

# On the $L_p$ -stability of quasi-hierarchical Powell-Sabin B-splines

*Hendrik Speleers*

*Paul Dierckx*

*Stefan Vandewalle*

*Report TW 492, May 2007*



Katholieke Universiteit Leuven  
Department of Computer Science

Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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Quasi-hierarchical Powell-Sabin splines are  $C^1$ -continuous quadratic splines defined on a locally refined hierarchical triangulation. They admit a compact representation in a normalized B-spline basis. We prove that the quasi-hierarchical basis is in general weakly  $L_p$ -stable, but for a broad class of hierarchical triangulations it is even strongly  $L_p$ -stable.

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**AMS(MOS) Classification :** Primary : 41A15 Secondary : 65D07

# On the $L_p$ -stability of quasi-hierarchical Powell-Sabin B-splines

Hendrik Speleers, Paul Dierckx and Stefan Vandewalle

*Department of Computer Science, Katholieke Universiteit Leuven  
Celestijnenlaan 200A, B-3001 Leuven, Belgium*

## Abstract

Quasi-hierarchical Powell-Sabin (QHPS) splines are  $C^1$ -continuous quadratic splines defined on a locally refined hierarchical triangulation. They admit a compact representation in a normalized B-spline basis. We prove that the QHPS basis is in general weakly  $L_p$ -stable, but for a broad class of hierarchical triangulations it is even strongly  $L_p$ -stable.

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## 1 Introduction

Powell-Sabin splines [9] are  $C^1$ -continuous quadratic splines defined on a conforming triangulation. They can be compactly represented in a stable normalized B-spline basis [1] with an intuitive geometric interpretation involving tangent control triangles. These splines are effective in many application domains, e.g., surface modelling and visualization [5], and data fitting [2, 8].

Recently, we developed a hierarchical variant of the Powell-Sabin splines in a quasi-hierarchical representation, the so-called QHPS splines [11]. A hierarchical basis is constructed such that the basis functions of the coarser spaces are retained in the basis of the finer space. In a quasi-hierarchical basis some of the coarse-level basis functions are replaced by finer-level basis functions [4]. Similar to classical hierarchical Powell-Sabin splines, the QHPS splines admit a local refinement in a straightforward way. In addition, they have some other advantages. The basis functions form a convex partition of unity. With each B-spline a control point can be associated such that the QHPS surface lies inside the convex hull of these points. The support of the basis functions is also smaller than in case of a hierarchical basis. Therefore, the associated Gram matrix will be a lot sparser.

In this paper we consider the  $L_p$ -stability of the QHPS basis. A basis  $\{\varphi_i\}_{i=1}^n$  is called  $L_p$ -stable if for all choices of the coefficient vector  $c$  it holds that

$$k_1 \|c\|_p \leq \left\| \sum_i c_i \varphi_i \right\|_p \leq k_2 \|c\|_p, \quad (1.1)$$

where  $\|v\|_p$  stands for the  $l_p$ -norm in case  $v$  is a vector, and for the  $L_p$ -norm in case  $v$  is a function. A (quasi-) hierarchical basis is weakly  $L_p$ -stable if the constants  $k_1^{-1}$  and  $k_2$  in (1.1) have at most a polynomial growth in the number of levels in the hierarchical mesh. The basis is strongly  $L_p$ -stable if these constants are independent of the number of levels. The ratio  $k_2/k_1$  gives an upper bound for the  $L_p$ -norm condition number  $\kappa_p$  of the basis. It reflects the influence of a change in the coefficients on the magnitude of the corresponding spline measured in the  $L_p$ -norm.

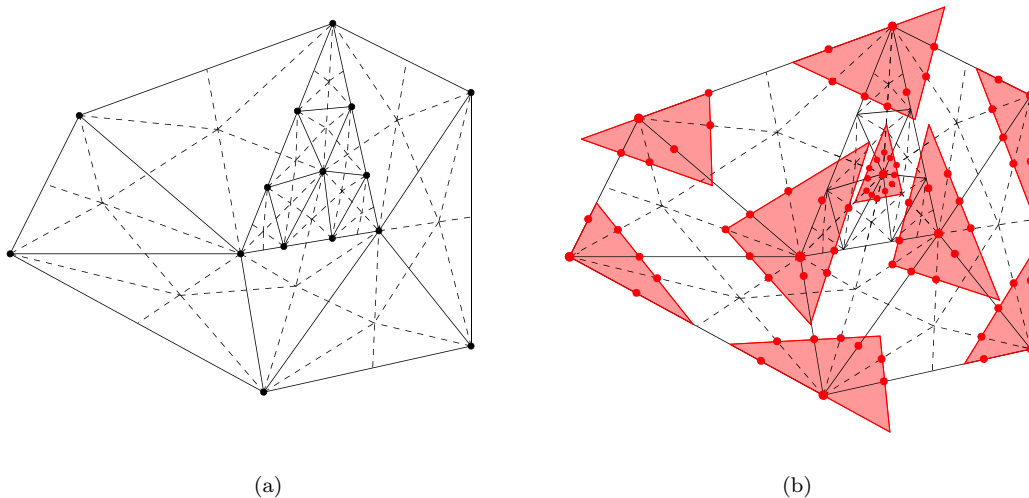


Figure 1: (a) A HPS refinement  $\Delta_H^*$  (dashed lines) of a given hierarchical triangulation  $\Delta_H$  (solid lines); (b) the QHPS points (bullets) and a set of suitable QHPS triangles (shaded).

The paper is organized as follows. Section 2 recalls the definition of the QHPS spline space, and the construction of its basis. In section 3 we consider some  $L_p$ -norm inequalities for polynomials on a triangle. Section 4 is devoted to the  $L_p$ -stability of the QHPS B-splines. The basis functions need to be scaled with respect to their support. In general, the scaled QHPS basis is weakly  $L_p$ -stable. Yet, in section 5 we will show that the triangulation can be locally adapted such that the basis on the new triangulation is strongly  $L_p$ -stable. We formulate some conclusions in section 6.

## 2 Quasi-hierarchical Powell-Sabin splines

Consider a simply connected subset  $\Omega \subset \mathbb{R}^2$  with polygonal boundary, and assume a conforming triangulation  $\Delta^0$  of  $\Omega$  is given. A triangulation is conforming if no triangle contains a vertex different from its own three vertices. We construct a hierarchical triangulation  $\Delta_H$  on  $\Omega$  by partitioning successively subsets of triangles with a triadic split, starting from the initial triangulation  $\Delta^0$ . An example of such a triangulation is drawn in Figure 1(a) with solid lines. The hierarchical triangulation has a total of  $n$  vertices. Of these vertices,  $n_{nc}$  are non-conforming (or hanging) vertices. They are located on interiors of triangle edges. The remaining ones, i.e.  $n_c = n - n_{nc}$ , are called conforming vertices. In Figure 1(a),  $\Delta^0$  consists of 8 triangles and 8 (conforming) vertices. The hierarchical triangulation in the figure consists of 16 triangles and 15 vertices ( $n_c = 9$ ,  $n_{nc} = 6$ ).

To each hierarchical triangulation  $\Delta_H$  we can associate a hierarchical mesh structure  $\mathbf{\Delta}_H$ . It is the set of triangles  $\rho^k$  that are generated during the construction of  $\Delta_H$ . The superscript  $k$  of a triangle  $\rho^k$  refers to the refinement level of that triangle, i.e., the minimal number of refinement steps needed to construct the triangle. The triangles in  $\mathbf{\Delta}_H$  that are part of  $\Delta_H$  are called leaf triangles. We will denote  $\mathbf{\Delta}_H^l$  as the subset of  $\mathbf{\Delta}_H$  containing all triangles whose level is  $l$  or lower, and let  $\Delta_H^l$  be its corresponding hierarchical triangulation. Note that these mesh structures are nested

$$\Delta^0 \subset \mathbf{\Delta}_H^1 \subset \mathbf{\Delta}_H^2 \subset \dots \subset \mathbf{\Delta}_H. \quad (2.1)$$

The hierarchical Powell-Sabin (HPS) refinement  $\Delta_H^*$  is a particular partition of  $\Delta_H$ , where every triangle is split into six smaller triangles with a common vertex (see [11] for details on the con-

struction). In Figure 1(a) such a HPS refinement is drawn in dashed lines. Analogous to (2.1), the HPS refinement yields a nested structure of sets of triangles

$$\Delta^{*,0} \subset \Delta_H^{*,1} \subset \Delta_H^{*,2} \subset \dots \subset \Delta_H^*. \quad (2.2)$$

Let  $\Pi_d$  be the space of bivariate polynomials of degree  $d$ . The space of piecewise quadratic polynomials on  $\Delta_H^*$  with global  $C^1$ -continuity is called the hierarchical Powell-Sabin spline space:

$$S_{2,H}^1(\Delta_H^*) = \left\{ s_H \in C^1(\Omega) : s_H|_{\rho_j^*} \in \Pi_2, \rho_j^* \in \Delta_H^* \right\}. \quad (2.3)$$

In [11] is proved that the following interpolation problem at each conforming vertex  $V_k$

$$s_H(V_k) = f_k, \quad D_x s_H(V_k) = f_{x,k}, \quad D_y s_H(V_k) = f_{y,k}, \quad (2.4)$$

has a unique solution  $s_H(x, y) \in S_{2,H}^1(\Delta_H^*)$  for any set of  $n_c$  triplets  $(f_k, f_{x,k}, f_{y,k})$ . It follows that the dimension of the hierarchical Powell-Sabin spline space is equal to  $3n_c$ .

We presented a geometric method in [11] to construct a normalized quasi-hierarchical basis for the spline space  $S_{2,H}^1(\Delta_H^*)$ . A hierarchical Powell-Sabin spline in its quasi-hierarchical representation is called a QHPS spline, and it is denoted as

$$s_{QH}(x, y) = \sum_{i=1}^{n_c} \sum_{j=1}^3 c_{i,j} B_{i,QH}^j(x, y). \quad (2.5)$$

To find the QHPS basis functions, we associate with each conforming vertex  $V_i$  in the hierarchical triangulation three linearly independent triplets  $(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})$ ,  $j = 1, 2, 3$ . These triplets are determined as follows:

1. For each conforming vertex  $V_i$  in the hierarchical triangulation  $\Delta_H$ , identify the corresponding QHPS points. Let  $k$  be the smallest level of all triangles in  $\Delta_H$  that contain the vertex  $V_i$ . The QHPS points of  $V_i$  are defined as the midpoints of all edges in the HPS refinement  $\Delta_H^{*,k}$  ending in  $V_i$ . The vertex  $V_i$  itself is also a QHPS point. In Figure 1(b) the QHPS points are indicated as bullets.
2. For each conforming vertex  $V_i$ , find a triangle  $t_i(Q_{i,1}, Q_{i,2}, Q_{i,3})$  containing all the QHPS points of  $V_i$ . The triangles  $t_i$ ,  $i = 1, \dots, n_c$ , are called QHPS triangles. Note that these triangles are not uniquely defined. Figure 1(b) shows a possible set of QHPS triangles.
3. The three linearly independent triplets  $(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})$ ,  $j = 1, 2, 3$ , are derived from the QHPS triangle  $t_i$  of a vertex  $V_i$  as follows:
  - $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3})$  are the barycentric coordinates of  $V_i$  with respect to  $t_i$ ,
  - $\beta_i = (\beta_{i,1}, \beta_{i,2}, \beta_{i,3})$  and  $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}, \gamma_{i,3})$  are the coordinates of the unit barycentric directions, in  $x$ - and  $y$ -direction respectively, with respect to  $t_i$ .

The QHPS B-spline  $B_{i,QH}^j(x, y)$  is defined as the unique solution of interpolation problem (2.4) with all  $(f_k, f_{x,k}, f_{y,k}) = (0, 0, 0)$  except for  $k = i$ , where  $(f_i, f_{x,i}, f_{y,i}) = (\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}) \neq (0, 0, 0)$ .

These quasi-hierarchical basis functions satisfy some interesting properties. The QHPS B-splines have a local support. We denote the support of the B-splines associated with the conforming vertex  $V_i$  as  $S_i$  and its area as  $A_{S_i}$ . Let  $k$  be the smallest level of any triangle in the hierarchical triangulation  $\Delta_H$  that contains vertex  $V_i$ . The molecule  $M_i^k$  of vertex  $V_i$  is defined as the union of all triangles  $\rho^k \in \Delta_H^k$  containing the vertex. The superscript  $k$  of  $M_i^k$  is called the level number of the molecule in  $\Delta_H$ . It is easy to check that  $B_{i,QH}^j(x, y)$  is zero outside the molecule  $M_i^k$ , or  $S_i \subseteq M_i^k$ .

The proposed basis also forms a convex partition of unity on  $\Omega$ , i.e.,

$$B_{i,QH}^j(x,y) \geq 0, \quad \text{and} \quad \sum_{i=1}^{n_c} \sum_{j=1}^3 B_{i,QH}^j(x,y) = 1. \quad (2.6)$$

We find QHPS control points as  $\mathbf{c}_{i,j} = (Q_{i,j}, c_{i,j})$ . These points define QHPS control triangles  $T_i(\mathbf{c}_{i,1}, \mathbf{c}_{i,2}, \mathbf{c}_{i,3})$ , which are tangent to the spline surface  $z = s_{QH}(x,y)$  at the conforming vertices  $V_i$ . The projection of the control triangles  $T_i$  in the  $(x,y)$ -plane are the QHPS triangles  $t_i$ .

If the hierarchical triangulation is obtained by global triadic splits, i.e., the final triangulation is conforming, then the QHPS B-splines are just the classical normalized Powell-Sabin B-splines [1].

### 3 Inequalities for polynomials on triangles

In this section we consider some inequalities for the  $L_p$ -norm of polynomials on a triangle  $\rho \subset \mathbb{R}^2$ . This norm will be denoted by the symbol  $\|\cdot\|_{p,\rho}$ . The  $l_p$ -norm of vectors will be denoted by  $\|\cdot\|_p$ . We will need the inequalities later on to prove the  $L_p$ -stability of the QHPS basis. The following  $L_p$ -norm equivalence is shown in [7].

**Lemma 3.1.** *Let  $f \in L_\infty(\rho)$ , then for any  $p, q$  with  $1 \leq q \leq p \leq \infty$  it holds that*

$$C^{1/p-1/q} \|f\|_{p,\rho} \leq \|f\|_{q,\rho} \leq A_\rho^{1/q-1/p} \|f\|_{p,\rho}, \quad (3.1)$$

where  $A_\rho$  is the area of  $\rho$ , and the constant  $C$  satisfies  $\|f\|_{\infty,\rho} \leq C \|f\|_{1,\rho}$ .

Denote the unit triangle as  $\rho_s = \{(x,y) : 0 \leq x, y \leq 1; x+y \leq 1\}$ . To estimate the constant  $C$  in (3.1) for  $f$  being a polynomial on  $\rho_s$ , we use the following Markov inequality from [6].

**Lemma 3.2.** *Let  $f \in \Pi_d(\rho_s)$  and define  $\|\nabla f\|_{\infty,\rho_s} = \|\max\{D_v f : \|v\|_2 = 1\}\|_{\infty,\rho_s}$ . Then,*

$$\|\nabla f\|_{\infty,\rho_s} \leq K_s \|f\|_{\infty,\rho_s}, \quad (3.2)$$

with  $K_s = (4d^2 - 2d)\sqrt{2}$ .

Later on, we are primarily interested in the case  $d = 2$ , for which  $K_s = 12\sqrt{2}$ . The proof of the next theorem is inspired by [7], but the result is improved.

**Lemma 3.3.** *For any  $f \in \Pi_d(\rho_s)$  it holds that*

$$\|f\|_{\infty,\rho_s} \leq 12 K_s^2 \|f\|_{1,\rho_s}. \quad (3.3)$$

*Proof.* Define  $a = (a_x, a_y)$  as the point in  $\rho_s$  where the max-norm of polynomial  $f$  is attained, i.e.,  $\|f\|_{\infty,\rho_s} = |f(a)|$ . Using the Mean Value Theorem, Lemma 3.2 and the vector norm inequality  $\|\cdot\|_2 \leq \|\cdot\|_1$ , we obtain for each  $b \in \rho_s$  that

$$|f(a)| - |f(b)| \leq |f(a) - f(b)| \leq \|\nabla f\|_{\infty,\rho_s} \|a - b\|_2 \leq K_s |f(a)| \|a - b\|_1,$$

or

$$\|f\|_{\infty,\rho_s} (1 - K_s \|a - b\|_1) \leq |f(b)|.$$

For any subdomain  $\Omega_s \subseteq \rho_s$ , it follows that

$$\|f\|_{\infty,\rho_s} \int_{\Omega_s} (1 - K_s \|a - b\|_1) db \leq \|f\|_{1,\Omega_s} \leq \|f\|_{1,\rho_s}. \quad (3.4)$$

We now determine a lower bound for the integral in (3.4) on a particular subdomain  $\Omega_s$ . We are looking for a subdomain where this integral is easy to compute and on which the value of the integral is as large as possible. Indeed, the larger this value, the sharper the lower bound in (3.4).

We first consider the following sequence of triangular subdomains:

$$\hat{\rho}_s = \{a + \mu(\hat{b} - a) : \hat{b} \in \rho_s\}, \quad (3.5)$$

for some  $0 < \mu \leq 1$ . In this case, the considered integral is denoted as  $G(\mu)$ , and it is given by

$$G(\mu) = \int_{\hat{\rho}_s} (1 - K_s \|a - b\|_1) db = \mu^2 \int_{\rho_s} (1 - K_s \mu \|a - \hat{b}\|_1) d\hat{b}.$$

After some elementary calculations, we obtain that

$$G(\mu) = \frac{\mu^2}{6} (3 - K_s \mu (2 - 2a_x^3 + 6a_x^2 - 3a_x - 3a_y + 6a_y^2 - 2a_y^3)). \quad (3.6)$$

We are interested in the case where (3.6) is maximal. The maximum of this function is attained at  $\mu^*$ , and is given by

$$G(\mu^*) = \frac{2}{3} (K_s (2 - 2a_x^3 + 6a_x^2 - 3a_x - 3a_y + 6a_y^2 - 2a_y^3))^{-2}. \quad (3.7)$$

Since we do not know the exact position of the point  $a \in \rho_s$ , we consider the lowest value of  $G(\mu^*)$  on  $\rho_s$ :

$$G(\mu^*) \geq \frac{2}{27 K_s^2}. \quad (3.8)$$

This gives us a lower bound for the integral in (3.4). However, we can improve the result as follows. The equality in (3.8) is attained when the point  $a$  is situated at the corners  $(0, 1)$  and  $(1, 0)$  of the unit triangle  $\rho_s$ . By changing the shape of the triangle  $\hat{\rho}_s$  in the neighbourhood of these corners we will obtain a sharper bound. When we restrict  $a$  to the square  $\{(x, y) : 0 \leq x, y \leq 1/2\}$ , then (3.7) is bounded below by

$$G(\mu^*) \geq \frac{1}{6 K_s^2}. \quad (3.9)$$

If  $a$  belongs to the domain  $\{(x, y) : 1/2 \leq x; 0 \leq y; x + y \leq 1\}$ , we construct the following sequence of triangles. Let  $e_1 = (-1, 0)$  and  $e_2 = (-1, 1)$ , then

$$\tilde{\rho}_s = \{a + \mu_1 \tilde{b}_x e_1 + \mu_2 \tilde{b}_y e_2 : (\tilde{b}_x, \tilde{b}_y) \in \rho_s\}, \quad (3.10)$$

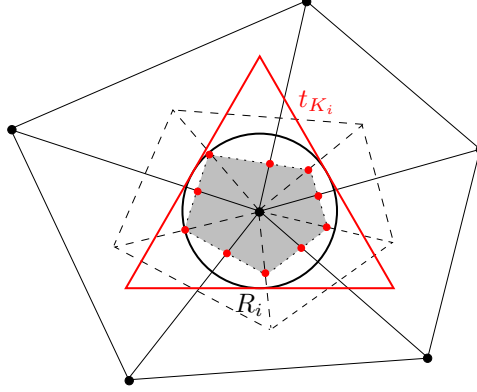
for some  $\mu_1 \leq a_x$  and  $\mu_2 \leq \mu_1$ , such that  $\tilde{\rho}_s \subseteq \rho_s$ . We now compute the integral in (3.4) on  $\tilde{\rho}_s$ . It is denoted as  $G(\mu_1, \mu_2)$ , and it holds

$$\begin{aligned} G(\mu_1, \mu_2) &= \int_{\tilde{\rho}_s} (1 - K_s \|a - b\|_1) db = \mu_1 \mu_2 \int_{\rho_s} (1 - K_s (\mu_1 \tilde{b}_x + 2\mu_2 \tilde{b}_y)) d\tilde{b} \\ &= \frac{\mu_1 \mu_2}{6} (3 - K_s (\mu_1 + 2\mu_2)). \end{aligned}$$

The maximal value of this function  $G(\mu_1, \mu_2)$  is obtained for

$$\mu_1^* = \frac{1}{K_s}, \quad \mu_2^* = \frac{1}{2K_s}, \quad G(\mu_1^*, \mu_2^*) = \frac{1}{12 K_s^2}. \quad (3.11)$$

This value is independent of the position of the point  $a$ . Analogously, we find the same bound for the corresponding integral if  $a$  is situated in  $\{(x, y) : 0 \leq x; 1/2 \leq y; x + y \leq 1\}$ . Taking the lowest bound of (3.9) and (3.11), and using it in (3.4), we obtain (3.3).  $\square$

Figure 2: The disk  $R_i$  and the equilateral triangle  $t_{K_i}$  for  $K_i = 1$ .

**Lemma 3.4.** For any  $f \in \Pi_d(\rho)$  we have

$$\|f\|_{\infty, \rho} \leq \left( \frac{6 K_s^2}{A_\rho} \right)^{1/p} \|f\|_{p, \rho}. \quad (3.12)$$

*Proof.* Suppose that the polynomial  $f$  on triangle  $\rho$  is written in its Bernstein-Bézier representation [3], and that  $g$  is the Bernstein-Bézier polynomial on the unit triangle  $\rho_s$  with the same coefficients as  $f$ . Then, it holds via a coordinate transformation and Lemma 3.3 that

$$\|f\|_{\infty, \rho} = \|g\|_{\infty, \rho_s} \leq 12 K_s^2 \|g\|_{1, \rho_s} = \frac{6 K_s^2}{A_\rho} \|f\|_{1, \rho}.$$

Using Lemma 3.1 we get the inequality (3.12).  $\square$

## 4 Stability for the $L_p$ -norm

From [11] we know that the QHPS B-spline functions form a stable basis for the  $L_\infty$ -norm. The choice of the QHPS triangles will be important, since the B-splines and hence the approximation constants in (1.1) will depend on these triangles. A constant  $K$  is introduced that reflects the size and shape of the QHPS triangles used in the definition of a QHPS B-spline basis.

**Definition 4.1.** Let  $R_i$  be the smallest disk with the conforming vertex  $V_i$  as center that contains all the QHPS points of  $V_i$ , as shown in Figure 2, and denote by  $r_i$  its radius. Define  $K_i$  as the smallest value such that there exists an equilateral triangle  $t_{K_i}$  with inradius  $K_i r_i$  which contains the actual QHPS triangle  $t_i$ . Let  $K$  be the maximum of all constants  $K_i$  for each conforming vertex  $V_i$  in  $\Delta_H$ .

**Theorem 4.1 ( $L_\infty$ -stability).** Denote  $\theta_{\Delta_H^*}$  as the smallest angle in  $\Delta_H^*$ . The QHPS basis is strongly  $L_\infty$ -stable, i.e., for any coefficient vector  $c$  in the QHPS spline representation (2.5) we have

$$k_{1, \infty} \|c\|_\infty \leq \|s_{QH}(x, y)\|_\infty \leq k_{2, \infty} \|c\|_\infty, \quad (4.1)$$

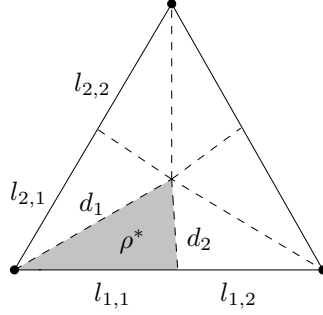


Figure 3: A triangle  $\rho^*$  in the HPS refinement  $\Delta_H^*$ , inside the triangle  $\rho \in \Delta_H$ .

with

$$k_{1,\infty} = \left( 1 + \frac{48\sqrt{3}KL}{\tan(\theta_{\Delta_H^*}/2)} \right)^{-1}, \quad k_{2,\infty} = 1, \quad (4.2)$$

and with  $L = \sin(\theta_{\Delta_H^*})^{-\pi/\theta_{\Delta_H^*}}$ . In some particular cases the constant  $L$  can be reduced, e.g., for the classical Powell-Sabin B-splines we can take  $L = 1$ .

To have an  $L_p$ -stable basis, the QHPS basis functions need to be scaled with respect to their support. The proof of the stability theorem will be similar to the one for classical PS splines [12]. However, our stability bounds will be somewhat sharper, and they contain no unspecified constants. We first formulate two lemmas with respect to some area relations.

**Lemma 4.2.** *Suppose that  $\rho^* \in \Delta_H^*$  is a subtriangle of  $\rho \in \Delta_H$ . The areas of both triangles are related as*

$$A_{\rho^*} \leq A_{\rho} \leq K_{a^*} A_{\rho^*}, \quad (4.3)$$

where  $K_{a^*} = 2 \sin(\theta_{\Delta_H^*})^{-3}$  with  $\theta_{\Delta_H^*}$  the smallest angle in  $\Delta_H^*$ .

*Proof.* The lower bound is trivial. In order to derive the upper bound, we first recall that by the law of sines it holds that

$$\sin(\theta_{\rho}) l_1 \leq l_2, \quad (4.4)$$

where  $l_1$  and  $l_2$  are the lengths of two sides of a triangle, and  $\theta_{\rho}$  is the smallest angle in the triangle. Now, consider triangles  $\rho^*$  and  $\rho$  in Figure 3. By relation (4.4) it follows that the lengths in the figure are related as

$$l_{1,1} \leq \sin(\theta_{\Delta_H^*})^{-1} d_2, \quad l_{1,2} \leq \sin(\theta_{\Delta_H^*})^{-1} d_2, \quad \text{and} \quad l_{2,1} + l_{2,2} \leq \sin(\theta_{\Delta_H^*})^{-1} d_1, \quad (4.5)$$

with  $\theta_{\Delta_H^*}$  the smallest angle in  $\Delta_H^*$ . Using

$$\frac{d_1 d_2}{2} \sin(\theta_{\Delta_H^*}) \leq A_{\rho^*} \leq \frac{d_1 d_2}{2}, \quad (4.6)$$

and the inequalities in (4.5), we obtain that

$$A_{\rho} \leq \frac{(l_{1,1} + l_{1,2})(l_{2,1} + l_{2,2})}{2} \leq d_1 d_2 \sin(\theta_{\Delta_H^*})^{-2} \leq 2 \sin(\theta_{\Delta_H^*})^{-3} A_{\rho^*}.$$

□

**Lemma 4.3.** *Let  $k$  be the smallest level of any triangle in  $\Delta_H$  that contains the conforming vertex  $V_i$ , and let  $\rho^k$  be such a triangle. The area of the molecule  $M_i^k$  of  $V_i$  is related to the area of the triangle  $\rho^k$  as*

$$A_{\rho^k} \leq A_{M_i^k} \leq K_a A_{\rho^k}, \quad (4.7)$$

where  $K_a = \frac{2\pi}{\theta_{\Delta_H}} \sin(\theta_{\Delta_H})^{-\frac{2\pi}{\theta_{\Delta_H}} - 1}$  with  $\theta_{\Delta_H}$  the smallest angle in  $\Delta_H$ .

*Proof.* The lower bound is trivial. Let  $\hat{\rho}^k \in \Delta_H^k$  be any triangle in the molecule  $M_i^k$ . Note that  $\rho^k$  also lies in  $M_i^k$ . Denote  $l_1$  and  $l_2$  as the lengths of the two edges of  $\rho^k$  ending in vertex  $V_i$ , and denote  $\hat{l}_1$  and  $\hat{l}_2$  as the lengths of the edges of  $\hat{\rho}^k$  ending in  $V_i$ . By the bounds (4.6) for the area of a triangle and by repeated application of the relation (4.4) for two neighbouring edges, we get

$$A_{\hat{\rho}^k} \leq \frac{\hat{l}_1 \hat{l}_2}{2} \leq \sin(\theta_{\Delta_H})^{-2\lfloor m/2 \rfloor} \frac{l_1 l_2}{2} \leq \sin(\theta_{\Delta_H})^{-2\lfloor m/2 \rfloor - 1} A_{\rho^k},$$

with  $m$  the number of triangles in the molecule. It follows that

$$A_{M_i^k} \leq m \max_{\hat{\rho}^k \in M_i^k} A_{\hat{\rho}^k} \leq m \sin(\theta_{\Delta_H})^{-2\lfloor m/2 \rfloor - 1} A_{\rho^k}.$$

Since the number  $m$  can be bounded by  $m \leq 2\pi/\theta_{\Delta_H}$ , the proof is completed.  $\square$

We now introduce a constant  $D$ , called the level disparity. It will reflect the influence of the number of levels in the triangulation on the  $L_p$ -stability constants.

**Definition 4.2.** *Let  $D_i$  be the largest difference between the levels of the triangles  $\rho \in \Delta_H$  that lie in the support  $S_i$  of the QHPS  $B$ -splines associated with a conforming vertex  $V_i$ . Define the level disparity  $D$  as the maximum of all constants  $D_i$ .*

When the hierarchical triangulation  $\Delta_H$  is obtained by global triadic splits, then  $D = 0$ . The level disparity in Figure 1(a) equals  $D = 1$ . Using many successive local refinements in the hierarchical triangulation may result in a large level disparity. Further on, we discuss this notion in more detail.

**Lemma 4.4.** *Each triangle  $\rho \in \Delta_H$  appears in the support of at most  $3(D+1)$  conforming vertices.*

*Proof.* Suppose that the triangle  $\rho$  is of level  $l$ . By Definition 4.2 the triangle  $\rho$  can only lie inside the support  $S_i$  of the vertices  $V_i$ , for which the smallest level number  $k$  of any triangle of  $\Delta_H$  in  $S_i$  is bounded by  $l - D \leq k \leq l$ . We also know that if  $\rho$  is situated outside the molecule  $M_i^k$  of a vertex  $V_i$ , then it lies outside  $S_i$ . Since a triangle can belong to at most three molecules of level  $k$ , and  $k$  can only vary between  $l - D \leq k \leq l$ , the proof is completed.  $\square$

We now come to the main theorem.

**Theorem 4.5 ( $L_p$ -stability).** *For  $1 \leq p \leq \infty$ , the spline representation in (2.5) satisfies*

$$k_{1,p} \|c\|_{p,w} \leq \|s_{QH}(x, y)\|_p \leq k_{2,p} \|c\|_{p,w}, \quad (4.8)$$

where

$$\|c\|_{p,w} = \begin{cases} \left( \sum_{i=1}^{n_c} \sum_{j=1}^3 |c_{i,j}|^p A_{S_i} \right)^{1/p}, & 1 \leq p < \infty, \\ \max_{i,j} |c_{i,j}|, & p = \infty, \end{cases} \quad (4.9)$$

and

$$k_{1,p} = k_{1,\infty} [54 K_{a^*} K_a K_s^2 (D+1)]^{-1/p}, \quad (4.10a)$$

$$k_{2,p} = [9(D+1)]^{1-1/p}. \quad (4.10b)$$

The constants  $k_{1,\infty}$ ,  $K_s$ ,  $K_{a^*}$  and  $K_a$  are defined in Theorem 4.1, and Lemmas 3.2, 4.2 and 4.3, respectively. It follows that the scaled QHPS B-splines  $\{B_{i,QH}^j(x,y) A_{S_i}^{-1/p}\}$  form an  $L_p$ -stable basis (1.1). If the hierarchical triangulation is constructed such that the level disparity  $D$  remains bounded independently of the number of levels in the triangulation, then the scaled QHPS basis is strongly  $L_p$ -stable. Otherwise, it is weakly  $L_p$ -stable.

*Proof.* We only treat the  $L_p$ -stability for the case  $1 \leq p < \infty$ . For  $p = \infty$ , the constants (4.10) simplify to  $k_{1,p} = k_{1,\infty}$  and  $k_{2,p} = 9(D+1)$ . In this case, the inequalities in (4.8) hold because of Theorem 4.1.

We first show the lower bound in (4.8). Since by Lemma 4.4 each  $\rho \in \Delta_H$  appears in the support  $S_i$  of at most  $3(D+1)$  conforming vertices  $V_i$ , we obtain

$$\|s_{QH}\|_p^p = \sum_{\rho \in \Delta_H} \|s_{QH}\|_{p,\rho}^p \geq \frac{1}{3(D+1)} \sum_{i=1}^{n_c} \sum_{\rho \subseteq S_i} \|s_{QH}\|_{p,\rho}^p. \quad (4.11)$$

Referring to Figure 3, we suppose that the max-norm of a QHPS spline on triangle  $\rho$  is attained at a point in subtriangle  $\rho^*$ . Applying inequalities (3.12) and (4.3), we obtain

$$\|s_{QH}\|_{\infty,\rho}^p = \|s_{QH}\|_{\infty,\rho^*}^p \leq \frac{6K_s^2}{A_{\rho^*}} \|s_{QH}\|_{p,\rho^*}^p \leq \frac{6K_{a^*}K_s^2}{A_\rho} \|s_{QH}\|_{p,\rho}^p. \quad (4.12)$$

Filling (4.12) in (4.11), we get

$$\begin{aligned} \|s_{QH}\|_p^p &\geq \frac{1}{9(D+1)} \sum_{i=1}^{n_c} \sum_{j=1}^3 \sum_{\rho \subseteq S_i} \|s_{QH}\|_{p,\rho}^p \\ &\geq \frac{1}{9(D+1)} \sum_{i=1}^{n_c} \sum_{j=1}^3 \sum_{\rho \subseteq S_i} \frac{A_\rho}{6K_{a^*}K_s^2} \|s_{QH}\|_{\infty,\rho}^p. \end{aligned} \quad (4.13)$$

Let  $k$  be the smallest level of any triangle in  $\Delta_H$  that contains the conforming vertex  $V_i$ . By the definition of the QHPS B-splines there is at least one triangle  $\rho_i^k \in \Delta_H$  of level  $k$  in the support  $S_i$  associated with vertex  $V_i$ . On such a triangle  $\rho_i^k$  the QHPS spline  $s_{QH}$  can be represented by a classical PS spline, denoted as  $s^k$  with coefficient vector  $c^k$ . It is shown in [11] that the corresponding three spline coefficients associated with vertex  $V_i$  in the representation (2.5) of  $s_{QH}$  and  $s^k$  have the same value. Theorem 4.1 with  $L = 1$  is valid on  $\rho_i^k$ , such that

$$\|s_{QH}\|_{\infty,\rho_i^k} \geq k_{1,\infty} \|c^k\|_\infty \geq k_{1,\infty} |c_{i,j}|. \quad (4.14)$$

We select a subset of terms in the summation of (4.13), and we apply inequality (4.14) to obtain

$$\begin{aligned} \|s_{QH}\|_p^p &\geq \frac{1}{9(D+1)} \sum_{i=1}^{n_c} \sum_{j=1}^3 \frac{A_{\rho_i^k}}{6K_{a^*}K_s^2} \|s_{QH}\|_{\infty,\rho_i^k}^p \\ &\geq \frac{1}{9(D+1)} \sum_{i=1}^{n_c} \sum_{j=1}^3 \frac{A_{\rho_i^k}}{6K_{a^*}K_s^2} k_{1,\infty}^p |c_{i,j}|^p. \end{aligned}$$

Using (4.7) and  $S_i \subseteq M_i^k$ , the lower bound (4.10a) is found.

We now show the upper bound. Since each basis function  $0 \leq B_{i,QH}^j \leq 1$ , it follows

$$\|s_{QH}\|_p^p = \sum_{\rho \in \Delta_H} \int_{\rho} \left| \sum_{i|\rho \subseteq S_i} \sum_j c_{i,j} B_{i,QH}^j \right|^p d\rho \leq \sum_{\rho \in \Delta_H} \int_{\rho} \left( \sum_{i|\rho \subseteq S_i} \sum_j |c_{i,j}| \right)^p d\rho.$$

Let  $1/p + 1/q = 1$ , then by Hölder's inequality for sums and by Lemma 4.4, we have

$$\begin{aligned} \|s_{QH}\|_p^p &\leq \sum_{\rho \in \Delta_H} \left( \sum_{i|\rho \subseteq S_i} \sum_j |c_{i,j}|^p \right)^{p/q} \left( \sum_{i|\rho \subseteq S_i} \sum_j 1 \right)^{p/q} \int_{\rho} d\rho \\ &\leq \sum_{\rho \in \Delta_H} \sum_{i|\rho \subseteq S_i} \sum_j |c_{i,j}|^p (9(D+1))^{p/q} A_{\rho}. \end{aligned}$$

We rearrange the terms in the summation to obtain

$$\begin{aligned} \|s_{QH}\|_p^p &\leq (9(D+1))^{p/q} \sum_i \sum_j \sum_{\rho \subseteq S_i} A_{\rho} |c_{i,j}|^p \\ &= (9(D+1))^{p/q} \sum_i \sum_j A_{S_i} |c_{i,j}|^p, \end{aligned}$$

which proves the upper bound in (4.8).  $\square$

In case of classical Powell-Sabin B-splines (where  $D = 0$ ) we can reduce the constant  $k_{1,p}$  in (4.10a). It can be seen as follows. In (4.13) we have selected a single triangle  $\rho_i^k \in \Delta_H$  of level  $k$  inside the support  $S_i$  for each  $i$ . However, when  $D = 0$  all triangles of  $\Delta_H$  inside the support  $S_i$  are of level  $k$ , and they all contain vertex  $V_i$ . In this case we can simplify (4.13) to

$$\begin{aligned} \|s_{QH}\|_p^p &\geq \frac{1}{9} \sum_{i=1}^{n_c} \sum_{j=1}^3 \sum_{\rho \subseteq S_i} \frac{A_{\rho}}{6 K_{a^*} K_s^2} \|s_{QH}\|_{\infty,\rho}^p \\ &\geq \frac{1}{9} \sum_{i=1}^{n_c} \sum_{j=1}^3 \sum_{\rho \subseteq S_i} \frac{A_{\rho}}{6 K_{a^*} K_s^2} k_{1,\infty}^p |c_{i,j}|^p \\ &\geq \frac{1}{9} \sum_{i=1}^{n_c} \sum_{j=1}^3 \frac{A_{S_i}}{6 K_{a^*} K_s^2} k_{1,\infty}^p |c_{i,j}|^p. \end{aligned}$$

Thus, we get as lower bound  $k_{1,p} = k_{1,\infty} [54 K_{a^*} K_s^2]^{-1/p}$  for classical Powell-Sabin B-splines.

The constant  $K$  in Theorem 4.1 describes the relation between the size of the QHPS triangles and the size of the triangles in the molecules of the vertices. Note that one can always find a set of QHPS triangles that satisfies  $K = 1$ . However, a common choice of (QH)PS triangles are the so-called optimal triangles [1], i.e., the triangles with minimal area. We now prove that the constant  $K$  for optimal QHPS triangles only depends on the smallest angle  $\theta_{\Delta_H^*}$  in  $\Delta_H^*$ .

**Corollary 4.6.** *The scaled QHPS basis in Theorem 4.5, constructed with optimal QHPS triangles, is  $L_p$ -stable, where the stability constants only depend on the level disparity  $D$  and the smallest angle  $\theta_{\Delta_H^*}$  in  $\Delta_H^*$ .*

*Proof.* Considering the stability constants (4.10), we only have to show that the constant  $K$  in  $k_{1,\infty}$  can be bounded in function of the smallest angle  $\theta_{\Delta_H^*}$ . Denote  $t_{R_i}$  as the equilateral triangle

depicted in Figure 2 with inradius  $r_i$ , and let  $t_i$  be the optimal QHPS triangle of vertex  $V_i$ . The triangle  $\rho_i^*$  is one of the triangles in the polygon spanned by the QHPS points of  $V_i$  with a side length equal to  $r_i$ . The inradius of a triangle  $\rho$  is denoted as  $r_\rho$ , the length of its longest side as  $l_\rho$ , and its smallest angle as  $\theta_\rho$ . Since it holds that

$$r_\rho \geq \tan\left(\frac{\theta_3}{2}\right) \frac{l_1 + l_2 - l_3}{2} \geq \tan\left(\frac{\theta_\rho}{2}\right) \frac{l_\rho}{2},$$

with  $l_i$  the side lengths of  $\rho$  and  $l_3$  corresponding to the side opposite to angle  $\theta_3$ , we obtain

$$r_{t_i} \geq r_{\rho_i^*} \geq \tan\left(\frac{\theta_{\rho_i^*}}{2}\right) \frac{l_{\rho_i^*}}{2} \geq \tan\left(\frac{\theta_{\Delta_H^*}}{2}\right) \frac{r_i}{2}. \quad (4.15)$$

By the optimality of the QHPS triangle  $t_i$  we know that  $A_{t_i} \leq A_{t_{R_i}}$ . It follows that

$$l_{t_i} r_{t_i} \leq A_{t_i} \leq A_{t_{R_i}} = 3\sqrt{3} r_i^2,$$

and with the aid of (4.15) we have

$$l_{t_i} \leq 3\sqrt{3} \frac{r_i^2}{r_{t_i}} \leq 6\sqrt{3} \tan\left(\frac{\theta_{\Delta_H^*}}{2}\right)^{-1} r_i. \quad (4.16)$$

The triangle  $t_{K_i}$  in Definition 4.1 is the minimal equilateral triangle that contains  $t_i$ . It follows that its side length  $l_{t_{K_i}}$  can be bounded below by  $l_{t_i}$ , and its height  $h_{t_{K_i}}$  must be smaller than  $l_{t_i}$ . The latter is true because the equilateral triangle with height  $h = l_{t_i}$  contains the circle segment with radius  $l_{t_i}$  and angle  $\pi/3$ , which is larger than any triangle with longest side length  $l_{t_i}$ . By Definition 4.1 it holds that  $l_{t_{K_i}} = 2\sqrt{3} K_i r_i$  and  $h_{t_{K_i}} = 3 K_i r_i$ , and we get the inequalities

$$\frac{l_{t_i}}{2\sqrt{3} r_i} \leq K_i \leq \frac{l_{t_i}}{3 r_i}.$$

Using (4.16) and  $l_{t_i} \geq r_i$ , it follows that

$$\frac{1}{2\sqrt{3}} \leq K_i \leq 2\sqrt{3} \tan\left(\frac{\theta_{\Delta_H^*}}{2}\right)^{-1}. \quad (4.17)$$

Since the bounds in (4.17) hold for each  $K_i$ , they are also valid for  $K$ .  $\square$

## 5 Hierarchical triangulations with bounded level disparity

We now investigate how we can construct the hierarchical triangulation such that the level disparity  $D$  will be bounded independently of the number of levels in the triangulation. Define a vertex-centered refinement as a refinement where all triangles in the triangulation containing the same vertex are split. Such a local refinement scheme is illustrated in Figure 4. The initial triangulation in Figure 4(a) is taken from [2, 10]. Note that if we apply this scheme successively to the same vertex, as in Figure 4(b), the level disparity remains  $D = 1$ . This is true because the triangles, that are intended to be split in each refinement step, only contain conforming vertices. The supports of the QHPS B-splines associated with these vertices have no disparity ( $D_i = 0$ ) before the refinement. Thus, after the refinement they become at most  $D_i = 1$ . The other B-splines are zero on the refined domain, such that their level disparity will not change. More generally, if one wants to refine a triangle  $\rho$  of level  $k$  in a hierarchical triangulation with  $D \leq 1$ , and one ensures that the 1-ring of triangles around  $\rho$  are also (made) at least of level  $k$ , then the level disparity will remain  $D \leq 1$ . The 1-ring of triangles is defined as the set of triangles that have a vertex in common with  $\rho$ .

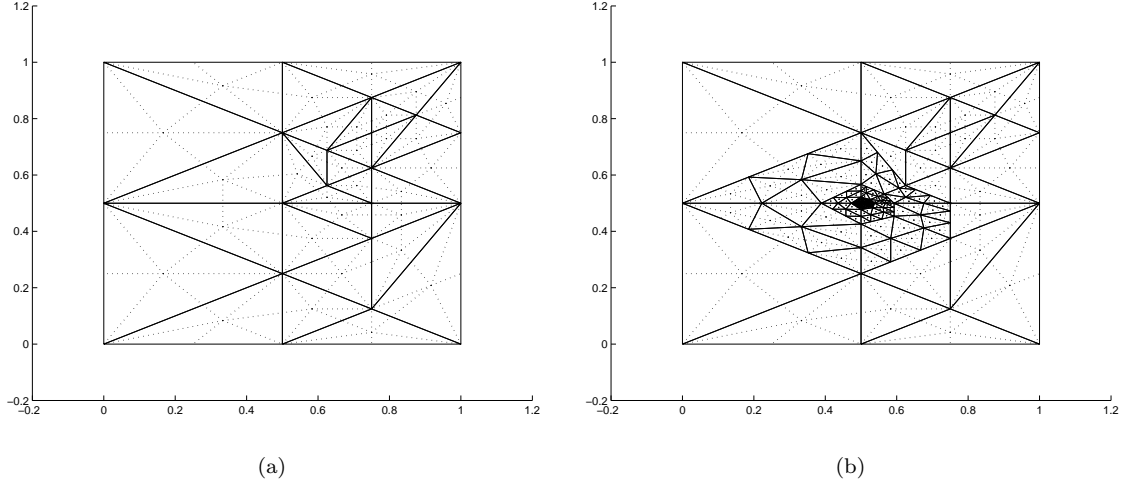


Figure 4: (a) Initial triangulation with its HPS refinement indicated in dotted lines ( $D = 0$ ). (b) A vertex-centered locally refined hierarchical triangulation, applied three times to the vertex  $V(0.5, 0.5)$ . The level disparity in the triangulation equals  $D = 1$ .

Refining every triangle in the hierarchical triangulation with the following algorithm, the level disparity will remain bounded by a given constant  $D_{\max} \geq 0$ . The general principle is to refine some extra surrounding triangles when the level disparity could exceed  $D_{\max}$ .

**Algorithm 5.1.** Let  $\Delta_H$  be a hierarchical triangulation and  $\mathbf{\Delta}_H$  be its corresponding structure. We suppose that the level disparity of  $\Delta_H$  is bounded by  $D_{\max}$ . The algorithm `qhps_refine_triangle`( $\rho$ ,  $\Delta_H$ ,  $D_{\max}$ ) produces a locally refined hierarchical triangulation with a level disparity  $D \leq D_{\max}$  and where the triangle  $\rho \in \Delta_H$  is split.

```

function qhps_refine_triangle(triangle  $\rho$ , hierarchical_triangulation  $\Delta_H$ , disparity  $D_{\max}$ )
  1. Refine  $\rho$  triadically.
  2. if  $k \geq D_{\max}$ , with  $k$  the level number of  $\rho$ :
    (a) Let  $\hat{\rho}$  be the triangle in  $\mathbf{\Delta}_H$  on level  $(k - D_{\max})$  such that  $\rho \subseteq \hat{\rho}$ .
    (b) while  $\rho$  belongs to any support  $S_i$  associated with one of the corner vertices of  $\hat{\rho}$ 
        that are conforming vertices in  $\Delta_H$ :
      Split the triangles  $\tilde{\rho} \in \Delta_H$  with the lowest level number in the 1-ring of triangles
      around  $\rho$  by qhps_refine_triangle( $\tilde{\rho}$ ,  $\Delta_H$ ,  $D_{\max}$ ).
    endwhile
  endif
end

```

One can intuitively verify the correctness of the algorithm as follows. Step 2b ensures that the level disparity of the new hierarchical triangulation remains below the bound  $D_{\max}$ . When the refinement of  $\rho$  leads to a level disparity larger than  $D_{\max}$ , it will refine some surrounding triangles. It is not needed to check the disparities  $D_i$  of all supports  $S_i$  in  $\Delta_H$ . Since the support of each B-spline is local ( $S_i \subseteq M_i^k$ ), we know that for most of them the triangle  $\rho$  will be situated outside  $S_i$ . Refining the triangles  $\tilde{\rho}$  in step 2b will decrease the level disparity locally around  $\rho$  till  $D \leq D_{\max}$ .

Note that it is better to refine triangles in the 1-ring around  $\rho$ , than to refine triangles in the supports  $S_i$  where the disparity  $D_i$  will become too large. In the former case, the propagation of the refined triangles will be more local.

## 6 Conclusion

In this paper we considered the  $L_p$ -stability of the QHPS B-splines. These splines are defined on a triadically refined hierarchical triangulation. In order to obtain a stable basis, the B-splines need to be scaled with respect to their support. The stability constants for the  $L_p$ -norm are only dependent on the smallest angle in the triangulation, the size of the QHPS triangles, and the level disparity of triangles in the support of the B-splines. By a particular construction of the hierarchical triangulation and using well-chosen QHPS triangles, the scaled QHPS B-splines are strongly  $L_p$ -stable.

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## References

- [1] P. Dierckx. On calculating normalized Powell-Sabin B-splines. *Comput. Aided Geom. Design*, 15(1):61–78, 1997.
- [2] P. Dierckx, S. Van Leemput, and T. Vermeire. Algorithms for surface fitting using Powell-Sabin splines. *IMA J. Numer. Anal.*, 12:271–299, 1992.
- [3] G. Farin. Triangular Bernstein-Bézier patches. *Comput. Aided Geom. Design*, 3(2):83–127, 1986.
- [4] E. Grinspun, P. Krysl, and P. Schröder. CHARMS: a simple framework for adaptive simulation. *ACM Trans. Graphics*, 21(3):281–290, 2002.
- [5] Y. He, M. Jin, X. Gu, and H. Qin. A  $C^1$  globally interpolatory spline of arbitrary topology. In N. Paragios, O.D. Faugeras, T. Chan, and C. Schnörr, editors, *Proc. 3rd IEEE Workshop on Variational, Geometric and Level Set Methods in Computer Vision*, pages 295–306, Beijing, China, 2005.
- [6] A. Kroó and S. Révész. On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies. *J. Approx. Theory*, 99:134–152, 1999.
- [7] T. Lyche and K. Scherer. On the  $p$ -norm condition number of the multivariate triangular Bernstein basis. *J. Comput. Appl. Math.*, 119:259–273, 2000.
- [8] C. Manni. Shape control in Powell-Sabin quasi-interpolation. In A. Iske and J. Levesley, editors, *Algorithms for Approximation*, pages 219–240, Heidelberg, 2006. Springer-Verlag.
- [9] M.J.D. Powell and M.A. Sabin. Piecewise quadratic approximations on triangles. *ACM Trans. Math. Softw.*, 3:316–325, 1977.
- [10] H. Speleers, P. Dierckx, and S. Vandewalle. Local subdivision of Powell-Sabin splines. *Comput. Aided Geom. Design*, 23(5):446–462, 2006.

- [11] H. Speleers, P. Dierckx, and S. Vandewalle. Quasi-hierarchical Powell-Sabin B-splines. Technical Report 472, Dept. Comp. Science, K.U. Leuven, 2006.
- [12] E. Vanraes, J. Maes, and A. Bultheel. Powell-Sabin spline wavelets. *Int. J. Wav. Multires. Inf. Proc.*, 2(1):23–42, 2004.