R}_\infty$ coincide. In this case $\tilde{\mathcal{R}}_\infty = \tilde{\mathcal{L}}_\infty = \mathcal{L}_\infty$ and thus the assumption above is automatically satisfied. Note also Remark 4.5, where the assumption is not needed. Thus our main result Theorem 4.4 does not depend on this assumption.

The theory of orthogonal rational functions with poles on the extended real line is equivalent to a theory of orthogonal rational functions with poles on the unit circle. See especially [8] and

[6].

For more details on the properties of orthogonal rational functions and rational moment problems that we have discussed so far, we refer to [2],[6],[7], [8, Chap 11], [9].

The convergence results and the parameterization results below were obtained by A. Almendral in [2].

Assume that $\Delta_\infty(z)$ is a proper disk (the *limit circle case* in contrast to the *limit point case*). Then the functions A_n, B_n, C_n, D_n converge locally uniformly in $\mathbb{C}_G$ to holomorphic functions A, B, C, D. We may then write

$$\begin{aligned}
A(z) &= H(z,x_0)\left[1+\sum_{k=1}^{\infty}\psi_k(x_0)\psi_k(z)\right]\\
B(z) &= H(z,x_0)\left[D(z,x_0)-\sum_{k=1}^{\infty}\psi_k(x_0)\varphi_k(z)\right]\\
C(z) &= H(z,x_0)\left[D(z,x_0)+\sum_{k=1}^{\infty}\varphi_k(x_0)\psi_k(z)\right]\\
D(z) &= H(z,x_0)\left[1+\sum_{k=1}^{\infty}\varphi_k(x_0)\varphi_k(z)\right].
\end{aligned}$$

The collection $\{A, B, C, D\}$ is called a *Nevanlinna matrix* for the problem.

The functions A, B, C, D appear in the following Nevanlinna type parameterization for an indeterminate rational moment problem.

Theorem 2.2 *Assume that the moment problem on $\mathcal{R}_\infty$ is indeterminate, and consider the formula*

$$\Omega_\mu(z) = -\frac{A(z)f(z) - C(z)}{B(z)f(z) - D(z)}. \tag{2.3}$$

Then

(i) For every Pick function f, there exists a $\mu \in \mathcal{M}(\mathcal{L}_\infty)$ such that (2.3) is satisfied.
(ii) For every $\mu \in \mathcal{M}(\mathcal{R}_\infty)$, there exists a Pick function f such that (2.3) is satisfied.

Remark 2.3 The correspondence between μ and Ω_μ is one-to-one. When Γ_α is infinite for all $\alpha \in G$, we have

$$\{\Omega_\mu(z) : \mu \in \mathcal{M}(\mathcal{L}_\infty)\} = \Delta_\infty(z).$$

Hence in this situation (2.3) establishes a one-to-one correspondence between Pick functions and solutions of the moment problem (on $\mathcal{L}_\infty$ or $\mathcal{R}_\infty$).

3 A Riesz type criterion

Let μ_1 and μ_2 be two distinct solutions of the moment problem on $\mathcal{R}_\infty$. The function $\Omega_{\mu_1}(z) - \Omega_{\mu_2}(z)$ is holomorphic in $\mathbb{C} \setminus [\mathbb{R} \cup \{-\mathrm{i}, \mathrm{i}\}]$, hence the zeros are isolated. It follows that there exist positive γ, $\gamma \neq 1$ such that $\Omega_{\mu_1}(\beta + \mathrm{i}\gamma) \neq \Omega_{\mu_2}(\beta + \mathrm{i}\gamma)$ for all $\beta \in \mathbb{R}$. Note that we then also have $S_{\mu_1}(\beta + \mathrm{i}\gamma) \neq S_{\mu_2}(\beta + \mathrm{i}\gamma)$ for all $\beta \in \mathbb{R}$. We choose a fixed γ with this property, and use the notation $\zeta_\beta = \beta + \mathrm{i}\gamma$.

Let us start by stating a general Poisson formula first.

Lemma 3.1 *Suppose ξ is in the lower half plane and $\zeta_\beta = \beta + \gamma\mathrm{i}$, $\beta \neq 0$, $\gamma > 0$, and $\alpha \in \mathbb{R}\setminus\{0\}$, then*

$$\frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{\ln\left|1 - \frac{t}{\xi}\right|}{|t - \zeta_\beta|^2} \, \mathrm{d}t = \ln\left|1 - \frac{\zeta_\beta}{\xi}\right|, \tag{3.1}$$

and

$$\frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{\ln\left|1 - \frac{t}{\alpha}\right|}{|t - \zeta_\beta|^2} \, \mathrm{d}t = \ln\left|1 - \frac{\zeta_\beta}{\alpha}\right|. \tag{3.2}$$

PROOF. This can be proved by standard complex analysis arguments. $\square$

Remark 3.2 Note that (3.1) also follows from [1, p.53] where in a footnote it is remarked that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln|1 - \frac{s}{c}|}{s^2 + 1} \, \mathrm{d}s = \ln\left|1 - \frac{\mathrm{i}}{c}\right|, \quad (\mathrm{Im}(c) < 0).$$

Change of variables $s = (t - \beta)/\gamma$ and $c = (\xi - \beta)/\gamma$ also yields (3.1).

In the following *positive function* shall always mean *strictly positive function.*

Proposition 3.3 *Let R be a function in $\mathcal{R}_\infty$ which is positive on $\mathbb{R}$. Then there exists a function $L \in \mathcal{L}_\infty$ such that*

$$\left|\frac{1}{x - \zeta_\beta} - L(x)\right| = \frac{\sqrt{R(x)}}{|x - \zeta_\beta|} \exp\left\{-\frac{\gamma}{2\pi} \int_{-\infty}^{\infty} \frac{\ln R(t)}{|t - \zeta_\beta|^2} \, \mathrm{d}t\right\}$$

for all x in $\mathbb{R}$.

PROOF. By dividing out possible common factors in the numerator and denominator of R we may write

$$R(z) = \frac{P(z)}{\left(1 - \frac{z}{\alpha_{k_1}}\right)^2 \cdots \left(1 - \frac{z}{\alpha_{k_p}}\right)^2}$$

where P is a polynomial of degree n, $P(x)$ positive for $x \in \mathbb{R}$. The polynomial P then has the form

$$P(z) = |A|^2 \left(1 - \frac{z}{\xi_1}\right) \cdots \left(1 - \frac{z}{\xi_n}\right) \left(1 - \frac{z}{\overline{\xi}_1}\right) \cdots \left(1 - \frac{z}{\overline{\xi}_n}\right)$$

for a suitable constant A and $\xi_1, \ldots, \xi_n$ in the lower half-plane.

We define

$$Q(z) = \frac{A\left(1 - \frac{z}{\xi_1}\right) \cdots \left(1 - \frac{z}{\xi_n}\right)}{\left(1 - \frac{z}{\alpha_{k_1}}\right) \cdots \left(1 - \frac{z}{\alpha_{k_p}}\right)}.$$

Then

$$|Q(z)|^2 = |A|^2 \frac{\left(1 - \frac{z}{\xi_1}\right) \cdots \left(1 - \frac{z}{\xi_n}\right) \left(1 - \frac{\overline{z}}{\overline{\xi}_1}\right) \cdots \left(1 - \frac{\overline{z}}{\overline{\xi}_n}\right)}{\left|1 - \frac{z}{\alpha_{k_1}}\right|^2 \cdots \left|1 - \frac{z}{\alpha_{k_p}}\right|^2}$$

and for $x \in \mathbb{R}$:

$$|Q(x)|^2 = \frac{|A|^2 \left(1 - \frac{x}{\xi_1}\right) \cdots \left(1 - \frac{x}{\xi_n}\right) \left(1 - \frac{x}{\overline{\xi}_1}\right) \cdots \left(1 - \frac{x}{\overline{\xi}_n}\right)}{\left|1 - \frac{x}{\alpha_{k_1}}\right|^2 \cdots \left|1 - \frac{x}{\alpha_{k_p}}\right|^2} = R(x).$$

We further define

$$L(z) = \frac{1 - \frac{Q(z)}{Q(\zeta_\beta)}}{z - \zeta_\beta}.$$

Note that $L \in \mathcal{L}_\infty$. We have

$$\frac{1}{z - \zeta_\beta} - L(z) = \frac{Q(z)}{(z - \zeta_\beta)Q(\zeta_\beta)},$$

hence for $x \in \mathbb{R}$:

$$\left|\frac{1}{z-\zeta_\beta} - L(x)\right| = \frac{|Q(x)|}{|x-\zeta_\beta||Q(\zeta_\beta)|} = \frac{\sqrt{R(x)}}{|x-\zeta_\beta||Q(\zeta_\beta)|}. \tag{3.3}$$

Using the identity

$$\int\limits_{-\infty}^{\infty} \frac{dt}{|t-\zeta|^2}\,\mathrm{d}t = \frac{\pi}{\gamma},$$

and Lemma 3.1 we get

$$\begin{aligned}
\ln|Q(\zeta_\beta)| &= \ln|A| + \sum_{j=1}^{n} \ln\left|1-\frac{\zeta_\beta}{\xi_j}\right| - \sum_{j=1}^{p} \ln\left|1-\frac{\zeta_\beta}{\alpha_{k_j}}\right| \\
&= \ln|A|\frac{\gamma}{\pi}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}t}{|t-\zeta_\beta|^2} + \sum_{j=1}^{n} \ln\left|1-\frac{\zeta_\beta}{\xi_j}\right| - \sum_{j=1}^{p} \ln\left|1-\frac{\zeta_\beta}{\alpha_{k_j}}\right| \\
&= \frac{\gamma}{\pi}\int\limits_{-\infty}^{\infty} \left\{\ln|A| + \sum_{j=1}^{n} \ln\left|1-\frac{\zeta_\beta}{\xi_j}\right| - \sum_{j=1}^{p} \ln\left|1-\frac{\zeta_\beta}{\alpha_{k_j}}\right|\right\} \frac{\mathrm{d}t}{|t-\zeta_\beta|^2}
\end{aligned}$$

and hence

$$\ln|Q(\zeta_\beta)| = \frac{\gamma}{\pi}\int\limits_{-\infty}^{\infty} \frac{\ln|Q(t)|}{|t-\zeta_\beta|^2}\,\mathrm{d}t = \frac{\gamma}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\ln R(t)}{|t-\zeta_\beta|^2}\,\mathrm{d}t. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\left|\frac{1}{x-\zeta_\beta} - L(x)\right| = \frac{\sqrt{R(x)}}{|x-\zeta_\beta|} \exp\left\{-\frac{\gamma}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\ln R(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2}\right\}.$$

which concludes the proof. □

Corollary 3.4 *For each non-negative integer n and each $\alpha \in G$ there exists an $L_n \in \mathcal{L}_\infty$ such that for $x \in \mathbb{R}$:*

$$\left|\frac{1}{x-\zeta_\beta} - L_n(x)\right| = \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} \exp\left\{-\frac{\gamma}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\ln[1+\omega_{\alpha,n}(t)]\,\mathrm{d}t}{|t-\zeta_\beta|^2}\right\}.$$

PROOF. The function $1+\omega_{\alpha,n}(x)$ is the restriction to $\mathbb{R}$ of the function $1+\sum_{k=1;\alpha_k\in\Gamma_\alpha}^{n-1} \varphi_k(z)^2$, which belongs to $\mathcal{R}_\infty$ and is positive on $\mathbb{R}$. Consequently the result follows from Proposition 3.3. □

Proposition 3.5 *There exists a finite constant* K_1 *such that for every* R *in* $\mathcal{R}_\infty$ *which is positive on* $\mathbb{R}$ *we have*

$$\exp\left\{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln R(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2}\right\} \leq K_1 \sup_{\mu\in\mathcal{M}(\mathcal{R}_\infty)}\left\{\int_{\mathbb{R}}\frac{\sqrt{R(x)}\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|}\right\},$$

where K_1 *is independent of* R.

PROOF. Recall that $S_{\mu_1}(\zeta_\beta) \neq S_{\mu_2}(\zeta_\beta)$, where μ_1, μ_2 are two different measures in $\mathcal{M}(\mathcal{R}_\infty)$, cf. the introduction to this section. Set $K_0 = S_{\mu_1}(\zeta) - S_{\mu_2}(\zeta) \neq 0$. Since $\int_{\mathbb{R}} L(x)\,\mathrm{d}\mu_1(x) = \int_{\mathbb{R}} L(x)\,\mathrm{d}\mu_2(x)$ for all $L \in \mathcal{L}_\infty$ we may write

$$K_0 = \int_{\mathbb{R}}\left(\frac{1}{x-\zeta_\beta} - L(x)\right)\mathrm{d}\mu_1(x) - \int_{\mathbb{R}}\left(\frac{1}{x-\zeta_\beta} - L(x)\right)\mathrm{d}\mu_2(x)$$

hence

$$|K_0| \leq \int_{\mathbb{R}}\left|\frac{1}{x-\zeta_\beta} - L(x)\right|\mathrm{d}\mu_1(x) + \int_{\mathbb{R}}\left|\frac{1}{x-\zeta_\beta} - L(x)\right|\mathrm{d}\mu_2(x),$$

and consequently

$$|K_0| \leq 2\sup\left\{\int_{\mathbb{R}}\left|\frac{1}{x-\zeta_\beta} - L(x)\right|\mathrm{d}\mu(x)\right\},$$

where the supremum is taken over all $\mu \in \mathcal{M}(\mathcal{R}_\infty)$.

Let R be an arbitrary function in $\mathcal{R}_\infty$ which is strictly positive on $\mathbb{R}$. Then we conclude from Proposition 3.3 that

$$|K_0| \leq 2\exp\left\{-\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln R(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2}\right\}\cdot\sup\left\{\int_{\mathbb{R}}\frac{\sqrt{R(x)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x)\right\},$$

hence

$$\exp\left\{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln R(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2}\right\} \leq K_1\cdot\sup\left\{\int_{\mathbb{R}}\frac{\sqrt{R(t)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x)\right\}$$

where $K_1 = 2/|K_0|$. $\square$

Corollary 3.6 *There exists a constant K_1 independent of the index n such that for every $\alpha \in G$,*

$$\exp\left\{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln[1+\omega_{\alpha,n}(t)]\,\mathrm{d}t}{|t-\zeta_\beta|^2}\right\} \le K_1\cdot\sup\left\{\int_{\mathbb{R}}\frac{\sqrt{1+\omega_{\alpha,n}(t)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x)\right\}$$

where the supremum is taken over all $\mu \in \mathcal{M}(\mathcal{R}_\infty)$.

PROOF. The function $1+\omega_{\alpha,n}(x)$ is the restriction to $\mathbb{R}$ of the function $1+\sum_{k=1;\alpha_k\in\Gamma_\alpha}^{n-1}\varphi_k(z)^2$, which belongs to $\mathcal{R}_\infty$ and is positive on $\mathbb{R}$. Thus the conditions of Proposition 3.5 are satisfied, and so the result follows from this proposition. $\square$

Lemma 3.7 *Let $\alpha \in G$, $\alpha \neq \infty$, $\beta \in \mathbb{R}$. Then the following inequality holds for $\alpha_k = \alpha$, $\mu \in \mathcal{M}(\mathcal{R}_\infty)$:*

$$\left|\int_{\mathbb{R}}\frac{(x-\alpha)(x-\beta)\varphi_k(x)^2}{|x-\zeta_\beta|^2}\,\mathrm{d}\mu(x)\right| \le \frac{|\alpha-\zeta_\beta|}{|1-\gamma^2|}\left[|\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)|+|\Omega_\mu(\zeta_\beta)\varphi_k(\zeta_\beta)^2|\right]. \tag{3.5}$$

PROOF. For any $\beta \in \mathbb{R}$ we may write

$$\begin{aligned}\frac{(1-\frac{x}{\alpha})\varphi_k(x)^2}{x-\zeta_\beta} &= (1-\frac{x}{\alpha})\frac{\varphi_k(x)-\varphi_k(\zeta_\beta)}{x-\zeta_\beta}\varphi_k(x) - \frac{\zeta_\beta+\frac{1}{\alpha}}{1+\zeta_\beta^2}\varphi_k(\zeta_\beta)\varphi_k(x) + \\ &+ \frac{1-\frac{\zeta_\beta}{\alpha}}{1+\zeta_\beta^2}\left[D(x,\zeta_\beta)\{\varphi_k(x)-\varphi_k(\zeta_\beta)\}\varphi_k(\zeta_\beta)+D(x,\zeta_\beta)\varphi_k(\zeta_\beta)^2\right].\end{aligned}$$

We observe that $(1-\frac{x}{\alpha})\frac{\varphi_k(x)-\varphi_k(\zeta_\beta)}{x-\zeta_\beta}$ belongs to $\mathcal{L}_{k-1}$. Thus the integral of the first term to the right vanishes by orthogonality. Also the integral of the second term vanishes by orthogonality. We then get

$$\int_{\mathbb{R}}\frac{(1-\frac{x}{\alpha})\varphi_k(x)^2}{x-\zeta_\beta}\,\mathrm{d}\mu(x) = \frac{1-\frac{\zeta_\beta}{\alpha}}{1+\zeta_\beta^2}\left[\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)+\varphi_k(\zeta_\beta)^2\Omega_\mu(\zeta_\beta)\right].$$

Hence by taking the real part of the equation we get

$$\left|\int_{\mathbb{R}}\frac{(x-\alpha)(x-\beta)\varphi_k(x)^2}{|x-\zeta_\beta|^2}\,\mathrm{d}\mu(x)\right| \le \frac{|\alpha-\zeta_\beta|}{|1+\zeta_\beta^2|}\left[|\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)|+|\varphi_k(\zeta_\beta)|^2|\Omega_\mu(\zeta_\beta)|\right].$$

We find that $|1+\zeta_\beta^2|^2 = (1+\beta^2-\gamma^2)^2+4\beta^2\gamma^2 \ge (1-\gamma^2)^2$, from which (3.5) now follows. $\square$

Lemma 3.8 *Let μ be a positive measure in $\mathbb{R}$ and let f be a non-negative function on $\mathbb{R}$. Let $[a,b]$ be a bounded interval and let $\beta \in \mathbb{R}$. Then there exist positive numbers $m(\beta)$ and $M(\beta)$ such that*

$$m(\beta)\int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2} \le \int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\alpha|^2} \le M(\beta)\int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}$$

for all $\alpha \in [a,b]$.

PROOF. The function $\phi(x) = |x-\zeta_\beta|^2/|x-\zeta_\alpha|^2$ has a positive minimum $m(\alpha,\beta)$ and a finite maximum $M(\alpha,\beta)$ since $\phi(x) \to 1$ as $x \to \pm\infty$. The values $m(\alpha,\beta)$ and $M(\alpha,\beta)$ are continuous functions of α, hence there exist a positive $m(\beta)$ and a finite $M(\beta)$ such that $m(\beta) \le m(\alpha,\beta)$, $M(\alpha,\beta) \le M(\beta)$ for all $\alpha \in [a,b]$. We may write

$$\int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\alpha|^2} = \int_{\mathbb{R}} \frac{f(x)}{|x-\zeta_\beta|^2}\cdot\frac{|x-\zeta_\beta|^2}{|x-\zeta_\alpha|^2}\,\mathrm{d}\mu(x),$$

hence

$$m(\beta)\int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2} \le \int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\alpha|^2} \le M(\beta)\int_{\mathbb{R}} \frac{f(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}$$

for all $\alpha \in [a,b]$. □

Proposition 3.9 *Assume that G is bounded, $\alpha \in G$, $\beta \in \mathbb{R}$. Then there exists a constant $K_2(\alpha,\beta)$ independent of the index n and the measure $\mu \in \mathcal{M}(\mathcal{R}_\infty)$ such that*

$$\int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x) \le K_2(\alpha,\beta). \tag{3.6}$$

PROOF. We know that the series $\sum_{k=1}^\infty |\varphi_k(\zeta_\beta)|^2$ and $\sum_{k=1}^\infty |\psi_k(\zeta_\beta)|^2$ converge, and by Schwarz' inequality then also the series $\sum_{k=1}^\infty |\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)|$ converges. Furthermore, $\Omega_\mu(\zeta_\beta) \in \Delta(\zeta_\beta)$, which implies that $\Omega_\mu(\zeta_\beta)$ is bounded independently of $\mu \in \mathcal{M}(\mathcal{R}_\infty)$.

It follows from Lemma 3.7 that

$$\int_{\mathbb{R}} \frac{(x-\alpha)^2\omega_{\alpha,n}(x)}{|x-\zeta_\alpha|^2}\,\mathrm{d}\mu(x) \le \frac{\gamma}{|1-\gamma^2|}\sum_{\substack{k=1\\ \alpha_k\in\Gamma_\alpha}}^{n-1}\left\{|\varphi_k(\zeta_\alpha)\psi_k(\zeta_\alpha)| + |\Omega_\mu(\zeta_\alpha)\varphi_k(\zeta_\alpha)^2|\right\}.$$

Taking into account Lemma 3.8 we then get

$$\int_{\mathbb{R}} \frac{(x-\alpha)^2\omega_{\alpha,n}(x)}{|x-\zeta_\beta|^2}\,\mathrm{d}\mu(x) \le$$

$$\frac{\gamma}{m(\beta)|1-\gamma^2|}\sum_{\substack{k=1\\ \alpha_k\in\Gamma_\alpha}}^{n-1}\left\{|\varphi_k(\zeta_\alpha)\psi_k(\zeta_\alpha)| + |\Omega_\mu(\zeta_\alpha)\varphi_k(\zeta_\alpha)^2|\right\}.$$

Thus there exists a constant $K_3(\alpha,\beta)$ independent of n and $\mu \in \mathcal{M}(\mathcal{R}_\infty)$ such that

$$\int_{\mathbb{R}} \frac{(x-\alpha)^2[1+\omega_{\alpha,n}(x)]}{|x-\zeta_\beta|^2}\,\mathrm{d}\mu(x) \le K_3(\alpha,\beta). \tag{3.7}$$

We may write

$$\frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} = \frac{|x-\alpha|\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|}\cdot\frac{1}{|x-\alpha|}.$$

Hence by Schwarz' inequality we get

$$\int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x) \le \left[\int_{\mathbb{R}} \frac{(x-\alpha)^2[1+\omega_{\alpha,n}(x)]}{|x-\zeta_\beta|^2}\,\mathrm{d}\mu(x)\right]^{1/2}\cdot\left[\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{(x-\alpha)^2}\right]^{1/2}. \tag{3.8}$$

The factor $1/(x-\alpha)^2$ belongs to $\mathcal{R}_\infty$, hence $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{(x-\alpha)^2}$ equals a finite constant K_4 (independent of μ). Setting $K_2 = \sqrt{K_3(\alpha,\beta)K_4}$, we obtain (3.6) from (3.7) and (3.8). ☐

Theorem 3.10 (Riesz type criterion) *Assume that the moment problem on $\mathcal{R}_\infty$ is indeterminate. Assume that G is finite and let $\beta\in\mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} \frac{\ln\omega(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2} < \infty \quad \textit{and} \quad \int_{-\infty}^{\infty} \frac{\ln\Omega(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2} < \infty. \tag{3.9}$$

PROOF. According to Proposition 3.9 we have

$$\sup_{\mu\in\mathcal{M}(\mathcal{R}_\infty)} \int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x) \le K_2(\alpha,\beta)$$

where $K_2(\alpha,\beta)$ is independent of n. Consequently there is a constant $K_2(\beta)$ such that

$$\sum_{\alpha\in G}\sup_{\mu\in\mathcal{M}(\mathcal{R}_\infty)} \int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|}\,\mathrm{d}\mu(x) \le K_2(\beta).$$

It then follows from Corollary 3.6 that there is a constant $K(\beta)$ such that

$$\int_{-\infty}^{\infty} \sum_{\alpha\in G} \frac{\ln[1+\omega_{\alpha,n}(t)]}{|t-\zeta_\beta|^2}\,\mathrm{d}t \le K(\beta) \quad \text{for all } n.$$

For any non-negative numbers $t_1, t_2, \ldots, t_N$ we have

$$\ln\left(1+\sum_{k=1}^{N} t_k\right) \le \sum_{k=1}^{N} \ln(1+t_k).$$

Consequently we may conclude from the fact that $\omega_n(t) = 1 + \sum_{\alpha\in G}\omega_{\alpha,n}(t)$:

$$\int_{-\infty}^{\infty} \frac{\ln \omega_n(t)}{|t-\zeta_\beta|^2}\,\mathrm{d}t \le \int_{-\infty}^{\infty} \frac{1}{|t-\zeta_\beta|^2}\left\{\sum_{\alpha\in G}\ln[1+\omega_{\alpha,n}(t)]\right\}\,\mathrm{d}t \le K(\beta)$$

for all n, from which the first inequality in (3.9) follows.

Similarly, since $\{\chi_n\}$ are the orthonormal functions associated with the indeterminate moment problem on $\tilde{\mathcal{R}}_\infty$, we find

$$\int_{-\infty}^{\infty} \frac{\ln \tilde{\omega}_n(t)}{|t-\zeta_\beta|^2}\,\mathrm{d}t \le \infty$$

Then from (2.1) and (2.2) we infer that also the second inequality in (3.9) is satisfied. $\square$

Remark 3.11 By considering the imaginary part in (3.7) when $G = \{\infty\}$, we get $\left|\int_{\mathbb{R}} \frac{\omega_n(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}\right| \le \frac{K_3}{\gamma}$ and hence by Schwartz' inequality

$$\int_{\mathbb{R}} \frac{\sqrt{\omega_n(x)}\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|} \le \left[\int_{\mathbb{R}} \frac{\omega_n(x)\,\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}\right]^{1/2} \le \sqrt{\frac{K_3}{\gamma}}.$$

It follows from Corollary 3.6 that in this case $\int_{-\infty}^{\infty} \frac{\ln\omega(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2} < \infty$.

4 Growth estimates in the finite case

We continue to assume that the moment problem on $\mathcal{R}_\infty$ is indeterminate. Let α be a fixed point in G. For the sake of simplicity we formulate the results and carry out the arguments only for the case $\alpha \ne \infty$. By adapting the arguments given in this section, estimates in appropriate form can be proved also in the case $\alpha=\infty$. In the following β shall denote an arbitrary point in G.

We set $m(z)=\max\{1,D(z,x_0)\}$, $p(z)=\max\{1,H(z,x_0)\}$,
$q=\max\left\{[\sum_{k=1}^{\infty}\varphi_k(x_0)^2]^{1/2},[\sum_{k=1}^{\infty}\psi_k(x_0)^2]^{1/2}\right\}$, and define

$$\Phi(t)=\ln\left[p(t)\left\{m(t)+q\sqrt{\omega(t)}\right\}\right],\quad \Psi(t)=\ln\left[p(t)\left\{m(t)+q\sqrt{\Omega(t)}\right\}\right].$$

It follows from Theorem 3.10 (the Riesz type criterion) that

$$\int_{-\infty}^{\infty}\frac{\Phi(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2}<\infty \quad\text{and}\quad \int_{-\infty}^{\infty}\frac{\Psi(t)\,\mathrm{d}t}{|t-\zeta_\beta|^2}<\infty \quad \text{for any } \beta\in\mathbb{R}. \tag{4.1}$$

Note that $\Phi(t)\geq 0$ and $\Psi(t)\geq 0$ for all $t\in\mathbb{R}$.

Let $\eta\in(0,\pi/2)$. We introduce the notation

$$\Delta(\alpha,\eta)=\{z\in\mathbb{C}:\eta\leq|\arg(z-\alpha)|\leq\pi-\eta\}.$$

As usual we set $z=x+y\mathrm{i}$.

Lemma 4.1 *Assume that f is a non-negative function on $\mathbb{R}$ satisfying $\int_{-\infty}^{\infty}\frac{f(t)\,\mathrm{d}t}{|t-\zeta_\alpha|^2}<\infty$ for some $\alpha\in\mathbb{R}$. Then for every $\varepsilon>0$ there exists a disk U_α with center at α such that*

$$\frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}<\frac{\varepsilon}{|z-\alpha|} \tag{4.2}$$

for $z\in U_\alpha\cap\Delta(\alpha,\varepsilon)$.

PROOF. We have $\frac{y^2 f(t)}{|t-z|^2}\leq f(t)$ a.e., and $\int_{|t-\alpha|\leq\gamma}f(t)\,\mathrm{d}t<\infty$ since $|t-z|^2$ is bounded for $|t-\alpha|\leq\gamma$. Hence it follows by Lebesgue's dominated convergence theorem, that

$$\frac{y^2}{\pi}\int_{|t-\alpha|\leq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}\underset{y\to 0}{\longrightarrow}0,\quad\text{hence}\quad \frac{y^2}{\pi}\int_{|t-\alpha|\leq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}<\frac{\varepsilon}{2}\sin\eta \tag{4.3}$$

for $|y|$ sufficiently small.

For $z\in\Delta(\alpha,\eta)$ we have $|t-z|^2\geq|t-\alpha|^2\sin^2\eta$. For $|t-x|\geq\gamma$ this implies $|t-x|^2\geq$

$\frac{1}{2}|t-z|^2\sin^2\eta$. Hence

$$\frac{y^2}{\pi}\int\limits_{|t-\alpha|\geq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}\leq\frac{2y^2}{\pi\sin^2\eta}\int\limits_{|t-\alpha|\geq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}.$$

Since $\int_{|t-\alpha|\geq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}<\infty$ we find that

$$\frac{y^2}{\pi}\int\limits_{|t-\alpha|\geq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}\underset{y\to 0}{\longrightarrow}0,\quad\text{hence}\ \frac{y^2}{\pi}\int\limits_{|t-\alpha|\leq\gamma}\frac{f(t)\,\mathrm{d}t}{|t-z|^2}<\frac{\varepsilon}{2}\sin\eta \tag{4.4}$$

for $|y|$ sufficiently small.

We have $|z-\alpha|\sin\eta<|y|$ when $z\in\Delta(\alpha,\eta)$, and so (4.2) follows from (4.3) and (4.4). $\square$

Proposition 4.2 *Assume that G is finite, and let $\alpha\in G$. Let V_α be a disk with center at α and let F denote any of the functions A, B, C, D. Then there exists for every $\varepsilon>0$ a constant $M_1(\varepsilon,\eta)$ such that*

$$|F(z)|\leq M_1(\varepsilon,\eta)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\} \tag{4.5}$$

for $z\in V_\alpha\cap\Delta(\alpha,\eta)$.

PROOF. Let H_n denote any of the functions B_n, D_n. It follows by Schwartz' inequality that $|H_n(t)|\leq p(t)[m(t)+q\sqrt{\omega(t)}]$, hence $\ln|H_n(t)|\leq\Phi(t)$ for $t\in\mathbb{R}$.

Recall that all the zeros and poles of H_n are real.

From Poisson's formula and Lemma 3.1, applied to the rational function $H_n(t)$ with all poles on $\mathbb{R}$, we find

$$\ln|H_n(z)|=\frac{|y|}{\pi}\int\limits_{-\infty}^{\infty}\frac{\ln|H_n(t)|\,\mathrm{d}t}{|t-z|^2},\quad\text{for } z\notin\mathbb{R}$$

and hence

$$\ln|H_n(z)|\leq\frac{|y|}{\pi}\int\limits_{-\infty}^{\infty}\frac{\Phi(t)\,\mathrm{d}t}{|t-z|^2}.$$

It now follows from Lemma 4.1 that $\ln|H_n(z)| \leq \frac{\varepsilon}{|z-\alpha|}$ for $z \in U_\alpha \cap \Delta(\alpha,\eta)$, where U_α is sufficiently small. In $(V_\alpha \setminus U_\alpha) \cap \Delta(\alpha,\eta)$ we have $\ln|H_n(z)| \leq M_1^*(\varepsilon,\eta)$ where the constant $M_1^*(\varepsilon,\eta)$ is independent of n, since H_n is uniformly convergent in $(V_\alpha \setminus U_\alpha) \cap \Delta(\alpha,\eta)$. Thus

$$\ln|H_n(z)| \leq M_1^*(\varepsilon,\eta) + \frac{\varepsilon}{|z-\alpha|}$$

in $V_\alpha \cap \Delta(\varepsilon,\eta)$ for all n.

Simiarly let G_n denote any of the functions A_n, C_n. Then $\ln|G_n(t)| \leq \Psi(t)$ for all $t \in \mathbb{R}$, and all the zeros and poles of G_n are real. It follows from (4.1) by the same kind of reasoning as above that there exists a constant $M_1^{**}(\varepsilon,\eta)$ such that

$$\ln|G_n(z)| \leq M_1^{**}(\varepsilon,\eta) + \frac{\varepsilon}{|z-\alpha|} \quad \text{for } z \in V_\alpha \cap \Delta(\alpha,\eta).$$

Setting $M_1(\varepsilon,\eta) = \max\{\exp[M_1^*(\varepsilon,\eta)], \exp[M_1^{**}(\varepsilon,\eta)]\}$, we obtain (4.5). ☐

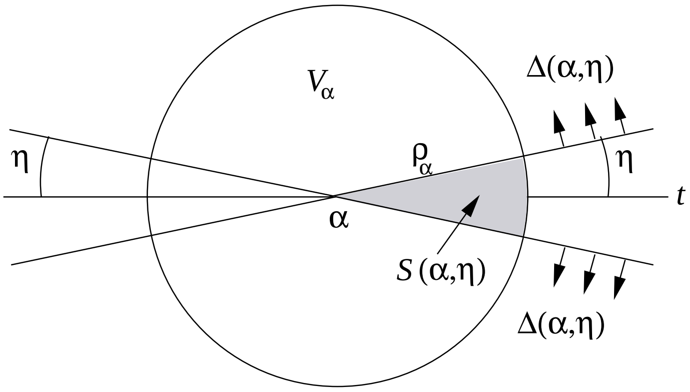

Fig. 1. Elements appearing in the proof of Proposition 4.3

Proposition 4.3 *Assume that G is finite, $\infty \notin G$. Let $\alpha \in G$ and let V_α be a disk with center at α containing no other point in G. Then for every $\varepsilon > 0$ there exists a constant $M_2(\varepsilon,\eta)$ such that*

$$|F(z)| \leq M_2(\varepsilon,\eta) \exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}$$

for $z \in V_\alpha \cap [\mathbb{C} \setminus \Delta(\alpha,\eta)]$ where F is any of the functions A, B, C, D.

PROOF. Let ρ_α denote the radius of V_α and let $S(\alpha,\eta)$ denote the sector of $V_\alpha \cap [\mathbb{C} \setminus \Delta(\alpha,\eta)]$ lying to the right of α. According to Proposition 4.2 there is for every $\varepsilon > 0$ a constant

$M_1(\varepsilon\cos\eta,\eta)$ such that $|F(z)| \le M_1(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\}$ for $z \in V_\alpha \cap \Delta(\alpha,\eta)$, hence a constant $M_2^*(\varepsilon\cos\eta,\eta)$ such that

$$|F_n(z)| \le M_2^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\} \tag{4.6}$$

for $z \in V_\alpha \cap \Delta(\alpha,\eta)$, where F_n is any of the functions A_n, B_n, C_n, D_n.

We now consider z in the closure $\overline{S}(\alpha,\eta)$ of the sector $S(\alpha,\eta)$. The function

$$Q_n(z) = F_n(z)\exp\left\{-\frac{\varepsilon}{z-\alpha}\right\}$$

is holomorphic in $\overline{S}(\alpha,\eta)\setminus\{\alpha\}$. We have

$$|Q_n(z)| = |F_n(z)|\exp\left\{-\varepsilon\ \mathrm{Re}\ \frac{1}{z-\alpha}\right\}, \tag{4.7}$$

hence by (4.6)

$$|Q_n(z)| \le M_2^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\}\cdot\exp\left\{-\varepsilon\frac{x-\alpha}{|z-\alpha|^2}\right\}. \tag{4.8}$$

Let z be a point on one of the line segments of the boundary $\partial S(\alpha,\eta)$. Then $x-\alpha = |z-\alpha|\cos\eta$, hence

$$|Q_n(z)| \le M_2^*(\varepsilon\cos\eta,\eta). \tag{4.9}$$

Next let z be a point on the circular arc of $\partial S(\alpha,\eta)$. Then we have

$$\begin{aligned}|Q_n(z)| &\le M_2^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{\rho_\alpha}\right\}\exp\left\{-\frac{\varepsilon(x-\alpha)}{\rho_\alpha^2}\right\}\\ &= M_2^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon}{\rho_\alpha^2}\left[\rho_\alpha\cos\eta-(x-\alpha)\right]\right\},\end{aligned}$$

hence

$$|Q_n(z)| \le M_2^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{\rho_\alpha}\right\}.$$

Thus there is a constant $M_3^*(\varepsilon\cos\eta,\eta)$ such that

$$|Q_n(z)| \le M_3^*(\varepsilon\cos\eta,\eta) \quad \text{for} \quad z \in \partial S(\alpha,\eta)\setminus\{\alpha\}.$$

Recall that F_n is a rational function. Therefore there exists for every $\varepsilon > 0$ a constant $k_n(\varepsilon)$ such that $|F_n(z)| \le k_n(\varepsilon)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\}$ for $z \in V_\alpha$. For $z \in S(\alpha,\eta)\setminus\{\alpha\}$ we have $|x-\alpha| \ge |z-\alpha|\cos\eta$. Hence

$$|Q_n(z)| \le k_n(\varepsilon)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\}\exp\left\{-\varepsilon\frac{x-\alpha}{|z-\alpha|^2}\right\} \le k_n(\varepsilon)$$

for $z \in \overline{S}(\alpha,\eta)\setminus\{\alpha\}$. It follows that

$$\limsup_{\substack{z\to\alpha \\ z\in S(\alpha,\eta)\setminus\{\alpha\}}} |Q_n(z)| \le k_n(\varepsilon) < \infty.$$

Then, according to a version of the maximum principle (see for example [15, Part II, p.208]) we have

$$|Q_n(z)| \le M_3^*(\varepsilon\cos\eta,\eta) \quad \text{for} \quad z \in S(\alpha,\eta),$$

and hence, according to (4.7)

$$|F_n(z)| \le M_3^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}, \quad \text{for} \quad z \in S(\alpha,\eta). \tag{4.10}$$

In the same way we find an estimate

$$|F_n(z)| \le M_3^{**}(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}, \tag{4.11}$$

for z in the sector of $V_\alpha \cap [\mathbb{C}\setminus\Delta(\alpha,\eta)]$ to the left of α.

Letting n tend to infinity in (4.10)-(4.11) and combining the resulting inequalities, the proof is completed. □

Theorem 4.4 *Assume that G is finite, $\infty \notin G$. Let $\alpha \in G$ and let V_α be a disk with center at α containing no other point of G. Then for every $\varepsilon > 0$ there exists a constant $M(\varepsilon)$ such that*

$$|F(z)| \le M(\varepsilon)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}$$

for all $z \in V_\alpha$, where F is any of the functions A, B, C, D.

PROOF. Choose a fixed $\eta = \eta_0$, and define $M(\varepsilon) = \max\{M_1(\varepsilon,\eta_0), M_2(\varepsilon,\eta_0)\}$. The result then follows from Proposition 4.3. □

Remark 4.5 For fixed $t \in \hat{\mathbb{R}}$, the rational function

$$T_n(z,t) = -\frac{A_n(z)t - C_n(z)}{B_n(z)t - D_n(z)}$$

has a partial fraction decomposition of the form

$$T_n(z,t) = \sum_{k=1}^{n} \lambda_{n,k}(t) \frac{1 + \xi_{n,k}(t)z}{\xi_{n,k}(t) - z},$$

with $\xi_{n,k} \in \mathbb{R}$ and $\lambda_{n,k} > 0$ for $k = 1, \ldots, n$, and $\sum_{k=1}^{n} \lambda_{n,k}(t) = 1$. (See [2, Sections 10-11], [8, Sction 11.10].) Since $|\xi_{n,k}(t) - z| \geq |y| \geq |z - \alpha| \sin \eta$ for $z \in \Delta(\alpha, \eta)$, we find $|T_n(z,t)| \leq \frac{1+|z|^2}{|z-\alpha|} + |z|$. In particular

$$\left|\frac{A_n(z)}{B_n(z)}\right| \leq \frac{1 + |z|^2}{|z - \alpha|} + |z|, \quad \left|\frac{C_n(z)}{D_n(z)}\right| \leq \frac{1 + |z|^2}{|z - \alpha|} + |z|,$$

for $z \in \Delta(\alpha, \eta)$. Since $\frac{1}{|z-\alpha|} \leq m \exp\{\frac{\varepsilon'}{|z-\alpha|}\}$ for arbitrary $\varepsilon' > 0$ and suitable m, we conclude that if B and D satisfy an estimate of the form $|F(z)| \leq M \exp\{\frac{\varepsilon}{|z-\alpha|}\}$ for $z \in \Delta(\alpha, \eta)$ for some M, then also A and C do. Hence the result of Proposition 4.2 can be obtained without making use of the Riesz type criterion $\int_{-\infty}^{\infty} \frac{\ln \Omega(t)}{|t-\zeta_\beta|^2} \,\mathrm{d}t < \infty$ (see (3.9)). The result in Theorem 4.4 can thus be established without use of this criterion.

Remark 4.6 When G consists of the only point ∞ (i.e. when $\alpha_k = \infty$ for all k), the functions F_n are polynomials with all zeros in $\mathbb{R}$. Hence $|F_n(z)|$ is an increasing function of y, and an estimate of the form $|F_n(z)| \leq M(\varepsilon, \eta) \exp\{\varepsilon |z|\}$ in $\{z \in \mathbb{C} : \eta \leq |\arg z| \leq \pi - \eta\}$ can easily be extended to an estimate of the same kind in the whole plane. See e.g. [1, Chap. 2]. An argument of this kind is not possible in the general case.

Remark 4.7 If α is an isolated point in G and there is only a finite number m of elements α_k in some Γ_α, then F has a pole of order m at α. Thus in a neighborhood V_α we have in this case the stronger estimate $|F(z)| \leq \tilde{M}(\varepsilon)|z - \alpha|^{-m}$.

References

[1] N.I. Akhiezer. *The classical moment problem and some related questions in analysis.* Hafner, New York, 1965.

[2] A. Almendral Vazquez. Nevanlinna parametrization of the solutions of some rational moment problems. *Analysis*, 23:107–124, 2003.

[3] C. Berg. Indeterminate moment problems and the theory of entire functions. *J. Comput. Appl. Math.*, 65:27–55, 1996.

[4] C. Berg and H.L. Pedersen. On the order and type of entire functions assiciated with an indeterminate Hamburger moment problem. *Ark. Math.*, 32:1–11, 1998.

[5] H. Buchwalter and C. Cassier. La paraméterization de Nevanlinna dans le problème des moments de Hamburger. *Expo. Math.*, 2:155–178, 1984.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions with poles on the unit circle. *J. Math. Anal. Appl.*, 182:221–243, 1994.

[7] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A rational moment problem on the unit circle. *Methods Appl. Anal.*, 4(3):283–310, 1997.

[8] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999. (407 pages).

[9] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions on the real half line with poles in $[-\infty, 0]$. *J. Comput. Appl. Math.*, 179(1-2):121–155, 2005.

[10] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. An indeterminate rational moment problem and Carathéodory functions. *J. Comput. Appl. Math.*, 219(2):359–369, 2008. Published on line 11-July-2008.

[11] H. Hamburger. Ueber eine Erweiterung des Stieltjesschen Moment Problems I. *Math. Ann.*, 81:235–319, 1920.

[12] H. Hamburger. Ueber eine Erweiterung des Stieltjesschen Moment Problems II. *Math. Ann.*, 82:120–164, 1921.

[13] H. Hamburger. Ueber eine Erweiterung des Stieltjesschen Moment Problems III. *Math. Ann.*, 82:168–187, 1921.

[14] W.B. Jones, O. Njåstad, and W.J. Thron. Orthogonal Laurent polynomials and the strong Hamburger moment problem. *J. Math. Anal. Appl.*, 98:528–554, 1984.

[15] A. Markushevich. *Theory of functions of a complex variable*, volume 2. Chelsea Publishing Co., New York, 1967.

[16] R. Nevanlinna. Über beschränkte analytische Funktionen. *Ann. Acad. Sci. Fenn. Ser. A.*, 32(7):75pp., 1929.

[17] O. Njåstad. Solutions of the strong Hamburger moment problem. *J. Math. Anal. Appl.*, 197:227–248, 1996.

[18] O. Njåstad. Riesz criterion for the strong Hamburger moment problem. *Comm. Anal. Th. Continued Fractions*, 7:34–50, 1999.

[19] O. Njåstad. Nevanlinna matrices for the strong Hamburger moment problem. *Rocky Mountain J. Math.*, 33(2):475–488, 2003.

[20] O. Njåstad. Orders of Nevanlinna matrices for strong moment problems. *East J. Approx.*, 13:467–481, 2007.

[21] O. Njåstad. Analytic functions associated with strong Hamburger moment problems. *Proc. Edinburgh Math. Soc.*, 52:181–187, 2009.

[22] M. Riesz. Sur le problème des moments I. *Ark. Mat. Astr. Fys.*, 16(12):1–21, 1922.

[23] M. Riesz. Sur le problème des moments II. *Ark. Mat. Astr. Fys.*, 16(19):1–23, 1922.

[24] M. Riesz. Sur le problème des moments III. *Ark. Mat. Astr. Fys.*, 17(16):1–52, 1923.

[25] J.A. Shohat and J.D. Tamarkin. *The problem of moments*, volume 1 of *Math. Surveys.* Amer. Math. Soc., Providence, R.I., 1943.

[26] M.H. Stone. *Linear transformations in Hilbert space and their applications to analysis*, volume 15 of *American Math. Soc. Coll. Publ.* Amer. Math. Soc., New York, 1932.