Note on "Electromagnetic Response of a Large Circular Loop Source on a Layered Earth: A New Computation Method"

Joris Van Deun and Ronald Cools

Report TW 449, February 2006

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Note on "Electromagnetic Response of a Large Circular Loop Source on a Layered Earth: A New Computation Method"

Joris Van Deun and Ronald Cools

Report TW449, February 2006

Department of Computer Science, K.U.Leuven

Abstract

In a paper recently published in Pure and Applied Geophysics analytic expressions for some integrals of products of Bessel functions were derived. The derivation however contains some flaws and consequently some of the results are wrong. In this note we correct the errors and simplify the results. In addition we announce a routine for numerical computation of such integrals.

Keywords : Bessel functions, integrals

Note on “Electromagnetic Response of a Large Circular Loop Source on a Layered Earth: A New Computation Method”

Joris Van Deun and Ronald Cools
Department of Computer Science, K.U.Leuven, Belgium
E-mail: {joris.vandeun–ronald.cools}@cs.kuleuven.be

Abstract

In a paper recently published in Pure and Applied Geophysics analytic expressions for some integrals of products of Bessel functions were derived. The derivation however contains some flaws and consequently some of the results are wrong. In this note we correct the errors and simplify the results. In addition we announce a routine for numerical computation of such integrals.

Keywords: Bessel functions, integrals

1 Introduction

Infinite range integrals of products of Bessel functions occur in a wide variety of problems from physics and engineering and are notoriously hard to evaluate. In [SM05] integrals of this type arise in the expressions of electromagnetic field components at a point on or above the surface of n-layered earth due to a circular loop. In the appendix to that article, the authors claim to derive analytical expressions for the integrals in question. Unfortunately, their derivation is mathematically flawed and some of the results they obtain are incorrect.

The purpose of this note is twofold. Firstly, we wish to correct the errors and simplify some of the (correct) results given in [SM05]. This simplification makes the computations more efficient. Secondly, we present an entirely different and efficient approach to compute integrals of this and a more general type, using an algorithm described in [VDC05, VDC06a, VDC06b], which we believe may be of interest to people in this field.

The three integrals which are studied in [SM05], are

$$
\begin{aligned}
I_1 &= \int_0^\infty tJ_1(at)J_0(rt)dt,\\
I_2 &= \int_0^\infty tJ_1(at)J_1(rt)dt,\\
I_3 &= \int_0^\infty J_1(at)J_1(rt)dt,
\end{aligned}
$$

where a is the radius of the loop, r is the source-receiver offset (these are obviously positive numbers), and $J_n(t)$ is the Bessel function of the first kind and order n. A first issue, which seems to have gone unnoticed by the authors, is that these integrals do not exist in the ordinary sense of improper Riemann integrals; they do not converge (for any value of a and r). Integrals of this kind are usually defined in the sense of *Abel summability.* Let a real function f be given such that the integral

$$I(c) = \int_0^\infty e^{-cx} f(x)dx, \quad c > 0$$

exists and is finite for every $c > 0$. Then f is said to be integrable in the sense of Abel summability if the limit

$$\lim_{c \to 0} I(c) = I$$

also exists and is finite. This limit I is then taken as the value of the integral of f. The integrals I_1, I_2 and I_3 are to be interpreted in this way.

2 Correct derivation and simplification

In [SM05] the authors start their derivation from the equation [Wat66, p. 390]

$$\int_0^\infty e^{-ct} t^{\mu-\nu} J_\mu(xt) J_\nu(yt) dt = \frac{\left(\frac{1}{2}x\right)^\mu \left(\frac{1}{2}y\right)^\nu \Gamma(2\mu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \frac{\sin^{2\nu}\phi \, d\phi}{[(c+iy\cos\phi)^2 + x^2]^{\mu+\frac{1}{2}}} \tag{1}$$

which is valid for $\Re(c \pm ix \pm iy) > 0$ and $\Re(\mu) > -1/2$, where i denotes the imaginary unit and $\Re(.)$ the real part. This is indeed the right equation to start from, but one cannot simply substitute $c = 0$, $x = a$, $y = r$, $\mu = 1$ and $\nu = 0$ to obtain an expression for I_1, as was done in [SM05]; this clearly violates the condition $\Re(c \pm ix \pm iy) > 0$.

Instead, the correct reasoning is as follows. The integrand on the right hand side of (1) is a meromorphic function of a which has simple poles at $c = -i(y\cos\phi \pm x)$. So if $x \neq y$, then the integrand is analytic in a neighbourhood of $c = 0$, for every $\phi \in [0, \pi]$. This means that the integral itself is an analytic function of c, and therefore continuous, so we may take the limit $c \to 0$ in (1) under the integral sign, and then proceed as in [SM05]. The result is of course the same.

In the case $x = y$, this technique can no longer be used, but now it follows from the asymptotic expansions of the Bessel functions that the integrals I_1, I_2 and I_3 are infinite. This is explained in more detail in [VDC06a].

So for the integral I_1, the results from [SM05] are correct, even though this was not immediately obvious from their derivation. However, the expressions given involve the classical Gauss hypergeometric function and can be simplified considerably, using relations between the hypergeometric function and the complete elliptic integrals given in [AS64, Chap. 17] (or proceeding directly from the integral expression and the definition of the complete elliptic integrals). The computations are rather straightforward and we only give the final result:

$$I_1 = \begin{cases} \dfrac{2E\left(\frac{r}{a}\right)}{\pi(a^2 - r^2)}, & r < a, \\ \dfrac{2}{\pi a}\left[\dfrac{1}{r}K\left(\dfrac{a}{r}\right) - \dfrac{r}{r^2 - a^2}E\left(\dfrac{a}{r}\right)\right], & a < r, \end{cases} \tag{2}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of modulus k of the first and second kind respectively. This is indeed a simplification, since these functions can be computed much more efficiently than the hypergeometric function, using the process of the arithmetic-geometric mean as described in [AS64, p. 598].

The last integral, I_3, is again evaluated using the same techniques and the formulas in [SM05] are correct. Here as well they involve the hypergeometric function and we can simplify them to

$$I_3 = \frac{2}{\pi r}\left[K\left(\frac{r}{a}\right) - E\left(\frac{r}{a}\right)\right], \quad r < a. \tag{3}$$

When $a > r$ their roles must be interchanged.

3 Incorrect result for I_2

For the integral I_2, the authors differentiate both sides of (1), then take the limit $c \to 0$ and substitute the right values for x, y, μ and ν. Because of the previous discussion, this is only allowed when $x \neq y$ (or $a \neq r$). Proceeding in this way in [SM05], they derive equation (A-10) which reads

$$\int_0^\infty tJ_1(xt)J_1(yt)dt = \\ \frac{3ixy^2}{\pi}\int_0^\pi \frac{\sin^2\phi\cos\phi\, d\phi}{(x^2 - y^2\cos^2\phi)^{5/2}} = \frac{6ixy^2}{\pi}\int_0^{\pi/2} \frac{\sin^2\phi\cos\phi\, d\phi}{(x^2 - y^2\cos^2\phi)^{5/2}}. \tag{A-10}$$

The last integral is then evaluated again in terms of hypergeometric functions to give the result

$$\int_0^\infty tJ_1(at)J_1(rt)dt = \frac{2i}{\pi}\frac{a^2}{r^2(r^2 - a^2)}F\left(\frac{3}{2}, 0; \frac{5}{2}; \frac{a^2}{r^2}\right), \quad a < r$$

where F is the hypergeometric function (also here, when $a > r$ their roles must be interchanged). Clearly there is something very wrong with this result: the left hand side is real, the right hand side is pure imaginary (the hypergeometric function is in fact identically equal to 1 because one of its numerator parameters is 0). The problem is that the second equality in equation (A-10) does *not* hold. The integrand

$$f(\phi) = \frac{\sin^2\phi\cos\phi\, d\phi}{(x^2 - y^2\cos^2\phi)^{5/2}}$$

satisfies $f(\pi/2 - \phi) = -f(\pi/2 + \phi)$ for any $\phi \in [0, \pi/2]$, from which it follows that the second integral in (A-10) is equal to zero. This proves the correct relation

$$I_2 = 0, \quad a \neq r, \tag{4}$$

which is quite different from the result in [SM05].

4 A useful program: `besselint`

At this point we would like to mention that the above analysis could in fact be avoided entirely, since all of the above integrals can be computed very efficiently using a recently developed algorithm described in [VDC05, VDC06a, VDC06b]. In fact, this method can serve to compute integrals of the form

$$I = \int_0^\infty \frac{e^{-cx}x^m}{dx^2 + 1}\prod_{j=1}^{k} J_{\nu_j}(a_j x)dx,$$

also when they are only defined in the sense of Abel summability (all the parameters involved are real, and neither m nor ν_j need be integer). However, only the case

$c = d = 0$ is described in the abovementioned articles and has been implemented in the publically available Matlab routine `besselint`. A complete implementation for $c \neq 0$ and $d \neq 0$ will be available shortly. Furthermore, the program can easily be translated into other programming languages, since it does not use any Matlab-specific features or toolboxes. The explicit expressions for the integrals I_1, I_2 and I_3, based on elliptic integrals, are probably more efficient than our program, but such explicit expressions are only very rarely available. Besides, the numerical result can help to detect errors in the analytic derivation. More importantly, our algorithm can easily compute the Hankel transform in equations (5), (8) and (9) in [SM05].

We are convinced that such a program will be of great use to researchers in many different fields, not only in geophysics. It allows them to compute this type of integrals without any difficulty or second thought and focus their attention on more important issues specific to their research, instead of having to engage in tedious mathematical analysis, which is all too often prone to errors, as has convincingly been shown in the case under consideration.

References

[AS64] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, volume 55 of *Applied Mathematics Series*. National Bureau of Standards, Washington, D.C., 1964.

[SM05] N. P. Singh and T. Mogi. Electromagnetic response of a large circular loop source on a layered earth: a new computation method. *Pure Appl. Geophys.*, 162:181–200, 2005.

[VDC05] J. Van Deun and R. Cools. Integrating products of Bessel functions using the incomplete Gamma function. In T. E. Simos, G. Psihoyios, and Ch. Tsitouras, editors, *International Conference on Numerical Analysis and Applied Mathematics 2005*, pages 668–671. Wiley–VCH, 2005.

[VDC06a] J. Van Deun and R. Cools. Algorithm 8XX: Computing infinite range integrals of an arbitrary product of Bessel functions. *ACM Trans. Math. Software*, 2006. To appear.

[VDC06b] J. Van Deun and R. Cools. A Matlab implementation of an algorithm for computing integrals of products of Bessel functions. 2006. Submitted.

[Wat66] G. N. Watson. *A treatise on the Theory of Bessel Functions.* Cambridge University Press, 1966.