

# Local solutions to high frequency 2D scattering problems

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*Report TW 524, June 2008*



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We consider the solution of high-frequency scattering problems in two dimensions, modelled by an integral equation on the boundary of the scattering object. We devise a numerical method to obtain solutions on only parts of the boundary with little computational effort. The method incorporates asymptotic properties of the solution and can therefore attain particularly good results for high frequencies. Potential uses of such partial solutions for non-convex objects and multiple scattering configurations are presented and a brief error analysis is included. We show that in the simplest implementation of the method the integral equation reduces to an ordinary differential equation.

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# Local solutions to high frequency 2D scattering problems

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## Abstract

We consider the solution of high-frequency scattering problems in two dimensions, modelled by an integral equation on the boundary of the scattering object. We devise a numerical method to obtain solutions on only parts of the boundary with little computational effort. The method incorporates asymptotic properties of the solution and can therefore attain particularly good results for high frequencies. Potential uses of such partial solutions for non-convex objects and multiple scattering configurations are presented and a brief error analysis is included. We show that in the simplest implementation of the method the integral equation reduces to an ordinary differential equation.

## 1 Introduction

Scattering problems in the domain exterior to a scattering surface are common in the modelling of electromagnetic and acoustic wave phenomena [12, 28]. In these problems, the governing differential equation on the infinite domain surrounding the scattering obstacle can conveniently be replaced by an integral equation on the surface of the scatterer. The new computational domain is bounded and has lower dimension, two significant advantages for computational purposes. However, the numerical simulation of highly oscillatory phenomena remains a challenging problem. Traditional methods require a fine discretization in order to resolve the oscillations. Due to phase errors, the degrees of freedom per wavelength (per dimension) may grow even faster than linearly with the frequency [3]. This means that scattering problems rapidly become intractable as the frequency grows.

Although the situation improves with high-order methods for integral equations [7, 16], the efficient implementation of fast multipole methods [30] or other approaches (see, for example, the overview in [5]), the computational complexity of these methods still increases with increasing frequency. Asymptotic methods on the other hand, such as Geometrical Optics, Physical Optics and the Geometric Theory of Diffraction [24], typically improve

with increasing frequency [15]. They exhibit  $\mathcal{O}(k^{-n})$  error behaviour, where  $k$  is the wavenumber of the problem and  $n$  is a positive integer, though usually  $n = 1$ . Asymptotic methods have uncontrollable error for a fixed frequency however and, depending on the method, may exhibit breakdown near caustics or points of diffraction.

Recent research focuses on a combined, hybrid approach in the construction of converging boundary element methods that incorporate asymptotic properties of the solution (see [1, 6, 11, 13], reviewed in [9]). This approach is most successful for scatterers with convex shape, but extends to multiple scattering configurations [14, 2]. Some of the methods may be implemented with a computational complexity that appears to be independent of the wavenumber for single convex obstacles [6, 22] and multiple convex obstacles [17]. Although applicability is limited by the convexity requirements, the wavenumber independence motivates further research in this area.

Integral operators, being global operators, give rise to dense matrices upon discretisation. This is however not compatible with the localisation principle of high-frequency scattering problems, which states that the solution of the problem essentially depends on local properties of the scatterer. The work in this paper was motivated by the observation that the method in [22] gives rise to a discretization matrix that is largely sparse. Moreover, this method achieves high asymptotic order of accuracy in large parts of the computational domain. These two features are unique among the hybrid methods currently described in literature and represent a numerical manifestation of the localisation principle. The result was obtained through a discretization based on Filon-type quadrature for highly oscillatory integrals [23] and does not require the construction of asymptotic expansions.

In this paper, we significantly extend the results and analysis of [22]. We restrict the computational domain to only a part of the boundary and we thereby relax convexity requirements. It turns out that, perhaps surprisingly, the integral equation in this setting reduces to an ordinary differential equation that is a singular perturbation problem. This equation can be solved rapidly and to essentially arbitrary high asymptotic order of accuracy. The main achievement of the method is the computation of single reflections very efficiently and to high accuracy. This is by no means competitive with more general methods capable of treating arbitrary scattering configurations. Yet, owing to its simplicity and efficiency we believe it to be valuable in a range of applications, specified in more detail in §5. Moreover, we believe the reduction of a two-dimensional partial differential equation to a lower-dimensional integral equation and subsequently to a univariate ordinary differential equation to be interesting in its own right.

The layout of the paper is as follows. We start with a formal, non-theoretical description of the method in §2. The description differs from the method in [22] mostly in that the computational domain is truncated, which greatly simplifies computations. We analyze the asymptotic properties of

our approach in §3. We give numerical results of a number of scattering problems in §4 and we highlight a number of possible applications in §5. Finally, we end with some concluding remarks in §6.

## 2 Description of the method

### 2.1 Problem statement

Consider a scattering object  $\Omega \in \mathbb{R}^2$  with a smooth boundary  $\Gamma = \partial\Omega$ . Our aim is to solve the two-dimensional Helmholtz equation in the exterior space with Dirichlet boundary conditions,

$$\begin{aligned} \Delta u + k^2 u &= 0, & \mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}, \\ u(\mathbf{x}) &= 0, & \mathbf{x} \in \Gamma. \end{aligned} \quad (2.1)$$

Dirichlet boundary conditions correspond to a perfectly reflecting object, but Neumann boundary conditions can equally be considered with only minor modifications. The solution to the scattering problem (2.1) can be written as the sum of the incoming wave  $u^i$  and a scattered wave  $u^s$ ,  $u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x})$ . Due to the linearity of the problem, the function  $u^s$  itself satisfies the Helmholtz equation with the boundary condition

$$u^s(\mathbf{x}) = -u^i(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

We can represent the unknown scattered wave with the *single-layer potential*

$$u^s(\mathbf{x}) = (Sq)(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) ds_{\mathbf{y}}, \quad (2.2)$$

Here  $q$  is the single-layer potential *density* and

$$G(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \quad (2.3)$$

is the Green's function of the two-dimensional Helmholtz equation. It is well known that the density function can be found from the integral equation of the first kind [12, 28]

$$\int_{\Gamma} G(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) ds_{\mathbf{y}} = -u^i(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \quad (2.4)$$

Other integral formulations exist, and we note in particular the integral equation of the second kind

$$\frac{q(\mathbf{x})}{2} + \int_{\Gamma} \left( \frac{\partial G}{\partial n_x}(\mathbf{x}, \mathbf{y}) + i\eta G(\mathbf{x}, \mathbf{y}) \right) q(\mathbf{y}) ds_{\mathbf{y}} = \frac{\partial u^i}{\partial n}(\mathbf{x}) + i\eta u^i(\mathbf{x}), \quad (2.5)$$

with  $\eta \in \mathbb{R}$  a coupling parameter. Equation (2.5) is uniquely solvable for all values of the wavenumber  $k$  [12]. We proceed with equation (2.4) for simplicity of presentation, but the method was implemented for both equations. In the following, we assume a plane wave incidence or, more generally, an incoming wave of the form

$$u^i(\mathbf{x}) = -f(\mathbf{x})e^{ikg^i(\mathbf{x})}. \quad (2.6)$$

## 2.2 Phase extraction and high frequency formulation

As the wavenumber  $k$  increases the single-layer potential density  $q$  becomes increasingly oscillatory. In order to avoid a fine discretization grid, we will assume knowledge of the oscillatory behaviour of  $q$ . The main reason for the restriction to convex obstacles in literature is that the phase of the solution is then known to be the phase of the incoming wave. Let us write, for the time being indeed assuming a convex obstacle,

$$q(\mathbf{x}) = k q_s(\mathbf{x}) e^{ikg^i(\mathbf{x})}. \quad (2.7)$$

The function  $q_s$  is non-oscillatory in the part of the computational domain that is ‘lit’ by the incoming wave, and slowly oscillatory but rapidly decaying in the remainder [13]. Thus, a coarser discretization can be used for  $q_s$  than for  $q$ . We introduced the normalizing factor  $k$  such that  $q_s = \mathcal{O}(1)$ ,  $k \rightarrow \infty$ .

Assume we have a smooth periodic parametrisation of  $\Gamma$

$$\kappa(\tau) : [0, 1] \rightarrow \Gamma,$$

with  $|\nabla\kappa(\tau)| > 0$ . The integral equation (2.4) in terms of  $q_s$  becomes

$$(Kq_s)(\tau) = \int_0^1 K(t, \tau) q_s(\tau) d\tau = f(t), \quad t \in [0, 1], \quad (2.8)$$

with

$$K(t, \tau) = k G(\kappa(t), \kappa(\tau)) e^{ik[g^i(\kappa(\tau)) - g^i(\kappa(t))]} |\nabla\kappa(\tau)| \quad (2.9)$$

Note that in a slight abuse of notation we identified  $q_s(\kappa(\tau))$  with  $q_s(\tau)$  and  $f(\kappa(t))$  with  $f(t)$ .

The total phase of the integrand in (2.8) is given by the sum of the extracted phase of the solution and the phase of the Green’s function (2.3). It can be written as  $ikg_{tot}(\tau; t)$  with the *oscillator*  $g_{tot}$  given by

$$g_{tot}(\tau; t) = g^i(\kappa(\tau)) - g^i(\kappa(t)) + d(t, \tau), \quad (2.10)$$

with  $d(t, \tau) := |\kappa(t) - \kappa(\tau)|$  the Euclidian distance between the two corresponding points on  $\Gamma$  [22]. This oscillator function plays an important role later on.

### 2.3 Local solutions

The concept of local solutions of high frequency scattering problems is based on the fact that local reflections are based only on local data, as described by the localisation principle. First we give the precise definition of an admissible part of the boundary.

**Definition 2.1.** *Consider a simply connected open subset  $\tilde{\Gamma} \subset \Gamma$  and a given incoming wave  $u^i$ . We say that  $\tilde{\Gamma}$  admits a local solution if*

$$\frac{\partial g_{tot}}{\partial \tau}(\tau; t) \neq 0, \quad \forall t, \tau \in \kappa^{-1}(\tilde{\Gamma}). \quad (2.11)$$

The definition excludes so-called *stationary points* of the integral  $Ku$  (see [23] and §2.4 below). In practice, this excludes cases where waves are reflected from points in  $\tilde{\Gamma}$  onto  $\tilde{\Gamma}$ . It does not however exclude cases where waves are reflected from  $\Gamma \setminus \tilde{\Gamma}$  onto  $\tilde{\Gamma}$ . In the latter case, the phase of the solution is not given simply by the phase of the incoming wave. Yet, we can still define a local solution that has this property and that disregards double reflections. Such solutions are useful in certain applications, examples of which are given in §5.

Let the interval  $[a, b]$  be contained in  $\kappa^{-1}(\tilde{\Gamma})$ , where  $\tilde{\Gamma}$  admits a local solution according to definition 2.1. This will be our computational domain. Define a cut-off function  $\phi_{[a,b]}$  which is one on a neighbourhood of  $[a, b]$  and drops smoothly off to zero such that the support of  $\phi_{[a,b]}$  is contained in  $\tilde{\Gamma}^1$ . Define the integral operator  $\tilde{K}$  as follows,

$$(\tilde{K}u)(t) = \int_0^1 K(t, \tau) u(\tau) \phi_{[a,b]}(\tau) d\tau. \quad (2.12)$$

Now assume that a function  $\tilde{q}$  exists that satisfies

$$(\tilde{K}\tilde{q})(t) = f(t), \quad t \in [a, b]. \quad (2.13)$$

We will call such  $\tilde{q}$  a *local solution* to the integral equation (2.8) on  $[a, b]$ . If  $\Omega$  is a convex object, then the local solution is asymptotic to the true solution of (2.8) on the interval  $[a, b]$ , in the sense that

$$q(t) - \tilde{q}(t) = \mathcal{O}(k^{-M}), \quad \forall M, \quad t \in [a, b].$$

This is again due to the localization principle, as  $\tilde{K}$  preserves the local geometry of the scatterer.

For a non-convex obstacle the local solution does not necessarily agree with the true solution. However, the local solution can be thought of as asymptotic to the solution of a related scattering problem. In many cases

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<sup>1</sup>Note that this cut-off function is not actually implemented in the numerical method. It is merely introduced here for convenience of presentation.

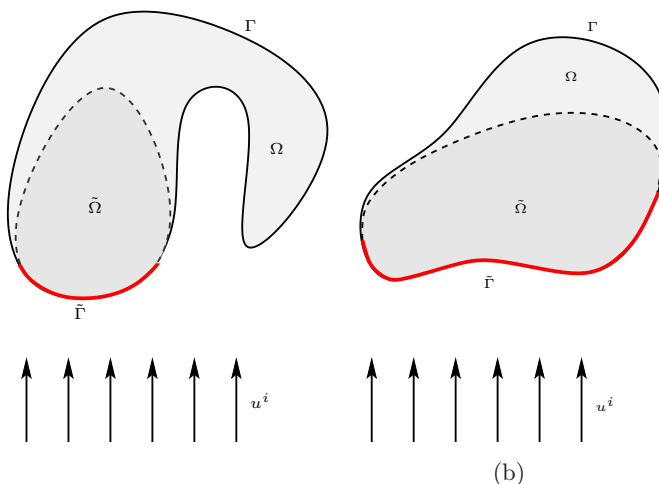


Figure 1: Examples of non-convex domains. The highlighted part  $\tilde{\Gamma}$  of the boundary  $\Gamma$  admits a local solution. The dashed curve indicates an extension of  $\tilde{\Gamma}$  to the boundary of an obstacle  $\tilde{\Omega}$ . The solution of the scattering problem involving  $\tilde{\Omega}$  is asymptotic to the local solution on  $\tilde{\Gamma}$ .

$\tilde{\Gamma}$  can be extended such that it becomes the boundary of a convex scatterer  $\tilde{\Omega}$  (see figure 1(a) for an illustration). But Definition 2.1 does not imply that this is always possible, because  $\tilde{\Gamma}$  itself is not necessarily convex. In the general case, consider an extension of  $\tilde{\Gamma}$  to a shape such that condition (2.11) holds in the illuminated part of  $\tilde{\Omega}$  (as illustrated in figure 1(b)). This construction excludes multiple reflections in the scattering problem. The phase of the solution can therefore be predicted by the phase of the incoming wave and the local solution is asymptotic to the true solution involving the scatterer  $\tilde{\Omega}$ , just as in the convex case.

## 2.4 Filon-type quadrature

Filon-type quadrature [23] is an effective quadrature approximation  $Q[f]$  for the evaluation of oscillatory integrals  $I[f]$  and has the general form

$$I[f] := \int_a^b f(x) e^{ikg(x)} dx \approx Q[f] := \sum_{m=1}^M \sum_{j=0}^{J_m} w_{m,j} f^{(j)}(x_m). \quad (2.14)$$

The quadrature points  $x_m$  include the endpoints  $a$  and  $b$  as well as the *stationary points*, these are all points  $\xi \in (a, b)$  such that  $g'(\xi) = 0$ . The method is based on polynomial interpolation of the non-oscillatory function  $f$ , which ensures possible convergence for any value of  $k$  simply by adding quadrature points. The use of derivatives of  $f$  at endpoints and stationary

points also ensures high asymptotic order of accuracy, in the sense that

$$I[f] - Q[f] = \mathcal{O}(k^{-d}), \quad k \rightarrow \infty,$$

where  $d > 0$  depends on the number of derivatives used. We refer the interested reader to [19] for a recent review of Filon-type quadrature and related methods.

The operator  $\tilde{K}$  yields an oscillatory integral of the form (2.14) for each value of  $t$ . It can therefore be approximated by Filon-type quadrature,

$$(\tilde{K}u)(t) \approx \sum_{m=1}^M \sum_{j=0}^{J_m} w_{m,j}(t) u^{(j)}(\tau_m(t)). \quad (2.15)$$

Both the weights and the quadrature points now depend on the value of  $t$ . This approach was suggested in [22], where it was shown that this quadrature approximation too has high asymptotic accuracy.

The approach simplifies considerably in the setting of local solutions. In particular, fewer quadrature points are necessary. Note that the integrand of  $\tilde{K}u$  is smooth except for the singularity of the Green's function when  $t = \tau$ . This point produces a contribution to the asymptotic expansion of the integral [31, 29] and should therefore be included as a quadrature point in Filon-type quadrature. The integrand vanishes identically at the endpoints, whence no contribution can come. Finally, stationary points are excluded by construction according to Definition 2.1: the derivative of the total oscillator is non-zero everywhere on  $[a, b]$ . Thus, the only contributing point is  $\tau = t$  and (2.15) simplifies to

$$(\tilde{K}u)(t) \approx (Q_n u)(t) := \sum_{j=0}^n w_j(t) \tilde{u}^{(j)}(t), \quad t \in [a, b]. \quad (2.16)$$

The quadrature weights or *moments*  $w_j(t)$  are defined as

$$w_j(t) = \frac{1}{j!} \tilde{K}[(t - \cdot)^j], \quad (2.17)$$

such that  $\tilde{K}u = Q_n u$  is exact whenever  $u$  is a polynomial of degree  $n$ . The moments are given by oscillatory integrals themselves, but they can be computed efficiently by a numerical steepest descent method [20]. The asymptotic convergence of the quadrature rule is already suggested by the size of the weights<sup>2</sup>

$$w_j(t) = \mathcal{O}(k^{-j}), \quad k \rightarrow \infty. \quad (2.18)$$

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<sup>2</sup>The asymptotic order of the weights as displayed here is a factor  $k$  larger than what is given in [22] due to the normalizing factor  $k$  in (2.7).

The weights corresponding to higher derivatives rapidly decrease in size.

We replace the integral operator  $\tilde{K}$  by its Filon-type quadrature approximation in equation (2.13). The local method consists of solving the equation

$$(Q_n q_F)(t) = f(t), \quad t \in [a, b], \quad (2.19)$$

where  $Q_n$  was defined by (2.16). We call a solution  $q_F$  to this problem a Filon solution.

## 2.5 Discretization

For the numerical solution of the scattering problem we introduce a discrete space and solve equation (2.19) based on collocation. Let  $V_h$  be a discrete space spanned by a set of  $N$  basis functions  $\{\phi_j\}_{j=1}^N$ . A function in  $V_h$  can be represented by a set of  $N$  coefficients  $\mathbf{d} = \{d_j\}_{j=1}^N$  through the operator

$$L_h : \mathbb{C}^N \rightarrow V_h, \quad L_h \mathbf{d} = \sum_{j=1}^N d_j \phi_j.$$

Furthermore, a set of  $N$  collocation points  $\{t_j\}_{j=1}^N$  is needed and we define the corresponding evaluation operator

$$\Pi_h : W \rightarrow \mathbb{C}^N, \quad \Pi_h f = \{f(t_j)\}_{j=1}^N.$$

With this notation, the discretization matrix is found to be

$$M = \Pi_h Q_n L_h. \quad (2.20)$$

An approximation to the solution of (2.19) is found by solving

$$M \mathbf{c} = \Pi_h f =: \mathbf{b}. \quad (2.21)$$

We denote by

$$q_h(t) = L_h \mathbf{c} \in V_h \quad (2.22)$$

the numerical solution. In our implementation we chose  $V_h$  to be the space of natural splines of order  $s$ . The nodes of the splines were taken as collocation points. We note that the Filon-type quadrature  $Q_n$  requires  $n$  derivatives, thus we require at least  $s \geq n$  in order to properly represent all derivatives in the discrete space.

### 3 Analysis

Numerical results of the method to compute local solutions will be given in §4. The results indicate that the accuracy of the computed solution  $q_h$  increases rapidly with increasing frequency throughout the computational domain. This behaviour is reminiscent of asymptotic expansions, but it is unlike traditional boundary element methods for oscillatory integral equations. In this section we will show that the computed solution indeed has a high asymptotic order of accuracy.

#### 3.1 Asymptotic properties of the solution

The asymptotic behaviour of the slowly oscillatory function  $q_s(t)$  that solves equation (2.8) is well understood, at least for convex domains and for points  $t$  that are in the illuminated region. The first order asymptotic term is commonly known as the Kirchoff approximation or Physical Optics approximation. In the case of an incident plane wave of the form  $e^{ik\mathbf{x}\cdot\hat{\mathbf{a}}}$ , and assuming a natural parameterization  $\kappa(t)$  of  $\Gamma$ , it is given explicitly by

$$q_s(t) \sim 2i\nu(\kappa(t)) \cdot \hat{\mathbf{a}}, \quad k \rightarrow \infty, \quad (3.1)$$

where  $\nu(\mathbf{x})$  is the exterior normal to  $\Gamma$  at  $\mathbf{x} \in \Gamma$ . See for example [13, §5] and references therein, in particular [27]. More generally, for smooth scatterers the solution  $q_s(t)$  has a full asymptotic expansion of the form

$$q_s(t) \sim \sum_{j=0}^{\infty} d_j(t) k^{-j}, \quad k \rightarrow \infty, \quad (3.2)$$

for all points  $t$  in the illuminated region. The functions  $d_j(t)$  are independent of  $k$ . They can be computed explicitly, at least in principle, for all sufficiently smooth scatterers. An example of such computation is given in [8]. It follows from this expansion that  $q_s(t)$  has *bounded derivatives*, in the sense that

$$q_s^{(j)}(t) = \mathcal{O}(1), \quad k \rightarrow \infty, \quad (3.3)$$

for all points  $t$  in the illuminated region [13].

#### 3.2 Existence of the local solution

Recall the property (2.13),

$$(\tilde{K}\tilde{q})(t) = f(t), \quad t \in [a, b],$$

that should be satisfied by the local solution  $\tilde{q}$ . It is not obvious that a solution of this equation exists, nor that it would be unique. The equation indeed seems to be ill-posed. However, it is sufficient for our purposes to obtain a solution that is only asymptotically close to satisfying this equation.

It follows from construction with the cut-off function that

$$(Ku)(t) - (\tilde{K}u)(t) = \mathcal{O}(k^{-M}), \quad \forall M, \quad t \in [a, b], \quad k \rightarrow \infty, \quad (3.4)$$

for any smooth and  $k$ -independent function  $u$ . We can replace pointwise estimates for  $t \in [a, b]$  by

$$\|Ku - \tilde{K}u\|_\infty = \mathcal{O}(k^{-M}),$$

with  $\|\cdot\|_\infty$  the maximum norm on  $[a, b]$ . Consider a convex obstacle and denote the truncation of expansion (3.2) after  $J$  terms by

$$q_s^A = \sum_{j=0}^{J-1} d_j(t) k^{-j}.$$

Summing term by term and using (3.4), we have

$$\|Kq_s^A - \tilde{K}q_s^A\|_\infty = \mathcal{O}(k^{-M}).$$

Moreover, from  $\|K\|_\infty = \mathcal{O}(k^{3/2})$  [10] and the corresponding bound of  $\tilde{K}[u] = K[\phi_{ab}u]$ , we have<sup>3</sup>

$$\|Kq_s^A - Kq_s\|_\infty = \mathcal{O}(k^{-J+3/2}), \quad \text{and} \quad \|\tilde{K}q_s^A - \tilde{K}q_s\|_\infty = \mathcal{O}(k^{-J+3/2}),$$

for any  $J$ , so that

$$\|\tilde{K}q_s - f\|_\infty = \mathcal{O}(k^{-M}), \quad \forall M, \quad k \rightarrow \infty.$$

It follows that the true solution of the convex scattering problem is asymptotically close to satisfying equation (2.13). For non-convex obstacles, in the following one can think of  $\tilde{q}$  as part of the true solution of a related scattering problem as illustrated in Figure 1.

We note that replacing  $K$  by  $\tilde{K}$  results in an asymptotic approximation, because the local solution is meaningful only for sufficiently large  $k$ . We will show in the numerical examples that onset of asymptotic behaviour is quite rapid, and that the wavenumber can be quite moderate.

### 3.3 Filon-type approximation of the integral operator

It is well known, from asymptotic analysis [31] and indirectly from Geometrical Optics computations [8], that for any fixed  $t$  the integral  $(Ku)(t)$  depends on derivatives of  $u$  at the contributing point  $\tau = t$ . Derivatives of higher order produce contributions of smaller asymptotic size. Indeed,

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<sup>3</sup>The bound given is a factor  $k$  larger than that in [10] due to the factor  $k$  appearing in (2.7). It holds for Lipschitz boundaries  $\Gamma$ . Other bounds hold for the operator in equation (2.5), but we only need here that the bounds are algebraic in  $k$  which is the case.

this is the basic observation underlying the rapid convergence of Filon-type quadrature using derivatives for integrals of the form  $I[f]$  as in (2.14) [23]. It means that we have

$$\|Ku - Q_n u\|_\infty = \mathcal{O}(k^{-n-1}), \quad k \rightarrow \infty. \quad (3.5)$$

Note that in  $Q_n$  the first discarded weight of higher order  $w_{n+1}(t)$  has size  $\mathcal{O}(k^{-n-1})$ , which agrees with (3.5). This estimate holds for functions  $u$  that are independent of  $k$ . By virtue of the known expansion (3.2), we can write

$$\|Kq_s^A - Q_n q_s^A\|_\infty = \mathcal{O}(k^{-n-1}),$$

by summing each of the  $J$  terms in  $q_s^A$  separately and noting that  $d_j(t)$  is independent of  $k$ . The operator  $Q_n$  is bounded *in the wavenumber*  $k$  because all functions  $w_j(t)$  are bounded, so that with  $J \geq n+1$  we obtain

$$\|Kq_s - Q_n q_s\|_\infty = \mathcal{O}(k^{-n-1}). \quad (3.6)$$

The local solution of the scattering problem is close to satisfying the Filon-type equation (2.19), up to asymptotic order  $n+1$ .

Sufficient conditions such that the converse is also true are supplied by the notion of bounded derivatives as the following lemma shows.

**Lemma 3.1.** *Assume that  $q_F(t) \in C^n[a, b]$  satisfies equation (2.19) and*

$$q_F^{(j)}(t) = \mathcal{O}(1), \quad k \rightarrow \infty. \quad (3.7)$$

*Then we have*

$$\|q_F^{(j)} - \tilde{q}^{(j)}\|_\infty = \mathcal{O}(k^{j-n-1}), \quad j = 0, \dots, n. \quad (3.8)$$

*Proof.* We have, for any  $M \in \mathbb{N}$ ,

$$\begin{aligned} (\tilde{K}\tilde{q})(t) - (Q_n \tilde{q})(t) &= f(t) - (Q_n \tilde{q})(t) + \mathcal{O}(k^{-M}) \\ &= Q_n(q_F - \tilde{q})(t) + \mathcal{O}(k^{-M}) \\ &= \sum_{j=0}^n w_j(t) \left[ q_F^{(j)}(t) - \tilde{q}^{(j)}(t) \right] + \mathcal{O}(k^{-M}) \\ &= \mathcal{O}(k^{-n-1}), \quad k \rightarrow \infty. \end{aligned}$$

Both  $q_F$  and  $\tilde{q}$  have  $\mathcal{O}(1)$  derivatives as  $k \rightarrow \infty$ . Thus we can compare terms in the sum in the latter equality. From (2.18),  $w_j = \mathcal{O}(k^{-j})$  implies

$$q_F^{(j)}(t) = \tilde{q}^{(j)}(t) + \mathcal{O}(k^{j-n-1}), \quad j = 0, \dots, n,$$

and the result follows.  $\square$

Any function that satisfies equation (2.19) and that has bounded derivatives is close to  $\tilde{q}$  up to asymptotic order  $n+1$ . Derivatives are accurate to lower asymptotic order.

### 3.4 Discretization error

We will employ basis functions  $\phi_j$  with the following property.

**Assumption 3.2.** For any given  $\varepsilon > 0$  and for any given  $\psi \in C^m[a, b]$ , there exists an  $N$  and a  $v \in V_h := \text{span}\{\phi_j\}_{j=1}^N$  such that

$$\inf_{v \in V_h} \|\psi^{(j)} - v^{(j)}\|_\infty < \varepsilon, \quad j = 0, \dots, n, \quad (3.9)$$

with  $N$  depending only on  $\|\psi^{(j)}\|_\infty$ ,  $j = 0, \dots, n$ .

This assumption holds for many basis functions of practical use, including polynomials and natural splines of sufficiently high degree. Typically  $\varepsilon = \mathcal{O}(h^\alpha)$  for some  $\alpha > 0$ , limited by  $n$ , where  $h \sim 1/N$  represents the grid-size.

The following lemma states that, subject to assumption (3.9) and certain additional conditions, the discretisation will introduce an error that can be made arbitrarily small and that is bounded in  $k$ .

**Theorem 3.3.** *Assume that the linear system of equations (2.21) is solvable for  $k \geq k_0$  and that the basis functions  $\phi_j$  satisfy condition (3.9). For any solution  $q_F$  of (2.19) that satisfies the condition (3.3) of bounded derivatives, we have*

$$\|q_h - q_F\|_\infty < \varepsilon R, \quad k \rightarrow \infty, \quad (3.10)$$

for any  $\varepsilon > 0$  and with  $R < \infty$  independent of  $k$ .

*Proof.* For any given  $\varepsilon > 0$ , denote by  $q_F^*$  an element in  $V_h$  that satisfies condition (3.9) with respect to  $q_F$ , i.e., there exists an  $N$  such that

$$\|q_F^{(j)} - q_F^{*(j)}\|_\infty < \varepsilon \quad j = 0, \dots, n.$$

Due to the property of bounded derivatives (3.3), the constant  $N$  can be chosen independently of  $k$ .

Denote by  $\mathbf{c}^*$  the coefficients of  $q_F^*$  in the basis of  $V_h$ ,  $q_F^* = L_h \mathbf{c}^*$ . For the numerical solution  $q_h$  with coefficients  $\mathbf{c}$ , we have

$$M\mathbf{c} = \mathbf{b} = \Pi_h f = \Pi_h Q_n q_F.$$

The latter equality follows because  $Q_n q_F = f$ . Similarly, we have

$$M\mathbf{c}^* = \Pi_h Q_n L_h \mathbf{c}^* = \Pi_h Q_n q_F^*.$$

Therefore, we have

$$\begin{aligned}
\|\mathbf{c} - \mathbf{c}^*\|_\infty &= \|M^{-1}\Pi_h Q_n(q_F - q_F^*)\|_\infty \\
&= \|M^{-1}\sum_{j=0}^n \Pi_h w_j (q_F^{(j)} - q_F^{*(j)})\|_\infty \\
&\leq \|M^{-1}\|_\infty \sum_{j=0}^n \|\Pi_h w_j\|_\infty \|q_F^{(j)} - q_F^{*(j)}\|_\infty \\
&\leq \varepsilon \|M^{-1}\|_\infty \sum_{j=0}^n \|\Pi_h w_j\|_\infty,
\end{aligned} \tag{3.11}$$

for  $k \geq k_0$ . The weights  $w_j$  are bounded in  $k$  and  $P_h$  is independent of  $k$ , so the sum in (3.11) is bounded. It remains to bound the norm of the inverse of  $M$ .

The entries of  $M$  consist of evaluations of the operator  $Q_n$  applied to the basis functions  $\phi_j$ , evaluated in the collocation points. Since the weights are bounded, each element has size  $\mathcal{O}(1)$ . We can write

$$M = M_0 + \frac{1}{k}\hat{M},$$

where  $M_0$  is a constant matrix and  $\hat{M} = \mathcal{O}(1)$  elementwise,  $k \rightarrow \infty$ . The solution is known to exist asymptotically, so the matrix  $M_0$  is invertible and the Neumann series of  $M^{-1}$  is

$$M^{-1} = (I - \frac{1}{k}M_0^{-1}\hat{M} + \mathcal{O}(k^{-2}))M_0^{-1}.$$

It follows that  $M^{-1} = \mathcal{O}(1)$ . This means that

$$\|\mathbf{c} - \mathbf{c}^*\|_\infty < \varepsilon R^*,$$

where  $R^* < \infty$ . As  $L_h$  is independent of  $k$ , we have

$$\begin{aligned}
\|q_h - q_F\|_\infty &\leq \|q_h - q_F^*\|_\infty + \|q_F^* - q_F\|_\infty \\
&< \varepsilon + \varepsilon R^*,
\end{aligned}$$

and the result follows with  $R = R^* + 1$ .  $\square$

We can now describe the total error in the approximation of the computed solution  $q_h$  to the local solution  $\tilde{q}$ .

**Theorem 3.4.** *Assume that  $\tilde{q}$  satisfies*

$$(\tilde{K}\tilde{q})(t) = f(t) + \mathcal{O}(k^{-M}), \quad k \rightarrow \infty,$$

*and  $\tilde{q}$  has bounded derivatives. Then for any  $\varepsilon_d > 0$ , subject to the conditions of Theorem 3.3, we have*

$$\|q_h - \tilde{q}\|_\infty \leq \varepsilon_d + \mathcal{O}(k^{-n-1}).$$

*Proof.* Let  $q_F$  be a solution to (2.19) with bounded derivatives. We have

$$\begin{aligned}\|q_h - \tilde{q}\|_\infty &\leq \|q_h - q_F\|_\infty + \|q_F - \tilde{q}\|_\infty \\ &\leq \varepsilon R + \mathcal{O}(k^{-n-1}),\end{aligned}$$

and the result follows by choosing  $\varepsilon = \varepsilon_d/R$ .  $\square$

### 3.5 Interpretation as a singular perturbation problem

Equation (2.19) is given explicitly by

$$\sum_{j=0}^n w_j(t) q^{(j)}(t) = f(t), \quad t \in [a, b].$$

This is a one-dimensional ordinary differential equation of order  $n$ . One would expect therefore that at least  $n$  boundary conditions are required in order to have a unique solution. This is problematic, as suitable boundary conditions are not available. Yet, the asymptotic analysis in this section holds for *any* solution  $q_F$  to (2.19), as long as it has bounded derivatives.

This apparent contradiction can be explained by interpreting (2.19) as a singular perturbation problem (see, for example, [4]). The equation  $Q_n q = f$  is a singular perturbation problem of the form

$$\sum_{j=0}^n \varepsilon^j v_j(t) q^{(j)}(t) = f, \quad \varepsilon \ll 1,$$

with  $\varepsilon = 1/k$  and  $v_j(t) = \mathcal{O}(1)$ ,  $\varepsilon \rightarrow 0$ . The solution to such a problem can be written as the sum of a slowly and rapidly varying part,

$$q(t) = q^S(t) + q^R(t), \quad t \in [a, b].$$

The rapidly varying part manifests itself in *boundary layers* near the endpoints of the interval, but we have a smooth solution  $q(t) \sim f(t)/v_0(t)$  everywhere in the interior  $(a, b)$ . In our case we are only interested in the slowly varying part  $q^S$ . Then the *bounded derivative principle* [25] applies: if boundary conditions are chosen such that derivatives up to order  $p$  are bounded in  $\varepsilon$  at the boundary, then the boundary data is correct to order  $p$ . The function  $q^S$  can moreover be determined to any order in the interior.

This is precisely what happens when using a basis of natural splines. Natural splines satisfy homogeneous boundary conditions in high-order derivatives at the endpoints. Using natural splines of odd degree  $s$ , we have

$$\forall v \in V_h : v^{(j)}(a) = v^{(j)}(b) = 0, \quad j = \frac{s+1}{2}, \dots, s-1.$$

Enforcing boundary conditions on higher derivatives independently of  $k$  ensures that lower derivatives are bounded and, hence, by the bounded derivative principle we find the correct solution to high asymptotic order. The

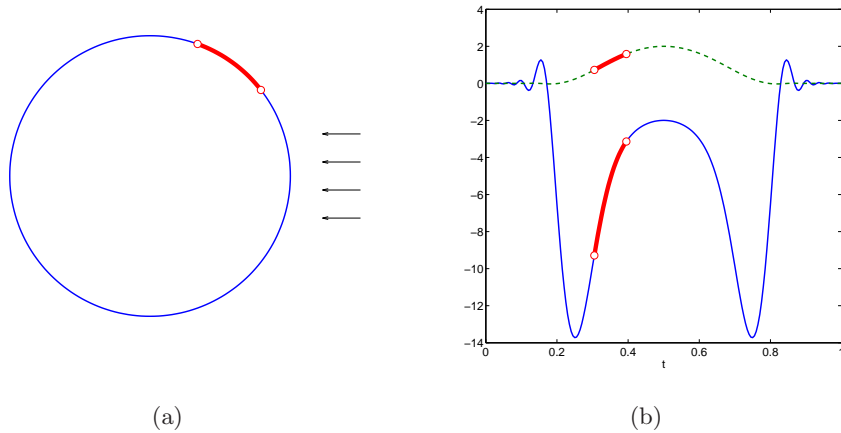


Figure 2: a) The circular obstacle. The part where we seek a local solution is emphasised. b) The exact solution for  $k = 100$  as a function of the parameter  $t \in [0, 1]$ , real part(continuous) and imaginary part, scaled by  $1/k$  (dashed).

solution in the interior has full asymptotic accuracy  $n + 1$  if  $s \geq n$ . We attain order  $n + 1$  near the boundary only if

$$\frac{s + 1}{2} > n \Rightarrow s > 2n - 1. \quad (3.12)$$

Condition (3.12) on the degree of natural splines is also required in order to satisfy assumption (3.9) on the basis functions.

## 4 Numerical results

In this section we describe a number of experiments illustrating the properties of our scheme. In §4.1 we illustrate that the scheme achieves the asymptotic order predicted by the theory and we show that the results are significantly more accurate than asymptotic expansions of comparable order. In §4.2 we illustrate that the scheme degrades gracefully close to the shadow boundaries, i.e., it does not break down. In §4.3 we consider a more general scattering problem. Finally, we illustrate the boundary layers, as discussed in §3.5, with a simple second order finite difference scheme in §4.4.

In all examples we efficiently computed the Filon-type quadrature weights (2.17) using a numerical steepest descent method [20] to treat the oscillatory nature of the integrand, combined with generalized Gaussian quadrature [26] to treat its singularities. We refer the interested reader to [18] for results on the rapid convergence of that approach for similar integrals (in particular §7, Example 2).

## 4.1 Asymptotic order in the computational domain

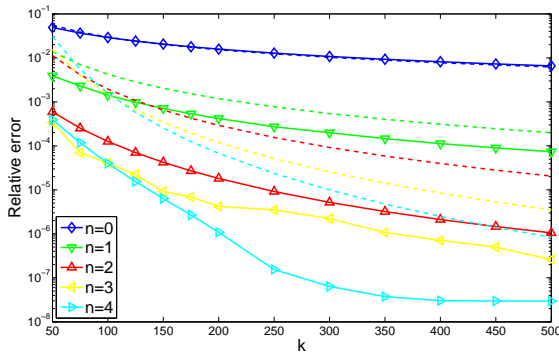


Figure 3: Relative error  $|q_h - q|$  measured at a point in the illuminated region ( $t = 0.37$ ) for increasing wavenumber  $k$  and for different numbers of quadrature points  $n$ . We show the Filon approximation (solid lines) and the asymptotic expansions truncated after  $n + 1$  terms (dashed lines).

We consider the scattering by a circular obstacle, because the solution of this problem is known in closed form. The boundary of the obstacle is parametrised by

$$\kappa(t) = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix}.$$

The incoming wave is a plane wave along the  $x$ -axis as shown in Figure 2. The illuminated region then corresponds to the range  $t \in [0.25, 0.75]$ . Inside the illuminated region we can compute local solutions and we expect these solutions to be close to the exact solution.

We compare the accuracy of the Filon-type method with the exact solution and with the (known) asymptotic expansion of the exact solution. In the computational domain  $[a, b] = [0.35, 0.4]$  from the example shown in Figure 2, we chose  $N = 26$  equidistant collocation points. The basis functions are natural splines of order  $s = 5$  with knots at the collocation points. This results in sparse  $26 \times 26$  discretization matrices. We show the error at a point in the interior of  $(a, b)$  in Figure 3. Observe that the Filon-type methods using  $n + 1$  quadrature points have the same asymptotic error decay as the asymptotic expansions truncated after  $n + 1$  terms. However, the accuracy of Filon-type methods is typically several orders of magnitude better. More than three digits of accuracy are already achieved at  $k = 50$  using only three quadrature points. The fifth order Filon-type method ( $n = 4$ ) improves on the corresponding truncated asymptotic expansion by a factor of 1000 in this example.

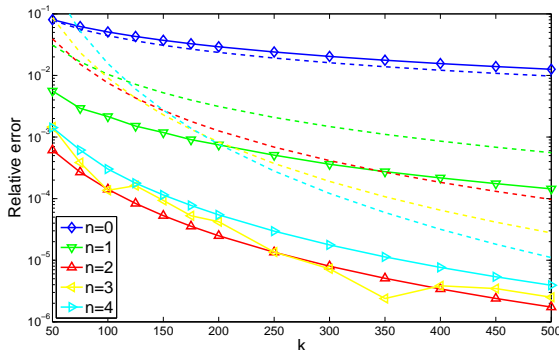


Figure 4: The relative error at the endpoint  $t = a = 0.35$  of the computational domain for different values of  $k$ . Increasing  $n$  beyond  $n = 2$  does not result in higher asymptotic order of the Filon approximation (solid lines). This is visible by comparing to the truncated asymptotic expansions (dashed lines).

The analysis reveals that a reduction in the asymptotic order is possible near the endpoints  $a$  and  $b$  of the computational domain. Indeed, a fifth order natural spline is not capable of representing derivatives to order higher than three at the boundary. Increasing the number of quadrature points  $n$  therefore does not always result in an increasing asymptotic order at the boundary, though it does result in higher asymptotic order in the interior of  $(a, b)$ . The reduction of order near the boundary (at  $t = a = 0.3$ ) is visible for  $n > 2$ . Condition (3.12) on the degree  $s = 5$  of the natural splines is not satisfied in this example if  $n > 2$ . Note that this is only a boundary effect: asymptotic order does increase in the interior as was shown in Figure 3.

## 4.2 Approaching the shadow boundary

It is well known that the asymptotic expansion (3.2) diverges as one approaches the shadow boundary. This blow-up is not present in the local solution. We computed a local solution using the whole illuminated side of the circle (excluding the shadow boundaries themselves) as the computational domain. The results are shown in Figure 5. The local method also exhibits lower accuracy near the shadow boundary, but there is no blow-up and some accuracy is even maintained (1–2 digits in this example).

The asymptotic order of Filon-type methods is not as good close to the shadow boundary as it is well into the illuminated region. We consider a point close to the shadow boundary ( $t = 0.27$ ). The results in Figure 6 show the error in the Filon approximation for varying  $n$  and increasing wavenumber  $k$ . All methods exhibit similar asymptotic order, though accu-

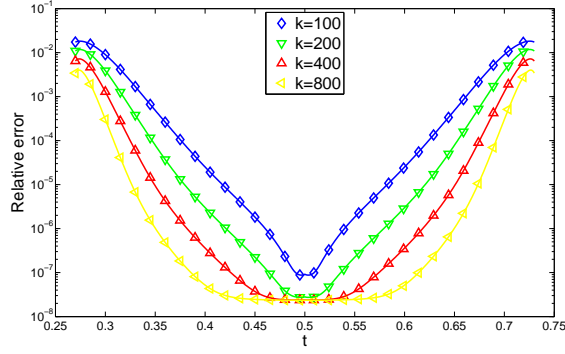


Figure 5: Error of the Filon approximation (with  $n = 2$ ) in the illuminated region for increasing wavenumber  $k$  (solid). The corresponding truncated asymptotic expansions (dashed) diverge near the shadow boundaries.

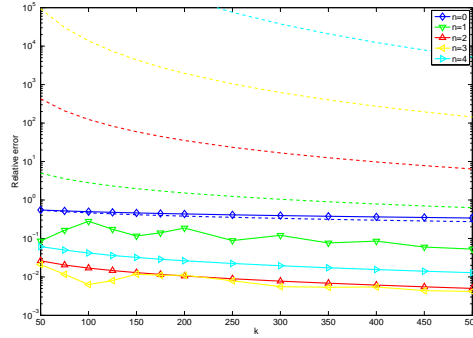


Figure 6: The error  $|q_h - q|$  at a point close to the shadow boundary ( $t = 0.27$ ) for increasing wavenumber  $k$ . The figure shows Filon approximations (solid) and truncated asymptotic expansions (dashed). The asymptotic expansions diverge, the Filon approximations achieve bounded error.

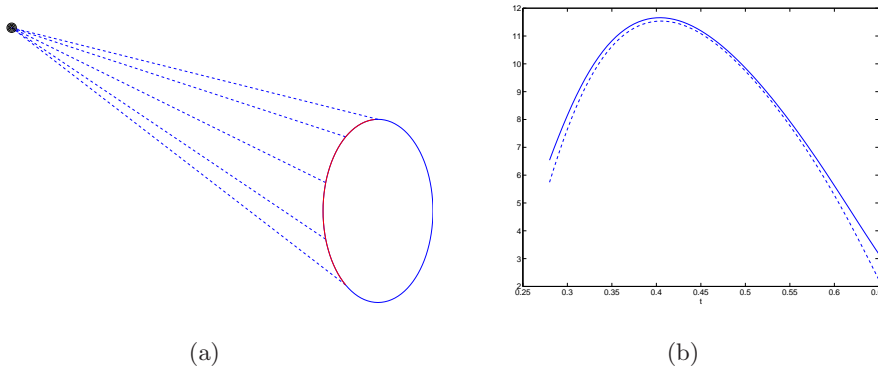


Figure 7: a) The ellipse-shaped obstacle. The part where we seek a local solution is marked in red. b) The computed solution for  $k = 200$  as a function of the parameter  $t$ , real (dashed) and imaginary (solid) part.

racy improves somewhat with increasing  $n$ .

### 4.3 More general scatterers

The proposed method can easily be adapted to other scatterers and boundary conditions than circles and plane waves. We demonstrate this with a circular wave impinging upon an ellipse-shaped scatterer. Let the ellipse be parametrised by

$$\kappa(t) = \begin{bmatrix} r_1 \cos(2\pi t) \\ r_2 \sin(2\pi t) \end{bmatrix},$$

with  $r_1 = 0.3$  and  $r_2 = 0.5$ . We impose a circular wave, modelled by  $u^i(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x} - \mathbf{x}_0|)$ , centered at  $\mathbf{x}_0 = (-2, 1)$  as shown in Figure 7(a). We computed an approximate solution for  $t \in [0.27, 0.65]$  using  $Q_2$  Filon-type quadrature in the space of natural cubic splines. The results are shown in Figure 7(b).

### 4.4 Boundary layers

It was shown in §3.5 that the Filon discretization  $Q_n$  of the integral operator  $K$  is a singular perturbation problem. Though no boundary conditions to the differential equation are known a priori, this does not prohibit the computation of solutions to high accuracy. In our final example we illustrate the effect of boundary layers when arbitrary boundary conditions are enforced.

To that end, we implemented a simple finite difference scheme using central differences for the second order ordinary differential equation

$$w_0(t)q(t) + w_1(t)q'(t) + w_2(t)q''(t) = f(t).$$

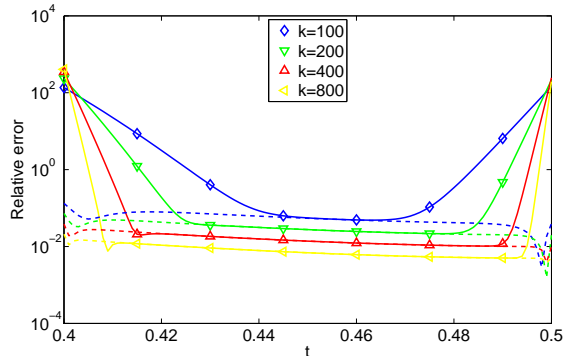


Figure 8: Error of a second order finite difference scheme for a circular obstacle ( $0.4 < t < 0.5$ ), imposing homogeneous Dirichlet conditions (solid lines) or Neumann conditions (dashed lines) at the endpoints.

This equation corresponds to  $Q_2q = f$ . We imposed homogeneous Dirichlet or Neumann boundary conditions. The results are shown in Figure 8 for the circular scatterer. It is clear that as the wavenumber increases, the boundary conditions are increasingly irrelevant. Convergence occurs to the same slowly varying solution in the interior, though a boundary layer effect is visible near the endpoints. Imposing vanishing derivatives (homogeneous Neumann conditions) at the endpoints results in considerably better results, due to the bounded derivatives principle.

## 5 Applications

Local solutions are useful in themselves for high accuracy computations, though they are limited to single reflections. A global solution can be obtained by augmenting the local solution with a standard collocation scheme outside the admissible part of the boundary. This is, in hindsight, the approach described in [22] for globally convex obstacles. Convergence for a fixed wavenumber can conceivably be achieved by adding quadrature points in the Filon-type quadrature rule, i.e., by using (2.15) rather than (2.16) in the discretization.

Local solutions can also be used as part of a larger computation. Consider as a first example a general scatterer and an incoming wave such that, from geometric considerations, parts of the domain only involve single reflections. In that case, one can write the solution as the sum of local reflections and a remainder,

$$u(x) = u_{\text{refl}}(x) + u_{\text{rem}}(x), \quad x \in \tilde{\Gamma}.$$

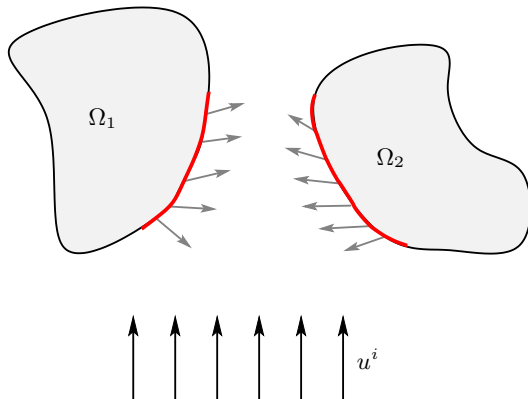


Figure 9: An iterative approach to a multiple scattering problem. Part of the incoming wave reflects off  $\Omega_1$  onto  $\Omega_2$ . A small portion of this wave reflects back onto  $\Omega_1$  and so on. Each step of the iteration requires the computation of a local solution on one of the obstacles.

The single reflection  $u_{\text{refl}}(x)$  is a local solution and can be computed separately. Any other method may be used to solve for the new unknown  $u(x) - u_{\text{refl}}(x)$  on  $\Gamma$ . This remainder is likely to be very small on  $\tilde{\Gamma}$ . The computational domain of the integral equation essentially reduces to the smaller domain  $\Gamma \setminus \tilde{\Gamma}$ . Thus, subtracting out local solutions may be a considerable optimization.

A more important use of local solutions however is in the computation of multiple scattering problems. We refer to results by Ecevit et al [14, 2] on the convergence of iterative schemes for multiple scattering problems in two and three dimensions. Each step of the iteration requires the solution of a single scattering problem involving just one of the obstacles. Part of the scattered wave may reflect onto another obstacle and this results in additional steps of the iteration. The process is illustrated in Figure 9. An important observation is that each step only requires a part of the solution of the previous step. This part corresponds precisely to a local solution as defined and computed in this paper. For example, for the first iteration in Figure 9 it is sufficient to compute a local solution on the red part in order to continue the iterative process. Whether the solution on other parts of the boundary is relevant or not depends on the application, but it plays no further role in the multiple scattering process.

## 6 Concluding remarks

We presented an efficient and simple scheme to compute single reflections in scattering problems to very high accuracy. We reduced the integral equation formulation of a two-dimensional Helmholtz problem to a univariate

ordinary differential equation and showed that even simple finite difference schemes can be used to solve this equation. The differential equation appeared due to the use of Filon-type quadrature in the discretization of the integral operator, as Filon-type quadrature employs derivatives of the integrand. The coefficients of the differential equation are given by the weights of the Filon-type quadrature.

Preliminary results were also obtained for three-dimensional scattering problems. A major difference compared to the two-dimensional case is the challenge of efficiently computing the weights of the Filon-type cubature rule. These weights are given by two-dimensional, singular and highly oscillatory integrals. Current research focuses on their numerical computation based on the results for multivariate oscillatory integrals in [21].

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