A Favard theorem for rational functions with complex poles

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Let $\{\varphi_n\}$ be a sequence of rational functions with arbitrary complex poles, generated by a certain three-term recurrence relation. In this paper we show that - under some mild conditions - the rational functions φ_n form an orthonormal system with respect to a Hermitian positive-definite inner product.

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A Favard theorem for rational functions with complex poles*

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Abstract

Let $\{\varphi_n\}$ be a sequence of rational functions with arbitrary complex poles, generated by a certain three-term recurrence relation. In this paper we show that - under some mild conditions - the rational functions φ_n form an orthonormal system with respect to a Hermitian positive-definite inner product.

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1 Introduction

In [7] a Favard theorem was given for Laurent polynomials. Later, several Favard theorems were determined for classes of rational functions with restrictions on the poles: first, the restriction that the poles are complex and outside the extended real line (or, using an inverse Cayley Transformation, outside the unit circle), see e.g. [1, 2, 4, 6]; afterwards, the restriction that the poles are all on the extended real line (or on the unit circle), see e.g. [3, 4]. Finally, in [5, Thm. 3.10] a Favard theorem was given for rational functions without restrictions on the poles. The complete proof of this last Favard type theorem was omitted in [5] because the outline of the proof was similar to the proof given in [4, Chapt. 11.9]. However, some adaptations needed to be made consistently throughout the proof given in [4, Chapt. 11.9], so that the proof also holds when there are no restrictions on the poles.

The aim of this paper is to give a complete proof for Theorem 3.10 in [5]. The outline is as follows. We start with an overview of the necessary theoretical preliminaries in Section 2. Next, we prove two extended recurrence relations in Section 3. Finally, Section 4 contains a complete proof of the Favard theorem that has been formulated in [5, Thm. 3.10].

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2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$, while the extended real line will be denoted by $\overline{\mathbb{R}}$. Further, we represent the positive real line by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, and the unit circle by $\mathbb{T} = \{z : |z| = 1\}$. If the value $a \in X$ is omitted in the set X, this will be represented by X_a; e.g.

$$\mathbb{C}_0 = \mathbb{C} \setminus \{0\}.$$

Let $c = a + \mathbf{i}b$, where $a, b \in \mathbb{R}$. Then we denote the real part of c by $\Re\{c\} = a$ and the imaginary part by $\Im\{c\} = b$.

Whenever we consider a summation or product of the form

$$\sum_{k=i}^{j} g_k(x), \quad \text{respectively} \quad \prod_{k=i}^{j} g_k(x), \quad j \geq i-1,$$

we will assume for the special case in which $j = i - 1$ that

$$\sum_{k=i}^{i-1} g_k(x) \equiv 0, \quad \text{respectively} \quad \prod_{k=i}^{i-1} g_k(x) \equiv 1.$$

Suppose a sequence of poles $\mathcal{A}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \overline{\mathbb{C}}_0$ is given. Define the factors

$$Z_k(x) = \frac{x}{1 - x/\alpha_k}, \quad k = 0, 1, \ldots$$

and the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = b_{k-1}(x) Z_k(x), \quad k = 1, 2, \ldots \quad .$$

Then the space of rational functions with poles in $\mathcal{A}_n$ is defined as

$$\mathcal{L}_n = \text{span}\{b_0, b_1, \ldots, b_n\}.$$

We denote with $\mathcal{P}_n$ the space of polynomials of degree less than or equal to n. Let π_n be given by

$$\pi_n(x) = \prod_{k=1}^{n} (1 - x/\alpha_k).$$

Then we may write equivalently

$$\mathcal{L}_n = \left\{ \frac{p_n}{\pi_n} : p_n \in \mathcal{P}_n \right\}.$$

In the remainder, we will use the notations $\pi_{n\setminus j}$ and $\mathcal{A}_{n\setminus j}$, with $n \geq j$, to denote respectively the polynomial

$$\pi_{n\setminus j} = \frac{\pi_n}{\pi_j} \in \mathcal{P}_{n-j}$$

and the reduced sequence of poles $\mathcal{A}_{n\setminus j} = \{\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n\}$. In the special case in which $j = 0$ or $j = n$ we have that $\pi_{n\setminus 0} = \pi_n$ and $\mathcal{A}_{n\setminus 0} = \mathcal{A}_n$, respectively $\pi_{n\setminus n} = \pi_0 = 1$ and $\mathcal{A}_{n\setminus n} = \mathcal{A}_0 = \emptyset$.

Note that the value $\alpha_\emptyset = 0$ represents a forbidden value for the poles α_k. Since we consider only a countable number of poles α_k, we can always find a point on $\mathbb{R}$ so that $\alpha_k \neq \alpha_\emptyset$ for all $k = 0, 1, \ldots$. A simple transformation can bring this $\alpha_\emptyset$ to any position that we would prefer. Therefore, this forbidden value $\alpha_\emptyset$ is not a real restriction on the sequence of poles, and we may assume it to be fixed by the value zero.

We define the involution operation or substar conjugate of a function $f \in \mathcal{L}_\infty$ as

$$f_*(x) = \overline{f(\overline{x})}.$$

This way we have that $f(x)$ has a pole in $x = \alpha$ iff $f_*(x)$ has a pole in $x = \overline{\alpha}$. With $\mathcal{L}_{n*}$ we then denote the space of rational functions given by $\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\}$.

Next, let us consider an inner product that is defined by a linear functional M:

$$\langle f, g\rangle = M\{fg_*\}, \quad f, g \in \mathcal{L}_\infty.$$

When $\langle f, f\rangle \neq 0$ for all $f \neq 0$ that are in $\mathcal{L}_\infty$, then the functional is called quasi-definite; moreover, when $\langle f, f\rangle > 0$ for all $f \neq 0$ that are in $\mathcal{L}_\infty$, it is called positive-definite. Finally, the functional is called Hermitian if for every $f \in \mathcal{L}_\infty \cdot \mathcal{L}_{\infty*}$ it holds that $M\{f_*\} = \overline{M\{f\}}$.

Suppose there exists a sequence of rational functions $\{\varphi_n\}$, with $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, so that the φ_n form an orthonormal system with respect to the Hermitian positive-definite linear functional M; i.e. $M\{\varphi_j\varphi_{k*}\} = \delta_{jk}$, where δ_{jk} denotes the Kronecker Delta. Further, assume that these rational functions are of the form $\varphi_n(x) = p_n(x)/\pi_n(x)$. We then call φ_n degenerate (respectively exceptional) iff $p_n(\overline{\alpha}_{n-1}) = 0$ (respectively $p_n(\alpha_{n-1}) = 0$). A zero of p_n at ∞ then means that the degree of p_n is less than n. If φ_n is not degenerate and not exceptional, it is called regular. In [4, Chapt. 11.1] (for all the poles on the extended real line), and [8, Thm. 2.1.1] and [5, Sec. 3] (for arbitrary complex poles), the following three-term recurrence relation has been proved.

Theorem 2.1. *Let $E_0 \in \mathbb{C}_0$, $\alpha_{-1} \in \overline{\mathbb{R}}_0$ and $\alpha_0 \in \overline{\mathbb{C}}_0$ be arbitrary but fixed in advance. Then the sequence of orthonormal rational functions $\{\varphi_n\}$ is regular iff for every $n \geq 1$ there exists a three-term recurrence relation of the form*

$$\varphi_n(x) = \left[E_n Z_n(x) + F_n \frac{Z_n(x)}{Z_{n-1}(x)}\right] \varphi_{n-1}(x) + C_n \frac{Z_n(x)}{Z_{n-2*}(x)} \varphi_{n-2}(x), \qquad (1)$$

with

$$\begin{aligned} E_n &\neq 0 \\ C_n = -\left[E_n + F_n/Z_{n-1}(\overline{\alpha}_{n-1})\right]/\overline{E}_{n-1} &\neq 0. \end{aligned}$$

The initial conditions are $\varphi_{-1}(x) \equiv 0$ and $\varphi_0(x) \equiv \frac{\eta}{\sqrt{M\{1\}}}$, where η is a unimodular constant; i.e. $\eta \in \mathbb{T}$.

The previous theorem starts from a system of rational functions $\{\varphi_n\}$ for which the φ_n are orthonormal with respect to a Hermitian positive-definite inner product, to prove the existence of a three-term recurrence relation iff the system is regular. In order to derive a Favard type theorem, however, we need to verify whether the statement holds in the opposite direction; i.e., starting from a regular system of rational functions $\{\varphi_n\}$ for which the φ_n are generated by the three-term recurrence relation (1), we have to prove whether there exists a Hermitian positive-definite inner product for which the φ_n form an orthonormal system. In the next section we start with two extensions of the above recurrence relation that we will need in the proof of this Favard type theorem. But first we need the following auxiliary results that have been proved in [4, Lem. 11.9.1].

Lemma 2.2.

1. *For all constants A and B, and for all integers j, k and n so that $\alpha_n \neq \alpha_k$, there exist unique constants a and b so that*

$$A + \frac{B}{Z_j(x)} = \frac{a}{Z_n(x)} + \frac{b}{Z_k(x)}.$$

2. *For all constants A and B, and for all integers j and k, there exist unique constants a and b so that*

$$A + \frac{B}{Z_j(x)} = a + \frac{b}{Z_k(x)}.$$

3 Extended recurrence relations

In order to prove the Favard theorem, we need the following extensions of the recurrence relation given in Theorem 2.1.

Lemma 3.1. *Let the sequence of rational functions $\{\varphi_n\}$ be generated by the three-term recurrence relation given by* (1)*. Consider an integer $n \geq j+2 \geq 1$, and assume that $\alpha_n \notin \mathcal{A}_{(n-1)\backslash j}$. In the special case in which $j = -1$, we define $\mathcal{A}_{(n-1)\backslash -1} = \mathcal{A}_{n-1} \cup \{\alpha_0\}$. Then there exist constants $a_1, a_2, \ldots, a_{n-(j+1)}, c_{n-(j+1)}, c_{n-(j+2)}$, all depending on n and j, so that*

$$\varphi_n(x) = \left\{ \sum_{k=1}^{n-(j+1)} a_k \varphi_{n-k}(x) \right\} + c_{n-(j+1)} \frac{Z_n(x)}{Z_{j+1}(x)} \varphi_{j+1}(x) + c_{n-(j+2)} C_{j+2} \frac{Z_n(x)}{Z_{j*}(x)} \varphi_j(x). \quad (2)$$

Proof. The proof is similar to the one in [4, Thm. 11.9.2]. For $n = j+2 \geq 1$, the formula reduces to

$$\varphi_n(x) = a_1 \varphi_{n-1}(x) + c_1 \frac{Z_n(x)}{Z_{n-1}(x)} \varphi_{n-1}(x) + c_0 C_n \frac{Z_n(x)}{Z_{n-2*}(x)} \varphi_{n-2}(x).$$

It follows from the first statement in Lemma 2.2, with $A = E_n$, $B = F_n$, $\alpha_j = \alpha_k = \alpha_{n-1} \neq \alpha_n$, that this is the three-term recurrence relation itself. So let us now proceed by induction. Assume that for $s = j - 1$, with $n \geq s + 3 \geq 2$, we have that

$$\varphi_n(x) - \left\{ \sum_{k=1}^{n-(s+2)} a_k \varphi_{n-k}(x) \right\} = c_{n-(s+2)} \frac{Z_n(x)}{Z_{s+2}(x)} \varphi_{s+2}(x) + c_{n-(s+3)} C_{s+3} \frac{Z_n(x)}{Z_{s+1*}(x)} \varphi_{s+1}(x). \quad (3)$$

Applying Theorem 2.1 to $\varphi_{s+2}(x)$ in the right hand side of (3) then gives

$$\begin{aligned} \varphi_n(x) &- \left\{ \sum_{k=1}^{n-(s+2)} a_k \varphi_{n-k}(x) \right\} \\ &= c_{n-(s+2)} \frac{Z_n(x)}{Z_{s+2}(x)} \left[\left(E_{s+2} + \frac{F_{s+2}}{Z_{s+1}(x)} \right) Z_{s+2}(x) \varphi_{s+1}(x) \right. \\ &\qquad \left. + C_{s+2} \frac{Z_{s+2}(x)}{Z_{s*}(x)} \varphi_s(x) \right] + c_{n-(s+3)} C_{s+3} \frac{Z_n(x)}{Z_{s+1*}(x)} \varphi_{s+1}(x) \\ &= \left[\hat{a} + \frac{\hat{b}}{Z_{s+1}(x)} + \frac{\hat{c}}{Z_{s+1*}(x)} \right] Z_n(x) \varphi_{s+1}(x) \\ &\qquad + c_{n-(s+2)} C_{s+2} \frac{Z_n(x)}{Z_{s*}(x)} \varphi_s(x), \end{aligned}$$

where

$$\hat{a} = c_{n-(s+2)} E_{s+2}, \quad \hat{b} = c_{n-(s+2)} F_{s+2} \quad \text{and} \quad \hat{c} = c_{n-(s+3)} C_{s+3}.$$

From the second statement in Lemma 2.2 it follows that there exist constants $\tilde{a}$ and $\tilde{b}$, so that

$$\hat{a} + \frac{\hat{b}}{Z_{s+1}(x)} + \frac{\hat{c}}{Z_{s+1*}(x)} = \tilde{a} + \frac{\tilde{b}}{Z_{s+1}(x)}.$$

Further, if $\alpha_n \neq \alpha_{s+1}$, we can apply the first statement of Lemma 2.2 to write

$$\tilde{a} + \frac{\tilde{b}}{Z_{s+1}(x)} = \frac{a_{n-(s+1)}}{Z_n(x)} + \frac{c_{n-(s+1)}}{Z_{s+1}(x)}.$$

Hence, it then holds that

$$\begin{aligned} \varphi_n(x) - \left\{ \sum_{k=1}^{n-(s+2)} a_k \varphi_{n-k}(x) \right\} &= a_{n-(s+1)} \varphi_{s+1}(x) \\ &+ c_{n-(s+1)} \frac{Z_n(x)}{Z_{s+1}(x)} \varphi_{s+1}(x) + c_{n-(s+2)} C_{s+2} \frac{Z_n(x)}{Z_{s*}(x)} \varphi_s(x), \end{aligned}$$

which proves the induction step. □

When $j \geq 0$ in (2), the three-term recurrence relation can be applied once more to give a second form of the extended recurrence relation.

Lemma 3.2. *Let the sequence of rational functions $\{\varphi_n\}$ be generated by the three-term recurrence relation given by* (1)*. Consider an integer $n \geq j+1 \geq 1$, and assume that $\alpha_n \notin \mathcal{A}_{(n-1)\setminus j}$. Then there exist constants $a_1, a_2, \ldots, a_{n-(j+1)}, a'_{n-j}, c'_{n-j}$, $c_{n-(j+1)}$, all depending on n and j, so that*

$$\varphi_n(x) = \left\{ \sum_{k=1}^{n-(j+1)} a_k \varphi_{n-k}(x) \right\} + a'_{n-j}\varphi_j(x) + c'_{n-j} Z_n(x)\varphi_j(x) + c_{n-(j+1)} C_{j+1} \frac{Z_n(x)}{Z_{j-1*}(x)} \varphi_{j-1}(x). \quad (4)$$

Proof. First, consider the case that $n = j+1 \geq 1$. From the second statement in Lemma 2.2 it follows that there exist constants a'_1 and c'_1, so that

$$E_n + \frac{F_n}{Z_{n-1}(x)} = c'_1 + \frac{a'_1}{Z_n(x)}.$$

From (1) we then deduce that

$$\varphi_n(x) = \left[a'_1 + c'_1 Z_n(x)\right] \varphi_{n-1}(x) + C_n \frac{Z_n(x)}{Z_{n-2*}(x)} \varphi_{n-2}(x),$$

which proves the case in which $n = j+1 \geq 1$.

Next, consider the case that $n \geq j+2 \geq 2$. The proof is then similar to that in [4, Thm. 11.9.3]. From Theorem 2.1 it follows that

$$\frac{\varphi_{j+1}(x)}{Z_{j+1}(x)} = \left(E_{j+1} + \frac{F_{j+1}}{Z_j(x)} \right) \varphi_j(x) + \frac{C_{j+1}}{Z_{j-1*}(x)} \varphi_{j-1}(x).$$

Hence, we find for the last two terms in (2) that

$$c_{n-(j+1)} \frac{Z_n(x)}{Z_{j+1}(x)} \varphi_{j+1}(x) + c_{n-(j+2)} C_{j+2} \frac{Z_n(x)}{Z_{j*}(x)} \varphi_j(x)$$
$$= \left[\hat{d} + \frac{\hat{e}}{Z_j(x)} + \frac{\hat{f}}{Z_{j*}(x)} \right] Z_n(x)\varphi_j(x) + c_{n-(j+1)} C_{j+1} \frac{Z_n(x)}{Z_{j-1*}(x)} \varphi_{j-1}(x),$$

where

$$\hat{d} = c_{n-(j+1)} E_{j+1}, \quad \hat{e} = c_{n-(j+1)} F_{j+1} \quad \text{and} \quad \hat{f} = c_{n-(j+2)} C_{j+2}.$$

Finally, it follows from the second statement in Lemma 2.2 that there exist constants a'_{n-j} and c'_{n-j}, so that

$$\hat{d} + \frac{\hat{e}}{Z_j(x)} + \frac{\hat{f}}{Z_{j*}(x)} = c'_{n-j} + \frac{a'_{n-j}}{Z_n(x)}.$$

$\square$

4 Favard type theorem

In this section we will assume that $\{\varphi_n\}_{n=0}^{\infty}$ is a sequence of rational functions in $\mathcal{L}_\infty$ and that the following assumptions are satisfied:

(A1) $\alpha_{-1} \in \overline{\mathbb{R}}_0$ and $\alpha_n \in \overline{\mathbb{C}}_0$, $n = 0, 1, \ldots$,

(A2) φ_n is generated by the three-term recurrence relation (1),

(A3) $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $n = 1, 2, \ldots$, and $\varphi_0 \in \mathbb{C}_0$,

(A4) let $\hat{F}_n$ be given by $\hat{F}_n = \frac{F_n}{E_n}$. Then for $n = 1, 2, \ldots$, it holds that $|\hat{F}_n| < \infty$ and

$$\left[\frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}|\hat{F}_n|^2 - \Im\{\hat{F}_n\}\right] = \frac{\frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} - \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{|E_{n-1}|^2}{|E_n|^2}}{\Delta_{n-1}}, \tag{5}$$

where

$$\Delta_n = |E_n|^2 - 4\frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} > 0, \tag{6}$$

(A5) $0 < |E_n| < \infty$, $n = 0, 1, \ldots$,

(A6) $C_n \overline{E}_{n-1} = -\left[E_n + F_n/Z_{n-1}(\overline{\alpha}_{n-1})\right] \neq 0$, $n = 1, 2, \ldots$.

The proof of the Favard theorem is then similar to the one in [4, p. 311–319]. First, we define M on $\mathcal{L}_\infty$ by setting

$$M\{\varphi_{0*}\} = 1/\varphi_0 \quad \text{and} \quad M\{\varphi_k\} = 0, \quad k = 1, 2, \ldots.$$

Note that the definition of $M\{\varphi_{0*}\}$ takes care of the normalization; i.e. $M\{|\varphi_0|^2\} = M\{\varphi_0\varphi_{0*}\} = 1$.

Next, we need to extend M to $\mathcal{R}_\infty = \mathcal{L}_\infty \cdot \mathcal{L}_{\infty*}$ so that $M\{\varphi_k\varphi_{l*}\} = 0$ for $k \neq l$.

4.1 Extension of M

Consider the following table of products:

column 0	**column** 1	$\ldots$	**column** n	$\ldots$
$\varphi_1\varphi_{0*}$				
$\varphi_2\varphi_{0*}$	$\varphi_2\varphi_{1*}$			
$\vdots$	$\vdots$	$\ddots$		
$\varphi_{n+1}\varphi_{0*}$	$\varphi_{n+1}\varphi_{1*}$	$\cdots$	$\varphi_{n+1}\varphi_{n*}$	
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ddots$

When M is applied to these entries, we should always get zero. Let us now define $\mathcal{B}_{-1} = \{\varphi_{0*}\}$ and

$$\mathcal{B}_n = \mathcal{B}_{n-1} \cup \{\varphi_k\varphi_{n*} : k = n+1, n+2, \ldots\}, \quad n \geq 0.$$

Note that this way $\mathcal{B}_n$ contains the first $n+1$ columns in the above table, together with φ_{0*}. Further, define

$$\mathcal{B}_{n,n*} = \mathcal{B}_{n-1} \quad \text{and} \quad \mathcal{B}_{k,n*} = \mathcal{B}_{k-1,n*} \cup \{\varphi_k \varphi_{n*}\}, \quad k > n.$$

These sets $\mathcal{B}_{k,n*}$ now generate the following spaces:

$$\mathcal{R}_{k,n*} = \text{span } \mathcal{B}_{k,n*}, \quad n = 0, 1, \ldots, \quad k = n, n+1, \ldots.$$

The strategy is then to obtain a definition of M on the successive spaces

$$\mathcal{R}_{1,1*}, \mathcal{R}_{2,1*}, \mathcal{R}_{3,1*}, \ldots; \mathcal{R}_{2,2*}, \mathcal{R}_{3,2*}, \mathcal{R}_{4,2*}, \ldots; \mathcal{R}_{3,3*}, \ldots,$$

so that we have the required orthogonality relations. Hence, each time we consider the next product $\varphi_k \varphi_{n*}$, we can define $M\{\varphi_k \varphi_{n*}\} = 0$ if $\varphi_k \varphi_{n*} \notin \mathcal{R}_{k-1,n*}$; otherwise, if $\varphi_k \varphi_{n*} \in \mathcal{R}_{k-1,n*}$, we need to prove that $M\{\varphi_k \varphi_{n*}\} = 0$.

Eventually, we will then have defined M on the subspace

$$\mathcal{R}'_\infty = \text{span } \{\cup \mathcal{B}_{k,n*} : n = 0, 1, \ldots; k = n, n+1, \ldots\} \subset \mathcal{R}_\infty$$

so that

$$M\{|\varphi_0|^2\} = 1 \quad \text{and} \quad M\{\varphi_k \varphi_{n*}\} = 0, \quad n = 0, 1, \ldots; k = n+1, n+2, \ldots.$$

It will, however, follow from assumption (A4) that the definition of M can easily be extended to the subspace $\mathcal{R}'_{\infty*} = \{f_* : f \in \mathcal{R}'_\infty\}$, in such a way that for every $f \in \mathcal{R}'_\infty + \mathcal{R}'_{\infty*}$ it holds that $M\{f_*\} = \overline{M\{f\}}$. Moreover, it will turn out that $\mathcal{R}'_\infty + \mathcal{R}'_{\infty*} = \mathcal{R}_\infty$, and with assumption (A6) we will then have that $M\{|\varphi_n|^2\} = M\{\varphi_n \varphi_{n*}\} = 1$ for $n = 0, 1, \ldots$. We will now elaborate these successive steps. Note, however, that $\mathcal{R}_{1,1*} = \mathcal{L}_\infty$; hence, the first column (column 0) has already been dealt with by our definition of M on $\mathcal{L}_\infty$.

Column $n = 1$

Initialization: $M\{\varphi_2 \varphi_{1*}\}$

If $\varphi_2 \varphi_{1*} \notin \mathcal{L}_\infty$, we define $M\{\varphi_2 \varphi_{1*}\} = 0$.
If $\varphi_2 \varphi_{1*} \in \mathcal{L}_\infty$, there exists an r so that

$$\varphi_2 \varphi_{1*} = \frac{q_r}{\pi_r}, \quad q_r \in \mathcal{P}_r. \tag{7}$$

Note that it should hold that $r > 2$; otherwise we would have that $p_1(\alpha_1) p_2(\overline{\alpha}_1) = 0$, while from our assumptions (A3), (A5) and (A6) it follows that both $p_1(\alpha_1)$ and $p_2(\overline{\alpha}_1)$ are nonzero. Consequently, we can rewrite (7) as

$$\pi_{r \backslash 2} p_2 p_{1*} = \pi_{1*} q_r.$$

It now follows that $\pi_{r\backslash 2}$ should have a zero at $\overline{\alpha}_1$. Let $m \geq 3$ be the smallest index so that $\alpha_m = \overline{\alpha}_1$. Note that, if $m > 3$, we have that $\alpha_m \notin \mathcal{A}_{(m-1)\backslash 2}$. We now use the extended recurrence relation given by (4) to write

$$\begin{aligned}\varphi_m &= \left\{\sum_{j=1}^{m-3} a_j \varphi_{m-j}\right\} + a'_{m-2}\varphi_2 + c'_{m-2} Z_m \varphi_2 + c_{m-3} C_3 \frac{Z_m}{Z_{1*}} \varphi_1 \\ &= \left\{\sum_{j=1}^{m-3} a_j \varphi_{m-j}\right\} + a'_{m-2}\varphi_2 + c'_{m-2} Z_{1*} \varphi_2 + c_{m-3} C_3 \varphi_1.\end{aligned}$$

Note that $c'_{m-2} \neq 0$; otherwise $\varphi_m \in \mathcal{L}_{m-1}$, contradicting assumption (A3). Next, applying M to the previous relation results in

$$M\{\varphi_m\} = \left\{\sum_{j=1}^{m-3} a_j M\{\varphi_{m-j}\}\right\} \\ + a'_{m-2} M\{\varphi_2\} + c'_{m-2} M\{Z_{1*}\varphi_2\} + c_{m-3} C_3 M\{\varphi_1\}.$$

Note that $M\{\varphi_j\} = 0$ for $j = 1, \ldots, m$. Thus, it holds that $M\{Z_{1*}\varphi_2\} = 0$ as well. Finally, because φ_{1*} can be written as

$$\varphi_{1*} = d_1 Z_{1*} + d_0,$$

it follows that

$$M\{\varphi_2 \varphi_{1*}\} = d_1 M\{Z_{1*}\varphi_2\} + d_0 M\{\varphi_2\} = 0.$$

Induction step for column 1

Under the assumption that $M\{\varphi_l \varphi_{i*}\} = 0$ for every $\varphi_l \varphi_{i*} \in \mathcal{R}_{k-1,1*}$ with $l \neq i$, we have to prove that $M\{\varphi_k \varphi_{1*}\} = 0$. The approach is the same for $n = 1$ as for $n > 1$. We therefore refer to the proof of the general step: induction step for column n (see below).

Column $n \geq 2$

Initialization: $M\{\varphi_{n+1}\varphi_{n*}\}$

If $\varphi_{n+1}\varphi_{n*} \notin \mathcal{R}_{n,n*}$, we define $M\{\varphi_{n+1}\varphi_{n*}\} = 0$.
If $\varphi_{n+1}\varphi_{n*} \in \mathcal{R}_{n,n*}$, there exists an r so that

$$\varphi_{n+1}\varphi_{n*} = \sum_{j=0}^{n-1} \varphi_{j*} \frac{q_{r,j}}{\pi_r}, \quad q_{r,j} \in \mathcal{P}_r. \tag{8}$$

Note that it should hold that $r > n+1$; otherwise we would have that $p_n(\alpha_n)p_{n+1}(\overline{\alpha}_n) = 0$, while from our assumptions (A3), (A5) and (A6) it follows that both $p_n(\alpha_n)$ and $p_{n+1}(\overline{\alpha}_n)$ are nonzero. Consequently, we can rewrite (8) as

$$\pi_{r\backslash(n+1)} p_{n+1} p_{n*} - \sum_{j=1}^{n-1} \pi_{n*\backslash j*} p_{j*} q_{r,j} = \pi_{n*} q_{r,0} \varphi_{0*}.$$

It now follows that $\pi_{r\backslash(n+1)}$ should have a zero at $\overline{\alpha}_n$. Let $m \geq n+2$ be the smallest index so that $\alpha_m = \overline{\alpha}_n$. Note that, in the case in which this $m > n+2$, we have that $\alpha_m \notin \mathcal{A}_{(m-1)\backslash(n+1)}$. We now use the extended recurrence relation given by (4) to write

$$\begin{aligned}\varphi_m &= \left\{\sum_{j=1}^{m-(n+2)} a_j \varphi_{m-j}\right\} + a'_{m-(n+1)}\varphi_{n+1} \\ &\qquad + c'_{m-(n+1)} Z_m \varphi_{n+1} + c_{m-(n+2)} C_{n+2} \frac{Z_m}{Z_{n*}} \varphi_n \\ &= \left\{\sum_{j=1}^{m-(n+2)} a_j \varphi_{m-j}\right\} + a'_{m-(n+1)}\varphi_{n+1} \\ &\qquad + c'_{m-(n+1)} Z_{n*} \varphi_{n+1} + c_{m-(n+2)} C_{n+2} \varphi_n .\end{aligned}$$

Note that $c'_{m-(n+1)} \neq 0$; otherwise $\varphi_m \in \mathcal{L}_{m-1}$, contradicting assumption (A3). Next, multiplying with $b_{n-1*}(x)$ and applying M results in

$$\begin{aligned}M\{b_{n-1*}\varphi_m\} = &\left\{\sum_{j=1}^{m-(n+2)} a_j M\{b_{n-1*}\varphi_{m-j}\}\right\} + a'_{m-(n+1)} M\{b_{n-1*}\varphi_{n+1}\} \\ &+ c'_{m-(n+1)} M\{b_{n*}\varphi_{n+1}\} + c_{m-(n+2)} C_{n+2} M\{b_{n-1*}\varphi_n\}.\end{aligned}$$

Because of the induction hypothesis all the terms vanish except for $M\{b_{n*}\varphi_{n+1}\}$, so that $M\{b_{n*}\varphi_{n+1}\} = 0$. Finally, because φ_{n*} can be written as

$$\varphi_{n*} = \sum_{j=0}^{n} d_j b_{j*},$$

it follows that

$$M\{\varphi_{n+1}\varphi_{n*}\} = \sum_{j=0}^{n} d_j M\{b_{j*}\varphi_{n+1}\} = 0.$$

Induction step for column n

We now consider $n \geq 1$ and $k \geq n+2$ for which we know that

$$\begin{aligned} M\{\varphi_l \varphi_{i*}\} &= 0, \quad i = 0, \ldots, n-1; \quad l \geq i+1, \\ M\{\varphi_j \varphi_{n*}\} &= 0, \qquad j = n+1, \ldots, k-1. \end{aligned}$$

We then have to prove that $M\{\varphi_k \varphi_{n*}\} = 0$. From (1) we get that

$$\varphi_k = Z_k \left[E_k + \frac{F_k}{Z_{k-1}}\right] \varphi_{k-1} + C_k \frac{Z_k}{Z_{k-2*}} \varphi_{k-2}.$$

Next, multiplying with $b_{n*}(x)/Z_k(x)$ and applying M then gives

$$M\left\{\frac{b_{n*}}{Z_k}\varphi_k\right\} = E_k M\left\{b_{n*}\varphi_{k-1}\right\} + F_k M\left\{\frac{b_{n*}}{Z_{k-1}}\varphi_{k-1}\right\} + C_k M\left\{\frac{b_{n*}}{Z_{k-2*}}\varphi_{k-2}\right\}. \quad (9)$$

Because of the induction hypothesis all the terms on the right hand side of (9) are zero, so that

$$M\left\{\frac{b_{n*}}{Z_k}\varphi_k\right\} = 0.$$

We now distinguish two cases: (A) $\alpha_k \neq \overline{\alpha}_n$ and (B) $\alpha_k = \overline{\alpha}_n$.

(A) $\alpha_k \neq \overline{\alpha}_n$.
From Lemma 2.2 it then follows that there exists a nonzero constant c so that $1/Z_k = 1/Z_{n*} + c$. Hence

$$0 = M\left\{\frac{b_{n*}}{Z_k}\varphi_k\right\} = M\left\{b_{n-1*}\varphi_k\right\} + cM\left\{b_{n*}\varphi_k\right\}.$$

The first term on the right hand side equals zero because of the induction hypothesis. Consequently, we find that $M\{b_{n*}\varphi_k\} = 0$, and thus also $M\{\varphi_k\varphi_{n*}\} = 0$.

(B) $\alpha_k = \overline{\alpha}_n$.
If $\varphi_k\varphi_{n*} \notin \mathcal{R}_{k-1,n*}$, we may define $M\{\varphi_k\varphi_{n*}\} = 0$.
If $\varphi_k\varphi_{n*} \in \mathcal{R}_{k-1,n*}$, there exist an r and constants c_i so that

$$\varphi_k\varphi_{n*} + \sum_{i=n+1}^{k-1} c_i\varphi_i\varphi_{n*} = \sum_{j=0}^{n-1} \varphi_{j*}\frac{q_{r,j}}{\pi_r}, \quad q_{r,j} \in \mathcal{P}_r. \quad (10)$$

Note that it should hold that $r > k$; otherwise we would have that $p_n(\alpha_n)\times p_{n+1}(\overline{\alpha}_n) = 0$, while from our assumptions (A3), (A5) and (A6) it follows that both $p_n(\alpha_n)$ and $p_{n+1}(\overline{\alpha}_n)$ are nonzero. Consequently, we can rewrite (10) as

$$\pi_{r\setminus k}p_k p_{n*} + \sum_{i=n+1}^{k-1} c_i\pi_{r\setminus i}p_i p_{n*} - \sum_{j=1}^{n-1} \pi_{n*\setminus j*}p_{j*}q_{r,j} = \pi_{n*}q_{r,0}\varphi_{0*}.$$

It now follows that $\pi_{r\setminus k}$ should have a zero at $\overline{\alpha}_n$. Let $m \geq k+1$ be the smallest index so that $\alpha_m = \overline{\alpha}_n = \alpha_k$. Note that, in the case in which this $m > k+1$, it holds that $\alpha_m \notin \mathcal{A}_{(m-1)\setminus k}$. We now use the extended recurrence relation given by (4) to write

$$\begin{aligned}\varphi_m &= \left\{\sum_{j=1}^{m-(k+1)} a_j\varphi_{m-j}\right\} + a'_{m-k}\varphi_k \\ &\qquad + c'_{m-k}Z_m\varphi_k + c_{m-(k+1)}C_{k+1}\frac{Z_m}{Z_{k-1*}}\varphi_{k-1} \\ &= \left\{\sum_{j=1}^{m-(k+1)} a_j\varphi_{m-j}\right\} + a'_{m-k}\varphi_k \\ &\qquad + c'_{m-k}Z_{n*}\varphi_k + c_{m-(k+1)}C_{k+1}\frac{Z_{n*}}{Z_{k-1*}}\varphi_{k-1}.\end{aligned}$$

Note that $\frac{Z_{n*}}{Z_{k-1*}}\varphi_{k-1} = \frac{Z_k}{Z_{k-1*}}\varphi_{k-1} \in \mathcal{L}_k \subseteq \mathcal{L}_{m-1}$ so that $c'_{m-k} \neq 0$; otherwise $\varphi_m \in \mathcal{L}_{m-1}$, contradicting assumption (A3). Next, multiplying with $b_{n-1*}(x)$ and applying M results in

$$\begin{aligned}M\{b_{n-1*}\varphi_m\} = &\left\{\sum_{j=1}^{m-(k+1)} a_{m-j}M\{b_{n-1*}\varphi_{m-j}\}\right\} + a'_{m-k}M\{b_{n-1*}\varphi_k\} \\ &+ c'_{m-k}M\{b_{n*}\varphi_k\} + c_{m-(k+1)}C_{k+1}M\left\{\frac{b_{n*}}{Z_{k-1*}}\varphi_{k-1}\right\}.\end{aligned}$$

Since $m \geq k+1 \geq n+3$ and $b_{n*}/Z_{k-1*} \in \mathcal{L}_{n*}$, it follows from the induction hypothesis that all the terms vanish except for $M\{b_{n*}\varphi_k\}$. Thus $M\{b_{n*}\varphi_k\} = 0$, and hence $M\{\varphi_k\varphi_{n*}\} = 0$ as well.

This concludes the induction step for column n.

4.2 The space $R'_{\infty*}$

Note that only assumptions (A1)–(A3), (A5) and (A6) were necessary and sufficient to define a functional M on $\mathcal{R}'_\infty$ giving the required orthogonality relations. The question is now whether this definition can be extended to $\mathcal{R}'_{\infty*}$ in such a way that we have the required orthogonality relations in $\mathcal{R}'_{\infty*}$ as well. To answer this question, we first need to consider the special case in which every pole is real, which has been treated in [4, Chapt. 11.9].

It is well-known that orthonormal rational functions are fixed up to a unimodular constant. In the special case in which every pole is real, it follows that $\mathcal{L}_n = \mathcal{L}_{n*}$ for every $n \geq 0$. Consequently, if the φ_n form an orthonormal system on $\mathcal{L}_\infty$, it must hold that

$$\forall n \geq 0 : \{\exists \varepsilon_n \in \mathbb{T} : \varphi_{n*} = \varepsilon_n\varphi_n\}. \tag{11}$$

It is easily verified that this is equivalent with the condition that

$$\hat{F}_n = \frac{F_n}{E_n} \in \mathbb{R}, \quad n = 1, 2, \ldots. \tag{12}$$

Not only is condition (11) (and hence, condition (12) as well) necessary, it is also sufficient to ensure that the functional M as defined before is Hermitian on $\mathcal{R}'_\infty$. The latter is necessary due to the fact that for this special case it holds that $\mathcal{R}'_\infty = \mathcal{R}'_{\infty *} = \mathcal{R}_\infty$.

In general, for the case of arbitrary complex poles, we have that $\mathcal{R}'_\infty \neq \mathcal{R}'_{\infty *}$, but this does not mean that their intersection

$$X = \mathcal{R}'_\infty \cap \mathcal{R}'_{\infty *}$$

is empty. Thus, when extending the definition of the functional M to $\mathcal{R}'_{\infty *}$, we have to take into account the possibility that one or more basis functions of $\mathcal{R}'_{\infty *}$ can be in X. Therefore, we need a condition similar to (12) to ensure that the functional M, as defined before, is already Hermitian on X.

In [5, Thm. 3.9] a relation has been proved between $|E_{n-1}|$, $|E_n|$ and $\hat{F}_n$. This relation, which is given by (5), holds for every Hermitian positive-definite inner product on a subset of the real line as long as the sequence of rational functions form an orthonormal system with respect to this inner product. In our assumption (A4) we have used this relation to extend (12) to the case of arbitrary complex poles. Note, however, that in [5, Thm. 3.9] it has not been assumed that $\Delta_n > 0$. Certainly, the relation given by Equation (5) is a necessary condition. However, at the moment of writing, no proof has been found that verifies that this relation, together with the assumption that $\Delta_n > 0$, is sufficient to ensure that the functional M is Hermitian on X. Nevertheless, we formulate it as a hypothesis and give the proof of the Favard theorem under the assumption that the hypothesis holds true. But first we will justify the inequality given in (6).

Note that for $\alpha_{n-1} \in \overline{\mathbb{R}}_0$, Equation (5) reduces to

$$\Im\{\hat{F}_n\} = \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{1}{|E_n|^2} - \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} \cdot \frac{1}{|E_{n-1}|^2}.$$

While for $\alpha_{n-1} \notin \overline{\mathbb{R}}$, we have that

$$\frac{2\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} = -\frac{\mathbf{i}}{Z_{n-1}(\overline{\alpha}_{n-1})},$$

so that Equation (5) is equivalent with

$$\Re\{\hat{F}_n\}^2 + \Big(\Im\{\hat{F}_n\} - \mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})\Big)^2 = [\mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})]^2 \, \frac{|E_{n-1}|^2}{|E_n|^2} \cdot \frac{\Delta_n}{\Delta_{n-1}}.$$

Because Equation (5) is a necessary condition, we may assume that $\Delta_n/\Delta_{n-1} \in \mathbb{R}^+$; otherwise the collection of (finite) $\hat{F}_n$ satisfying Equation (5) will be empty. Moreover, if $\Delta_n/\Delta_{n-1} = 0$, it follows that $\hat{F}_n = -Z_{n-1}(\overline{\alpha}_{n-1})$. Consequently,

$$C_n\overline{E}_{n-1} = E_n \left[1 + \hat{F}_n/Z_{n-1}(\overline{\alpha}_{n-1})\right] = 0,$$

contradicting assumption (A6). Therefore, we should have that

$$\Delta_n/\Delta_{n-1} \in \mathbb{R}_0^+. \tag{13}$$

Finally, note that $\Delta_0 > 0$. Thus, suppose that $\Delta_{n-1} > 0$. By induction it then follows from assumption (A5) or (13) that $\Delta_n > 0$, if respectively $\alpha_{n-1} \in \overline{\mathbb{R}}_0$ or $\alpha_{n-1} \notin \overline{\mathbb{R}}$.

We now have the following hypothesis.

Hypothesis 4.1. *Suppose assumptions (A1)–(A3), (A5) and (A6) are satisfied. Then it holds that the functional M as defined before is Hermitian on $X = \mathcal{R}'_\infty \cap \mathcal{R}'_{\infty *}$ iff $|E_{n-1}|$, $|E_n|$ and $\hat{F}_n$ do satisfy assumption (A4) for every $n > 0$.*

Supposing the functional M is Hermitian on X, we can easily extend the definition of the functional M to $\mathcal{R}'_{\infty *}$. Each time a next product $\varphi_{k*}\varphi_n$, with $k > n \geq 0$, is considered, we define $M\{\varphi_{k*}\varphi_n\} = 0$ if $\varphi_{k*}\varphi_n \notin \mathcal{R}_{k-1*,n} + \mathcal{R}'_\infty$; otherwise, if $\varphi_{k*}\varphi_n \in \mathcal{R}'_\infty + \mathcal{R}_{k-1*,n}$, it follows that there exist functions $f \in \mathcal{R}'_\infty$ and $h \in \mathcal{R}_{k-1*,n}$ so that $\varphi_{k*}\varphi_n = f + h$. Clearly, we then have that $f \in X$ so that $M\{f\} = \overline{M\{f_*\}}$. Because of the induction hypothesis we also have that $M\{h\} = \overline{M\{h_*\}}$. Consequently, it must hold that $M\{\varphi_{k*}\varphi_n\} = \overline{M\{\varphi_k\varphi_{n*}\}} = 0$.

4.3 The diagonal $M\{|\varphi_n|^2\} = M\{\varphi_n\varphi_{n*}\}$

So far we have defined M on $\mathcal{R}'_\infty + \mathcal{R}'_{\infty *}$ so that

$$M\{|\varphi_0|^2\} = 1 \quad \text{and} \quad M\{\varphi_k\varphi_{n*}\} = 0, \quad k \neq n.$$

Hence, it remains to verify whether $\mathcal{R}'_\infty + \mathcal{R}'_{\infty *} = \mathcal{R}_\infty = \mathcal{L}_\infty \cdot \mathcal{L}_{\infty *}$.

From the three-term recurrence relation given by (1) it follows that

$$\frac{Z_{1*}}{Z_2}\varphi_2 = E_2 Z_{1*}\varphi_1 + F_2\frac{Z_{1*}}{Z_1}\varphi_1 + C_2\frac{Z_{1*}}{Z_{0*}}\varphi_0.$$

Because of Lemma 2.2 we have that there exist constants a_i, with $i = 0, 1, 2$, so that

$$\frac{1}{Z_i} = a_i + \frac{1}{Z_{1*}}, \quad i = 1, 2, \quad \text{respectively} \quad \frac{1}{Z_{0*}} = a_0 + \frac{1}{Z_{1*}}.$$

It is easily verified that $a_i = 1/Z_i(\overline{\alpha}_1)$, while from assumption (A6) it follows that $E_2 + F_2/Z_1(\overline{\alpha}_1) \neq 0$. Consequently,

$$Z_{1*}\varphi_1 \in \text{span}\,\{\varphi_0, \varphi_1, \varphi_2, \varphi_{1*}, \varphi_2\varphi_{1*}\} \subset \mathcal{R}'_\infty + \mathcal{R}'_{\infty *}.$$

Thus $\varphi_1\varphi_{1*} \in \mathcal{R}'_\infty + \mathcal{R}'_{\infty *}$ as well. Similarly, from the three-term recurrence relation it follows for $n > 1$ that

$$\frac{b_{n*}}{Z_{n+1}}\varphi_{n+1} = E_{n+1}b_{n*}\varphi_n + F_{n+1}\frac{b_{n*}}{Z_n}\varphi_n + C_{n+1}\frac{b_{n*}}{Z_{n-1*}}\varphi_{n-1}.$$

We again have that there exist constants $a_i = 1/Z_i(\overline{\alpha}_n)$, with $i = n-1, n, n+1$, so that

$$\frac{1}{Z_i} = a_i + \frac{1}{Z_{n*}}, \quad i = n, n+1, \quad \text{respectively} \quad \frac{1}{Z_{n-1*}} = a_{n-1} + \frac{1}{Z_{n*}},$$

while from assumption (A6) it follows that $E_{n+1} + F_{n+1}/Z_n(\overline{\alpha}_n) \neq 0$. Hence, it follows that

$$b_{n*}\varphi_n \in \text{span}\,\{\varphi_{n-1}, \varphi_n, \varphi_{n+1}, \varphi_{n-1}\varphi_{n*}, \varphi_{n+1}\varphi_{n*}\} \subset \mathcal{R}'_\infty + \mathcal{R}'_{\infty*}.$$

Therefore, $\varphi_n\varphi_{n*} \in \mathcal{R}'_\infty + \mathcal{R}'_{\infty*}$ for every $n \geq 0$, which means that $\mathcal{R}'_\infty + \mathcal{R}'_{\infty*} = R_\infty = \mathcal{L}_\infty \cdot \mathcal{L}_{\infty*}$.

Finally, we have the following lemma.

Lemma 4.2. *Suppose that $M\{|\varphi_0|^2\} = 1$. Under the assumption given by (A6) it then holds that $M\{|\varphi_n|^2\} = 1$ for every $n \geq 0$.*

Proof. In [5, Thm. 3.1(1)] it has been proved that

$$M\{|\varphi_k|^2\} = \overline{\kappa}_k M\{\varphi_k b_{k*}\} + M\{\varphi_k f_{k-1*}\},$$

where $f_{k-1} \in \mathcal{L}_{k-1}$ and κ_k is the coefficient of b_k in the expansion of φ_k with respect to the basis $\{b_0, \ldots, b_k\}$. This equality has been used in the proof of [5, Thm. 3.2], under the assumptions that

$$M\{|\varphi_k|^2\} = 1, \quad k = n-2, n-1, \quad n \geq 2, \tag{14}$$

and

$$M\{\varphi_k\varphi_{j*}\} = 0, \quad k \neq j. \tag{15}$$

Rewriting this proof, keeping the assumption given by (15) but without the assumption given by (14), leads to

$$M\{|\varphi_{n-1}|^2\} = -\frac{C_n\overline{E}_{n-1}}{E_n + F_n/Z_{n-1}(\overline{\alpha}_{n-1})} M\{|\varphi_{n-2}|^2\}, \quad n = 2, 3, \ldots,$$

which proves the statement. □

4.4 Favard theorem

Provided that Hypothesis 4.1 holds, we now have proved the following Favard type theorem.

Theorem 4.3 (Favard)**.** *Let $\{\varphi_n\}$ be a sequence of rational functions, and assume that the following conditions are satisfied:*

(C1) $\alpha_{-1} \in \overline{\mathbb{R}}_0$ *and* $\alpha_n \in \overline{\mathbb{C}}_0$, $n = 0, 1, \ldots$,

(C2) φ_n *is generated by the three-term recurrence relation* (1),

(C3) $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $n = 1, 2, \ldots$, *and* $\varphi_0 \in \mathbb{C}_0$,

(C4) let $\hat{F}_n$ be defined as before. Then $|\hat{F}_n| < \infty$ and

$$\frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} - \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{|E_{n-1}|^2}{|E_n|^2} = \left[\frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}|\hat{F}_n|^2 - \Im\{\hat{F}_n\}\right] \cdot \left[|E_{n-1}|^2 - 4\frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} \cdot \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2}\right],$$

$n = 1, 2, \ldots,$

(C5) *max* $\left\{0, 4\frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}\right\} < |E_n|^2 < \infty$, $n = 0, 1, \ldots$,

(C6) $C_n\overline{E}_{n-1} = -\left[E_n + F_n/Z_{n-1}(\overline{\alpha}_{n-1})\right] \neq 0$, $n = 1, 2, \ldots$.

Then there exists a functional M *on* $\mathcal{L}_\infty \cdot \mathcal{L}_{\infty *}$ *so that*

$$\langle f, g \rangle = M\{fg_*\}$$

defines a Hermitian positive-definite inner product on $\mathcal{L}_\infty$ *for which the* φ_n *form an orthonormal system.*

Proof. Because of the previous analysis, the linear functional M is defined in such a way that the orthonormality is satisfied. For every $h \in \mathcal{R}_\infty$ there exist functions $f = \sum a_i\varphi_i \in \mathcal{L}_\infty$ and $g = \sum b_j\varphi_j \in \mathcal{L}_\infty$ so that $h = fg_*$. Consequently,

$$M\{h_*\} = M\left\{\sum \overline{a}_i\varphi_{i*} \cdot \sum b_j\varphi_j\right\} = \sum \overline{a}_i b_i = \overline{\sum \overline{b}_i a_i} = \overline{M\{h\}}.$$

Finally, the positivity is guaranteed by

$$M\{|f|^2\} = \sum |a_i|^2 > 0$$

for every $f \neq 0$. □

5 Conclusion

Let $\{\varphi_n\}$ be a sequence of rational functions with arbitrary complex poles, generated by a certain three-term recurrence relation. In this paper we have shown that - under some mild conditions - the rational functions φ_n form an orthonormal system with respect to a Hermitian positive-definite inner product.

The proof of this Favard type theorem involved a hypothesis with respect to one of the conditions given in the formulation of the theorem; i.e. we have assumed that the specific condition is necessary and sufficient to ensure the inner product is Hermitian. Certainly this specific condition is necessary. Moreover, in the special case in which every pole is real, the condition is sufficient as well. However, still open for investigation is whether the condition is also sufficient in the general case of arbitrary complex poles.

References

[1] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, *"Orthogonal rational functions similar to Szegő polynomials"*, Orthogonal polynomials and their applications, IMACS annals on computing and applied mathematics, Vol. 9, pp. 195–204, 1991.

[2] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, *"A Favard theorem for orthogonal rational functions on the unit circle"*, Numerical Algorithms, Vol. 3, pp. 81–90, 1992.

[3] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, *"A Favard theorem for orthogonal rational functions with poles on the unit circle"*, East Journal on Approximation, Vol. 3, pp. 21–37, 1997.

[4] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, *"Orthogonal Rational Functions"*, volume 5 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1999. (407 pages).

[5] K. Deckers and A. Bultheel, *"Recurrence and asymptotics for orthogonal rational functions on an interval"*, IMA Journal of Numerical Analysis, 2008. (Accepted)

[6] E. Hendriksen and O. Njåstad, *"A Favard theorem for rational functions"*, Journal of Mathematical Analysis and Applications, Vol. 142, No. 2, pp. 508–520, 1989.

[7] E. Hendriksen and H. van Rossum, *"Orthogonal Laurent polynomials"*, Nederl. Akad. Wetensch. Indag. Math., Vol. 48, No. 1, pp. 17–36, 1986.

[8] J. Van Deun and A. Bultheel, *"Orthogonal rational functions and quadrature on an interval"*, Technical Report TW322, Department of Computer Science, Katholieke Universiteit Leuven, Heverlee, Belgium, March 2001.