Powell–Sabin spline based multilevel preconditioners for the biharmonic equation

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The Powell–Sabin (PS) piecewise quadratic C^1 finite element on the PS 12-split of a triangulation is a common choice for the construction of a BPX-type preconditioner for the biharmonic equation. In this note we investigate the related Powell–Sabin element on the PS 6-split instead of the PS 12-split for the construction of such preconditioners. For the PS 6-split element multilevel spaces can be created using a $\sqrt{3}$-refinement scheme instead of the traditional dyadic scheme. Topologically $\sqrt{3}$-refinement has many advantages: it is a slower refinement than the dyadic split operation, and it alternates the orientation of the refined triangles. Therefore we expect a reduction of the amount of work when compared to the PS 12-split element BPX preconditioner, although both methods have the same asymptotical complexity. Numerical experiments confirm this statement.

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Abstract

The Powell–Sabin (PS) piecewise quadratic C^1 finite element on the PS 12-split of a triangulation is a common choice for the construction of a BPX-type preconditioner for the biharmonic equation. In this note we investigate the related Powell–Sabin element on the PS 6-split instead of the PS 12-split for the construction of such preconditioners. For the PS 6-split element multilevel spaces can be created using a $\sqrt{3}$-refinement scheme instead of the traditional dyadic scheme. Topologically $\sqrt{3}$-refinement has many advantages: it is a slower refinement than the dyadic split operation, and it alternates the orientation of the refined triangles. Therefore we expect a reduction of the amount of work when compared to the PS 12-split element BPX preconditioner, although both methods have the same asymptotical complexity. Numerical experiments confirm this statement.

This note is concerned with the construction of multilevel preconditioners for the efficient solution of the biharmonic equation with Dirichlet boundary conditions, i.e.,

$$\Delta^2 u = f \text{ on } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where n is the outward normal to the boundary $\partial\Omega$ of the polygonal domain Ω. We restrict our attention to Powell–Sabin (PS) splines ([8]) for discretizing the variational problem

$$a(u,v) = (f,v) \quad \text{for all } v \in H_0^2(\Omega), \tag{2}$$

corresponding to (1). Here (f,v) denotes the scalar product in $L_2(\Omega)$, and the bilinear form $a : H_0^2 \times H_0^2 \to \mathbb{R}$ is defined as $a(u,v) = (\Delta u, \Delta v)$.

An optimal BPX-type ([1, 6]) preconditioner for (2) was first established by Oswald in [5]. Similar constructions can also be found in [2]. The key ingredient for such constructions is a norm equivalence of the form

$$\|u\|_{H^2(\Omega)} \asymp \inf_{u=\sum_j v_j} \left(\sum_{j=0}^{\infty} \rho^{4j} \|v_j\|^2_{L_2(\Omega)} \right)^{1/2}, \quad v_j \in V_j, \quad \rho > 1, \tag{3}$$

where $V_0 \subset V_1 \subset \cdots \subset V_j \subset \cdots$ is a sequence of C^1 conforming finite element subspaces on partitions $\Delta_0 \to \Delta_1 \to \cdots \to \Delta_j \to \cdots$ with grid-size $h_j \asymp \rho^{-j}$. The interested reader is referred to [7] for the general theory concerning multilevel splittings and its application to multilevel preconditioning.

The most common conforming finite element space V_j for discretizing (2) is the spline space $S_2^1(\Delta_j^{12})$ of C^1 piecewise quadratic polynomials on the PS 12-split Δ_j^{12} of a given triangulation Δ_j. Any spline function in $S_2^1(\Delta_j^{12})$ is completely determined by its function value and first derivatives at the vertices

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of Δ_j, and cross-derivative at the midpoints of the edges of Δ_j. Figure 1 (a) displays the PS 12-split and the corresponding degrees of freedom. Obviously one has that $S_2^1(\Delta_0^{12}) \subset S_2^1(\Delta_1^{12}) \subset \cdots$, with $\Delta_0 \to \Delta_1 \to \cdots$ a sequence of triangulations obtained from Δ_0 by regular dyadic refinement. It is well known that (3) holds with $\rho \equiv 2$ and $V_j \equiv S_2^1(\Delta_j^{12})$, see [5].

We are particularly interested in the spline space $S_2^1(\Delta_j^6)$ of C^1 piecewise quadratic polynomials on the PS 6-split Δ_j^6 of Δ_j. Figure 2 (a) shows the PS 6-split of a triangulation and Figure 1 (b) shows the corresponding degrees of freedom that uniquely define a spline in $S_2^1(\Delta_j^6)$, that is, its function value and first derivatives at the vertices of Δ_j. We can create a multiresolution structure by applying a $\sqrt{3}$-refinement scheme ([4]) to the underlying triangulations. Figure 2 depicts the process. Obviously the spaces are nested: $S_2^1(\Delta_0^6) \subset S_2^1(\Delta_1^6) \subset \cdots$. One can easily check that this $\sqrt{3}$-type of refinement is always possible, but it remains an open problem whether the refinement process is regular, i.e, whether the diameters of the triangles in Δ_j remain proportional to $\sqrt{3}^{-j}$ as j increases. We have restricted ourselves to the numerical verification of the regularity of the refinement process, and in all our tests the refinement process gave a regular sequence of triangulations. For a practical implementation of the subdivision process we refer to [9]. If we assume that the sequence $\Delta_0^6 \to \Delta_1^6 \to \cdots$ obtained by $\sqrt{3}$-refinement is regular, the norm equivalence (3) with $\rho \equiv \sqrt{3}$ and $V_j \equiv S_2^1(\Delta_j^6)$ follows from standard arguments.

The purpose of this note is to show the numerical benefit gained from the $\sqrt{3}$-type refinement. We solve the biharmonic equation (1) for Ω the unit square. The initial triangulation Δ_0 is constructed by dividing Ω by its bissectrice in two triangles. The right hand side f is chosen such that the exact solution u is given by $(10x(1-x)y(1-y))^2$. We use a BPX preconditioned conjugate gradient method to solve the problem. More specifically we follow the approach outlined by Griebel in [3] to implement the BPX preconditioner. The starting vector for each iteration is the zero-vector. We stop the conjugate gradient iteration if the energy norm (H^2-norm) of the residual is proportional to the discretization error, i.e., $\|r_J\|_{H^2} \leq \epsilon_0 h_J$. Table 1 shows the results for $\epsilon_0 = 0.01$. The first column contains the maximum resolution level J. Then we distinguish between the results for the BPX preconditioner based on the PS 6-split element and the results for the BPX preconditioner based on the PS 12-split element. For each preconditioner we display the dimension and the spectral condition number κ of the system matrix for the linear system of equations that is solved, the H^2-norm of the residuals corresponding to the approximate solution, and the number of iterations that are needed to reach discretization error accuracy.

	$\sqrt{3}$ (6-split)				dyadic (12-split)			
J	dim	κ	residual	#iter	dim	κ	residual	#iter
2	18	7.4804	5.403207e-15	4	79	45.0098	1.571911e-3	12
3	84	15.7856	3.731378e-4	10	402	80.1263	1.150745e-3	24
4	276	28.1456	7.937759e-4	16	1813	107.6156	5.083393e-4	35
5	954	36.0941	4.605986e-4	20	7704	128.0993	3.100278e-4	43
6	2982	43.9239	3.153517e-4	24	31771	143.6639	1.438365e-4	52
7	9384	50.7505	1.775536e-4	28	129054	–	7.238170e-5	59
8	28584	54.5326	1.089472e-4	31				
9	87150	–	5.625347e-5	35				
10	262842	–	3.308055e-5	38				

Table 1: Iteration history for the biharmonic equation using a BPX preconditioner based on PS 6-split elements and a BPX preconditioner based on PS 12-split elements.

In practical situations one should use a nested iteration conjugate gradient method to solve the biharmonic equation, i.e., by means of an outer iteration loop going from a coarse resolution level to the finest resolution level J one computes the solution at each level with the conjugate gradient method and one uses the solution obtained at the previous coarser level as an initial guess. We do not use nested iteration in Table 1 because our intention here is to demonstrate the full power of the preconditioner itself. We also remark that computing the H^2-norm of the residual is easy. For an optimal

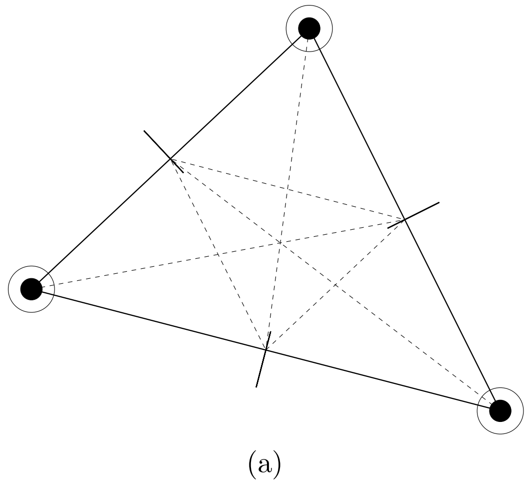

Figure 1: (a) The Powell–Sabin piecewise quadratic C^1 element on the PS 12-split of a triangulation has 12 degrees of freedom (function value and first derivatives at the vertices of the macrotriangle, and cross-derivative at the midpoints of the edges). (b) The Powell–Sabin piecewise quadratic C^1 element on the PS 6-split of a triangulation has 9 degrees of freedom (function value and first derivatives at the vertices of the macrotriangle).

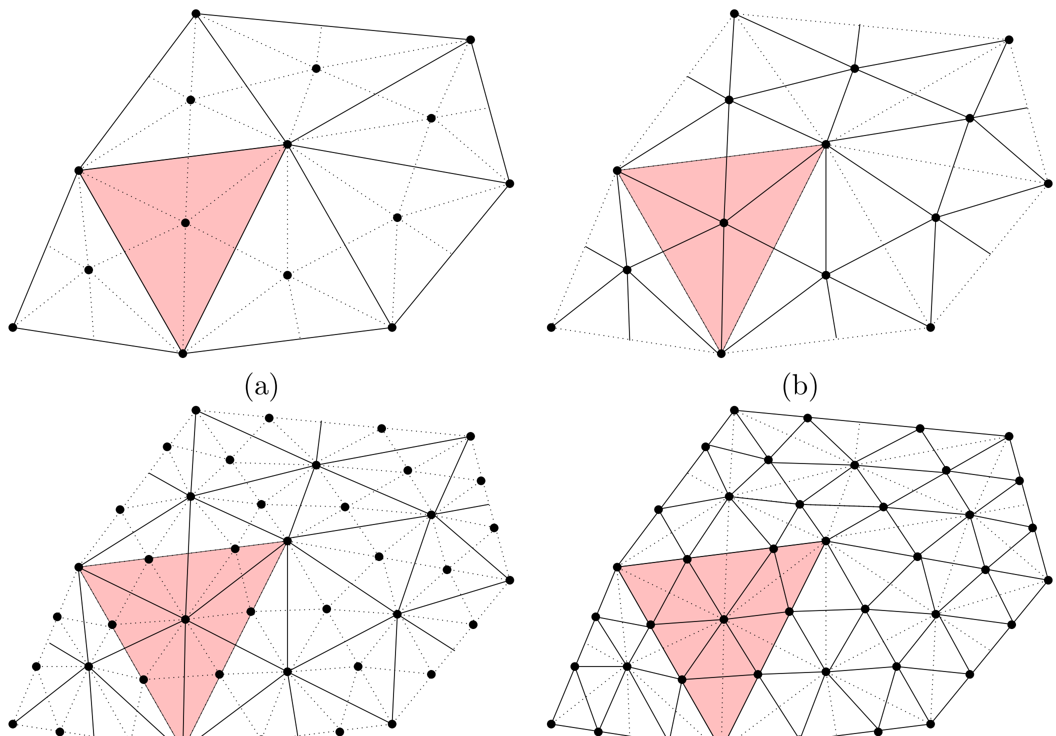

Figure 2: Principle of $\sqrt{3}$-refinement. Instead of splitting each edge in Δ_0 and performing a 1-to-4 split for each triangle, we compute a new vertex for each triangle and retriangulate the old and new vertices. Remark that the new edges in Δ_1 coincide with the lines of the PS 6-split Δ_0^6. In the new triangles new interior points must be chosen on the one line of the new PS 6-split Δ_1^6 that is already fixed, that is, the original edge that crosses the triangle. Two steps of $\sqrt{3}$ refinement give triadic refinement.

preconditioner such as the BPX preconditioner one can prove that the energy norm of the residual is equivalent to the ℓ_2 norm of the discrete residual of the discretized system.

The results from Table 1 clearly favor the PS 6-split based BPX preconditioner.

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