On the evaluation of highly oscillatory integrals by analytic continuation

Daan Huybrechs and Stefan Vandewalle

Report TW431, July 2005 (revised December 2005)

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The approach described in this paper is related to the Steepest Descent method, but it does not employ asymptotic expansions. It can be used for small or moderate frequencies as well as for very high frequencies. The approach is compared with the oscillatory integration techniques recently developed by Iserles and Nørsett.

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Abstract

We consider the integration of one-dimensional highly oscillatory functions. Based on analytic continuation, rapidly converging quadrature rules are derived for a fairly general class of oscillatory integrals with an analytic integrand. The accuracy of the quadrature increases both for the case of a fixed number of points and increasing frequency, and for the case of an increasing number of points and fixed frequency. These results are then used to obtain quadrature rules for more general oscillatory integrals, i.e., for functions that exhibit some smoothness but that are not analytic.

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1 Introduction

Consider the oscillatory integral

$$I := \int_a^b f(x) e^{i\omega g(x)} \, \mathrm{d}x \tag{1}$$

with $\omega > 0$ and with $f(x)$ and $g(x)$ smooth functions. Integrals of this form abound in mathematical models and computational algorithms for oscillatory phenomena in science and engineering. Recently, much progress has been made in numerical quadrature techniques for (1). Methods have been devised that compute an accurate approximation to the value of the integral with low computational complexity and with a number of operations that actually decreases as ω increases to infinity [12, 13, 14, 15, 16, 17, 18]. This is in contrast to most classical integration approaches, based on polynomial interpolation, that rapidly deteriorate in the presence of strong oscillations. In order to appreciate the inner workings of these methods, one should understand the asymptotic behaviour of the oscillatory integral (1) for large values of the frequency parameter ω.

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The value of I at large frequencies depends on the behaviour of the smooth functions f and g near the endpoints a and b, and near the so-called stationary points. The latter are the solutions to the equation $g'(x) = 0$ on $[a, b]$; they represent points in which the integrand locally does not oscillate. An intuitive justification of this property may be that, away from the endpoints and the stationary points, the oscillations of the integrand increasingly cancel out. Mathematically, the property is reflected in the asymptotic expansion of I. We say that a stationary point ξ has order r if $g^{(j)}(\xi) = 0$, $j = 1, \ldots, r$, but $g^{(r+1)}(\xi) \neq 0$, i.e., the first r derivatives of the oscillator vanish. Assuming one stationary point ξ of order r in the interval $[a, b]$, the asymptotic expansion of (1) has the form

$$I \sim \sum_{j=0}^{\infty} \frac{a_j}{\omega^{(j+1)/(r+1)}}, \tag{2}$$

where the coefficients a_j depend only on a finite number of function values and derivatives of f and g at the critical points a, b and ξ [20]. The coefficients are in general not easily obtained, although the leading order coefficient a_0 is given by the method of stationary phase. Still, the mere existence of the asymptotic expansion reveals a lot of information about I. For example, an immediate consequence is that $|I| = O(\omega^{-1/(r+1)})$.

A first efficient method is to simply truncate the asymptotic expansion (2) after a finite number of terms. By construction, the truncation error decays as a power of $1/\omega$. This *asymptotic method* was described by Iserles and Nørsett in [15]. The problem of the unknown coefficients in the presence of stationary points is solved by constructing a uniform asymptotic expansion, based on factoring out the moment $\mu_0 = \int_a^b e^{i\omega g(x)}\,\mathrm{d}x$, or similar higher order moments. The coefficients in this expansion can be computed explicitly, if the moments themselves are known a priori. A disadvantage of such an approach is that the error of an asymptotic expansion is essentially uncontrollable, since asymptotic expansions tend to diverge. This is especially true for smaller frequencies.

A different approach, proposed also in [15], is to extend Filon's method for oscillatory integrals (see [9, 5]) by considering Hermite interpolation of f. The result is a quadrature rule for I with a classical form, involving function values and derivatives of f. The error of this approach is controllable and may be very small. A disadvantage is that the weights of the rule are given by oscillatory integrals themselves, and they can not always be explicitly computed. We will revisit *Filon-type methods* in §6.

An entirely different approach was proposed by Levin in [17]. If the indefinite integral of the integrand is written as $F(x)e^{i\omega g(x)}$, then we immediately have $I = F(b)e^{i\omega g(b)} - F(a)e^{i\omega g(a)}$. It was observed in [17] that $F(x)$ is a smooth function, in the absence of stationary points. Moreover, it satisfies the non-oscillatory differential equation

$$F'(x) + i\omega g'(x) = f(x). \tag{3}$$

This system can be solved for $F(x)$ by collocation. The method was generalized in [8, 7] to more general oscillatory functions that satisfy a linear ordinary differential equation, for example Bessel functions. The accuracy of the methods improves with ω if the boundary points are included in the collocation. Recently, it was shown in [18] that collocating also the derivatives of f in the endpoints can arbitrarily increase the order of accuracy as a function of $1/\omega$. In some cases, the order can also be increased by adding internal points. This *Levin-type method* allows an accurate evaluation of the integral, without the need for moments. The accuracy is increased simply by solving the differential equation more accurately.

For the particular case of an oscillating factor of the form $\cos(\omega x)$ or $\sin(\omega x)$, specialized quadrature rules using first order derivatives were developed in [16]. An *exponentially fitting quadrature rule* with n points has an error of order ω^{-n}. The weights depend on ω and converge to zero.

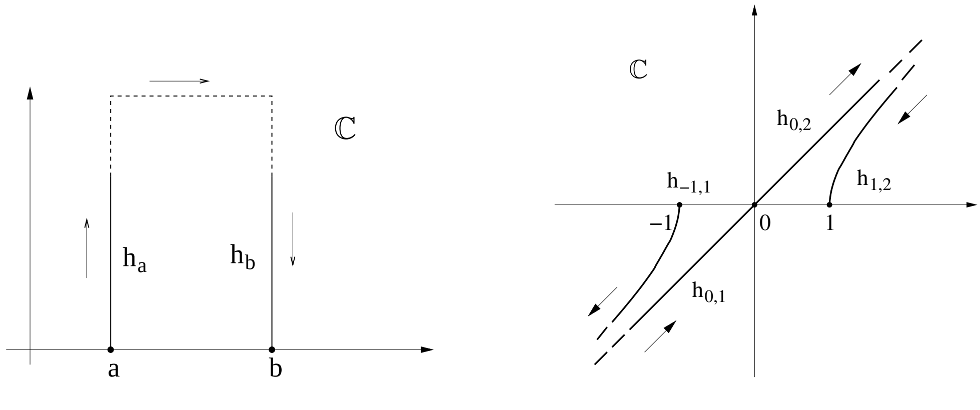
The approach taken in this paper achieves a similar high convergence rate as a function of ω. We will show that it solves some of the problems of the other methods, and introduces some peculiarities of its own, thus adding to the spectrum of available approaches that appear to complement each other. For example, we present a case that exhibits a significantly faster convergence rate for increasing ω. The method we describe for approximating (1) depends on two simple observations. First, the oscillatory function $e^{i\omega g(x)}$ decays exponentially fast for a complex $g(x)$ along a path with a growing imaginary part. Second, the oscillatory function $e^{i\omega g(x)}$ does *not* oscillate for complex $g(x)$ along a path with fixed real part. These observations are exploited numerically in combination with a corollary to Cauchy's Theorem, i.e., the value of a line integral of an analytic function along a path between two points in the complex plane does not depend on the exact path taken (see, e.g., [11]). The same observations also provide the foundation for the Steepest Descent method [1, 2]. In that method, an asymptotic expansion of the form (2) is developed for I. The method was used already by Cauchy and Riemann, and developed further by Debye [6]. Methods in the complex plane have been considered for oscillatory integrals several times since, in specific applications or for Laplace transforms (see, e.g., [21, 4, 3]). We will present a rather general implementation of the steepest descent method, that is also valid for small values of ω. We prove convergence estimates of the numerical scheme as a function of the frequency, and we extend the method to functions f and g that are not analytic. The implementation can be entirely numerical; hence we shall refer to the method as the *numerical steepest descent method*.

We start this paper in Section 2 with some practical and motivating examples that illustrate most of the theory described later. In Section 3 we describe and analyze the idealized setting that gives the best possible convergence. It is shown that a suitable n-point quadrature rule in that setting leads to a convergence of $O(\omega^{-2n-1})$. This setting comes with the most restrictions, but still covers many important applications. The first requirement is that the functions f and g in (1) are analytic in an (infinitely) large region of the complex plane containing the integration interval $[a, b]$. Further, it is assumed that there are no stationary points in $[a, b]$, and that the equation $g(x) = c$ should be "easily solvable". This rather vague description will be made more precise further on. We then proceed by relaxing the requirements one by one, until a more generally applicable method is obtained. This increase in generality will, at times, come with a loss in convergence rate. In Section 4 we will allow stationary points. We relax the "easy-solvability" requirement in Section 5. We drop the requirements that f and g should be analytic in Sections 6 and 7 respectively. Some final remarks conclude the paper in Section 8.

2 Some motivating examples

Consider the following integral, which frequently appears in Fourier analysis applications,

$$\int_a^b f(x) e^{i\omega x} \, \mathrm{d}x. \tag{4}$$



(a) $g(x) = x$ (b) $g(x) = x^2$

Figure 1: Illustration of the integration paths for $g(x) = x$ and $g(x) = x^2$.

This integral has the form of (1) with $g(x) = x$. An overview of solution techniques for large values of ω is given in [12]. In that case, the integrand is highly oscillatory along the real axis. An important observation is that the function $e^{i\omega x}$ decays rapidly for complex values of x with a positive imaginary part, since $e^{i\omega x} = e^{-\omega \Im x} e^{i\omega \Re x}$. The speed of the decay actually grows as the frequency parameter ω increases. Additionally, the function $e^{i\omega x}$ does not oscillate if the real part of the argument x remains fixed.

Based on these observations integral (4) can be reformulated in such a way that the difficulty - the highly oscillatory nature - is removed. To that end, the integration on interval $[a, b]$ is replaced by a path in the complex plane as illustrated in the left panel of Figure 1. The first, vertical part of the path is of the form $z = h_a(p) := a + ip$ for $p \in [0, P]$. The second part is horizontal and connects the points $h_a(P) := a + iP$ to the point $h_b(P) := b + iP$. Finally, the third part connects $h_b(P)$ to b with the vertical path $z = h_b(p)$ for $p \in [0, P]$. Now assume that f is analytic, and that f itself does not grow exponentially large in the complex plane. Letting P go to infinity, and using paths parameterized by $h_a(p)$ and $h_b(p)$, for $p \in [0, \infty)$, we can write (4) as

$$\int_a^b f(x) e^{i\omega x}\, \mathrm{d}x = e^{i\omega a} \int_0^\infty f(a + ip) e^{-\omega p}\, \mathrm{d}p - e^{i\omega b} \int_0^\infty f(b + ip) e^{-\omega p}\, \mathrm{d}p. \tag{5}$$

The integral along the path that connects the endpoints of $h_a(P)$ and $h_b(P)$ vanishes for $P = \infty$ and can therefore be discarded. Both integrals in the right hand side of (5) are well behaved. They can be evaluated efficiently by standard numerical integration techniques, e.g., by Gauss-Laguerre integration [5]. It can be expected from (5) that the accuracy of any numerical integration scheme will increase with increasing ω, thanks to the faster decay of the integrand. This expectation will be confirmed both theoretically and numerically in the subsequent sections. One also sees that, asymptotically, the behaviour of f around $x = a$ and $x = b$ completely determines the value of (4).

Next, we consider the function $g(x) = x^2$ and the corresponding integral

$$\int_{-1}^{1} f(x) e^{i\omega x^2} \, \mathrm{d}x. \tag{6}$$

Again, we can remove the integration difficulty by a careful selection of an integration path in the complex plane. The path is drawn in the right panel of Figure 1. The following notation is used for the parameterization: $h_{xj}(p) = (-1)^j \sqrt{x^2 + ip}$. Integrating along any such path for $p \in [0, \infty)$ leads to an integrand with the desired decay properties, since $e^{i\omega h_{xj}(p)^2} = e^{i\omega x^2} e^{-\omega p}$. One can see that, for general g, a similar result is obtained if the path satisfies $g(h_x(p)) = g(x) + ip$. This path can be found by using the inverse of g, if it exists, i.e., $h_x(p) = g^{-1}(g(x) + ip)$. Returning to the example function $g(x) = x^2$ however, we note that the inverse of $y = g(x)$ is multivalued: we have $x = -\sqrt{y}$ corresponding to the restriction $g_1 := g|_{[-1,0]}$, and $x = \sqrt{y}$ corresponding to $g_2 := g|_{[0,1]}$. The paths leaving -1 and arriving at 1 are uniquely determined by the requirement that $h_{xj}(0) = x$. Hence,

$$h_{-1,1}(p) = -\sqrt{1 + ip} \quad \text{and} \quad h_{1,2}(p) = \sqrt{1 + ip}.$$

Contrary to the first example, the integral along the path that connects the limiting endpoints of $h_{-1,1}(p)$ and $h_{1,2}(p)$ cannot be discarded. Since $h_{-1,1}(p)$ and $h_{1,2}(p)$ have opposite signs, any connecting path should cross the real axis. Additionally we require the connecting path to be such that the integrand along the path is non-oscillatory. The solution is to pass explicitly through the point $x = 0$, via two new paths

$$h_{0,1}(p) = -\sqrt{ip} \quad \text{and} \quad h_{0,2}(p) = \sqrt{ip}.$$

The point $x = 0$ is such that the paths corresponding to the two inverses coincide at $x = 0$. We can now rewrite (6) as

$$\begin{aligned}\int_{-1}^{1} f(x) e^{i\omega x^2} \, \mathrm{d}x =& e^{i\omega} \int_0^\infty f(h_{-1,1}(p)) e^{-\omega p} h'_{-1,1}(p) \, \mathrm{d}p - \int_0^\infty f(h_{0,1}(p)) e^{-\omega p} h'_{0,1}(p) \, \mathrm{d}p \\ &+ \int_0^\infty f(h_{0,2}(p)) e^{-\omega p} h'_{0,2}(p) \, \mathrm{d}p - e^{i\omega} \int_0^\infty f(h_{1,2}(p)) e^{-\omega p} h'_{1,2}(p) \, \mathrm{d}p.\end{aligned}$$

These four integrals are well behaved, although the derivatives $h'_{0,1}(p)$ and $h'_{0,2}(p)$ introduce a weak singularity of the form $1/\sqrt{p}$, for $p \to 0$. The integrands do not oscillate, and their decay is exponentially fast.

Note that $\xi = 0$ is a stationary point because $g'(\xi) = 0$. More general stationary points, where also higher order derivatives of g vanish, are handled in a similar way. Consider, e.g., $g(x) = x^3$ and its inverse $g^{-1}(y) = \sqrt[3]{y}$. The cubic root has three branches in the complex plane, and the optimal path $h_x(p) = g^{-1}(g(x) + ip)$ at the point x is found by taking the branch corresponding to the inverse of g that is valid at x, i.e., for which $h_x(0) = x$. At $\xi = 0$, we have that $g'(\xi) = g''(\xi) = 0$ and the three branches coincide. For this example, integral (1) can again be decomposed into 4 contributions, each of which corresponds to a non-oscillating integral. The integration path is drawn in Figure 2.

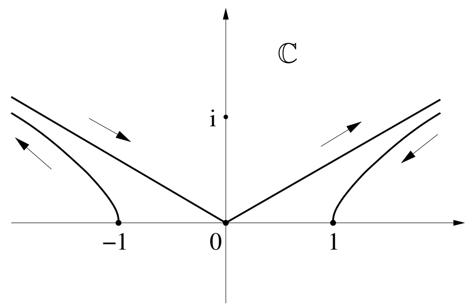


Figure 2: Illustration of the integration path for $g(x) = x^3$.

3 The ideal case: analytic integrand and no stationary points

3.1 An approximate decomposition of the oscillatory integral

The ideal setting for our approach has three conditions: both f and g are analytic functions, there are no stationary points in the integration interval $[a, b]$ (i.e., $g'(x) \neq 0$), and the equation $g(x) = z$ is easily solvable, preferably by analytical means. None of these conditions is crucial in order to obtain a convergent quadrature method, as we will relax all conditions later on. But, the ideal case leads to the highest convergence rate among all cases described, and is most suited to demonstrate our approach: the problem of evaluating (1) can be transformed into the problem of integrating two integrals on $[0, \infty)$ with a smooth integrand that does not oscillate, and that decays exponentially fast. This will be proved in this section in Theorem 3.3. First, we give a basic lemma for the approximation of an integral with an integrand that becomes small in some region S of the complex plane.

Lemma 3.1. *Assume u is analytic in a simply connected complex region $D \subset \mathbb{C}$ with $[a, b] \subset D$, and there exists a bounded and connected region $S \subset D$ such that $|u(z)| \leq \epsilon$, $\forall z \in S$. If the shortest distance between any two points p and q of S along a curve that lies in S can be bounded from above by a constant $M > 0$, then there exists a function $F(x)$, $x \in [a, b]$, such that the integral of u can be approximated by*

$$\int_a^x u(z)\,\mathrm{d}z \approx F(a) - F(x) \tag{7}$$

with an error e that satisfies $|e| \leq M\epsilon$. The function F is of the form

$$F(x) = \int_{\Gamma_x} u(z)\,\mathrm{d}z \tag{8}$$

with Γ_x any path in D that starts at x and ends in S.

Proof. Let Γ_x be a curve in D from x to an arbitrary point in S, denoted by $q(x)$, and Γ_a a curve in D from a to $q(a) \in S$. Choose κ as the shortest path in S that connects $q(a)$ and $q(x)$. Since u is analytic in D, the integration path between a and x may be chosen as the concatenation of Γ_a, κ and $-\Gamma_x$. The integral can be written as

$$\int_a^x u(z)\,\mathrm{d}z = F(a) + \int_\kappa u(z)\,\mathrm{d}z - F(x), \quad \text{with} \quad \left|\int_\kappa u(z)\,\mathrm{d}z\right| \leq M\epsilon.$$

This proves the result. ∎

Note that F is not completely determined by the conditions of this Lemma. In particular, the endpoint $q(x)$ of Γ_x may be an arbitrary function of x.

If g is analytic, then the oscillating function $e^{i\omega g(x)}$ in the integrand of (1) is also analytic as a function of x. This function is small in absolute value if

$$|e^{i\omega g(x)}| \leq \epsilon \iff e^{-\omega \Im g(x)} \leq \epsilon \iff \Im g(x) \geq \frac{-\log(\epsilon)}{\omega}.$$

Hence, if the inverse of g exists, we can find a suitable region S that is required for Lemma 3.1 with points given by $g^{-1}(c + id)$, for $d \geq d_0 := \frac{-\log(\epsilon)}{\omega}$. Note that, in general, the inverse of an analytic function may be multi-valued. Each single-valued branch of the inverse has branch points that are located at the points ξ where $g'(\xi) = 0$, and it is discontinuous across branch cuts that extend from one branch point to another, or from a branch point to infinity. By explicitly excluding the presence of branch points locally, a single-valued branch of the inverse can be found that is analytic in a neighbourhood of $[a, b]$. We can then characterize the error of the decomposition given in Lemma 3.1 for the particular case of integral (1) as a function of ω.

Theorem 3.2. *Assume f and g are analytic in a bounded and open complex neighbourhood D of $[a, b]$, and assume $g'(z) \neq 0$, $z \in D$. Then there exists an approximation of the form (7) for (1), with an error that has order $O(e^{-\omega d_0})$ as a function of ω, for a real constant $d_0 > 0$.*

Proof. Define $S := \{z : \Im g(z) \geq d_0\} \cap D$ with $d_0 > 0$. A positive constant d_0 can always be found such that S is non-empty because g is analytic. In order to prove this, consider a point $x \in [a, b]$. Since g is analytic at x, the equation $g(z) = g(x) + id_0$ always has a solution z for sufficiently small $d_0 > 0$ [11]. Additionally, d_0 can be chosen small enough such that $z \in D$, because D contains an open neighbourhood of x. The necessary geometrical conditions on S required by Lemma 3.1 follow from the continuity properties of g. We have

$$\forall x \in S : |f(x)e^{i\omega g(x)}| \leq |f(x)|e^{-\omega d_0}.$$

Since S is finite (because D is bounded), there exists a constant $C > 0$ such that $|f(x)| \leq C$, $x \in S$. The result is established by Lemma 3.1 with $u(x) = f(x)e^{i\omega g(x)}$ and $\epsilon = Ce^{-\omega d_0}$. ∎

Theorem 3.2 shows that the error in the approximation $I \approx F(a) - F(b)$ for (1) decays exponentially fast as the frequency parameter ω increases. It only requires that f and g are analytic in a finite neighbourhood of $[a, b]$. The function F is given by an integral along a curve that originates in x, and leads to a point z such that $g(z)$ has a positive imaginary part. The result follows from the exponential decay of the integrand, which is the first of the two observations about the integrand made in the introduction.

3.2 An exact decomposition of the oscillatory integral

Next, we will take the second observation into account: $e^{i\omega g(x)}$ does not oscillate along a path where $g(x)$ has a fixed real part. This will lead to a particularly useful choice for the path Γ_x in the definition (8) of F.

Let $h_x(p)$ be a parameterization for Γ_x, $p \in [0, P]$, then we find a suitable path as the solution to

$$g(h_x(p)) = g(x) + ip, \quad x \in [a, b].$$

If the inverse of g exists, we have the unique solution $h_x(p) = g^{-1}(g(x)+ip)$. The path $h_x(p)$ is also called the path of Steepest Descent [1, 2]. This can be understood as follows. Define $k(x, y) := ig(z) = u(x, y) + iv(x, y)$, with $z = x + iy$. Then we have $e^{i\omega g(z)} = e^{\omega k(x,y)}$. It can be shown that the path is such that $v(x, y) = v(x_0, y_0)$ is constant, and that the descent of $u(x, y)$ is maximal. In particular, the direction of steepest descent coincides with $-\nabla u$ at each point $z = x + iy$.

Using this path in the definition of F, the decomposition for (1) becomes

$$\begin{aligned}\int_a^x f(z)e^{i\omega g(z)}\,\mathrm{d}z &\approx F(a) - F(x)\\ &= e^{i\omega g(a)} \int_0^P f(h_a(p))e^{-\omega p}h_a'(p)\,\mathrm{d}p - e^{i\omega g(x)} \int_0^P f(h_x(p))e^{-\omega p}h_x'(p)\,\mathrm{d}p.\end{aligned}$$

The integrands in the right hand side do not oscillate, and they decay exponentially fast as the integration parameter p or the frequency parameter ω increases.

In the following Theorem, we will consider the limit case $P \to \infty$ in which the error of the approximation vanishes. This will require stronger analyticity conditions for both f and g. Additionally, the function f can no longer be assumed to be bounded. The result of the theorem will hold if f does not grow faster than polynomially in the complex plane along the suggested integration path.

Theorem 3.3. *Assume that the functions f and g are analytic in a simply connected and sufficiently (infinitely) large complex region D containing the interval $[a, b]$, and that the inverse of g exists on D. If the following conditions hold in D:*

$$\exists m \in \mathbb{N} : |f(z)| = O(|z|^m), \quad \text{and} \tag{9}$$

$$\exists \omega_0 \in \mathbb{R} : |g^{-1}(z)| = O(e^{\omega_0|z|}), \quad |z| \to \infty, \tag{10}$$

then there exists a function $F(x)$, for $x \in [a, b]$, such that

$$\int_a^x f(z)e^{i\omega g(z)}\,\mathrm{d}z = F(a) - F(x), \qquad \forall \omega > (m + 1)\omega_0, \tag{11}$$

where $F(x)$ is of the following form,

$$F(x) := \int_{\Gamma_x} f(z)e^{i\omega g(z)}\,\mathrm{d}z, \tag{12}$$

with Γ_x a path that starts at x. A parameterization $h_x(p)$, $p \in [0, \infty)$, for Γ_x exists such that the integrand of (12) is $O(e^{-\omega p})$.

Proof. In this proof, we will use $u(z)$ to denote the integrand of (1). Using the fact that $|u(z)| = |f(z)e^{i\omega g(z)}| = |f(z)|e^{-\omega \Im g(z)}$, and conditions (9) and (10), we can state

$$c + id \in D \Rightarrow |u(g^{-1}(c + id))| = O(e^{(m\omega_0-\omega)d}), \quad d \to \infty. \tag{13}$$

If $\omega > m\omega_0$, then (13) characterizes the exponential decay of the integrand in the complex plane. We will now choose an integration path from the point a to the region where the integrand becomes small, and from that region back to the point $x \in [a, b]$. We will show that the contribution along the line that connects both paths can be discarded. This will establish the existence of Γ_a and Γ_x in (12), and the independence of Γ_a and Γ_x.

Assume an integration path for I that consists of three connected parts, parameterized as $h_a(p)$ and $h_x(p)$ with $p \in [0, P]$, and $\kappa(p)$ with $p \in [a, x]$. The parameterizations can be chosen differentiable and satisfy $h_a(0) = a$, $h_x(0) = x$, $h_a(P) = \kappa(a)$ and $h_x(P) = \kappa(x)$. We have

$$\int_a^x u(z)\,\mathrm{d}z = \int_0^P u(h_a(p))h_a'(p)\,\mathrm{d}p + \int_a^x u(\kappa(p))\kappa'(p)\,\mathrm{d}p - \int_0^P u(h_x(p))h_x'(p)\,\mathrm{d}p. \qquad (14)$$

Since the inverse function g^{-1} exists, we can choose the points $h_a(P)$ and $h_x(P)$ as follows: $h_a(P) = g^{-1}(g(a) + iP)$ and $h_x(P) = g^{-1}(g(x) + iP)$. Hence, by (13),

$$|u(h_a(P))| = O(e^{(m\omega_0-\omega)P}) \quad \text{and} \quad |u(h_x(P))| = O(e^{(m\omega_0-\omega)P}).$$

We will now show that, as $P \to \infty$, the second integral vanishes. Equation (14) is then of the form (11), with Γ_a and Γ_x parameterized by $h_a(p)$ and $h_x(p)$ respectively, $p \in [0, \infty)$.

The contribution of the integral along $\kappa(p)$ is bounded by

$$\left| \int_a^x u(\kappa(p))\kappa'(p)\,\mathrm{d}p \right| \le \max_{p\in[a,x]} |u(\kappa(p))| \max_{p\in[a,x]} \left|\kappa'(p)\right| |x - a|. \qquad (15)$$

By selecting the path $\kappa(p) = g^{-1}(g(p) + iP)$ we have by (13): $|u(\kappa(p))| = O(e^{(m\omega_0-\omega)P})$, $p \in [a, x]$. We can write the second factor in the bound (15) as

$$\kappa'(p) = \frac{\partial g^{-1}}{\partial y}(g(p) + iP)\frac{\mathrm{d}g}{\mathrm{d}p}(p).$$

The derivative of $g(p)$ with respect to p is bounded on $[a, b]$ because g is analytic. The factor $\frac{\partial g^{-1}}{\partial y}(g(p) + iP)$ is bounded by $O(e^{\omega_0 P})$. Combining the asymptotic behaviour of the factors in (15), the second term in (14) vanishes for $P \to \infty$ and for all $x \in [a, b]$ if $\omega > (m + 1)\omega_0$. This proves the result. □

Remark 3.4. Note that f and g should be analytic in a simply connected region D that contains the paths h_a, h_b and $\kappa(p)$ in order to apply Cauchy's Theorem. The unique existence of the inverse of g is a necessary condition: if $g'(z) = 0$ with $z \in D$, then the point z is a branch point of the inverse function. The path $\kappa(p)$ may cross the branch cut that originates at z, and Cauchy's Theorem cannot be applied.

Remark 3.5. Conditions (9) and (10) are sufficient but not necessary. For example, the limit case also applies when $f(x) = e^x$ and $g(x) = x$. If however $f(x) = e^{-x^2}$ and $g(x) = x$, the integrand always diverges at infinity along the steepest descent path, regardless of the size of ω. In that case, the path should be truncated at a finite distance from the real axis. The accuracy of the decomposition is then described by Theorem 3.2, i.e., the error decays exponentially fast.

3.3 Evaluation of $F(x)$ by Gauss-Laguerre quadrature

Next, we consider the evaluation of $F(x)$ as defined by (12). The parameterization of the path $h_x(p)$ is such that it solves the equation

$$g(h_x(p)) = g(x) + ip. \tag{16}$$

The integrand of (1) along this path is non-oscillatory and exponentially decaying,

$$f(h_x(p))e^{i\omega g(h_x(p))} = f(h_x(p))e^{i\omega g(x)}e^{-\omega p}.$$

In the simplest, yet important case $g(x) := x$ the suggested path is given by $h_x(p) = x + ip$. An efficient approach for infinite integrals with exponentially decaying integrand is Gauss-Laguerre quadrature [5]. Laguerre polynomials are orthogonal w.r.t. e^{-x} on $[0, \infty]$. A Gauss-Laguerre rule with n points is exact for polynomials up to degree $2n - 1$. The integral $F(x)$ with the suggested path can be written as

$$\begin{aligned} F(x) &= \int_0^\infty f(h_x(p))e^{i\omega(g(x)+ip)}h_x'(p)\,\mathrm{d}p \\ &= e^{i\omega g(x)} \int_0^\infty f(h_x(p))h_x'(p)e^{-\omega p}\,\mathrm{d}p \\ &= \frac{e^{i\omega g(x)}}{\omega} \int_0^\infty f(h_x(q/\omega))h_x'(q/\omega)e^{-q}\,\mathrm{d}q \end{aligned}$$

with $q = \omega p$ in the last expression. Applying a Gauss-Laguerre quadrature rule with n points x_i and weights w_i yields a quadrature rule

$$F(x) \approx Q_F[f, g, h_x] := \frac{e^{i\omega g(x)}}{\omega} \sum_{i=1}^{n} w_i f(h_x(x_i/\omega))h_x'(x_i/\omega). \tag{17}$$

The rule requires the evaluation of f in a complex neighbourhood of x.

Theorem 3.6. *Assume functions f and g satisfy the conditions of Theorem 3.3. Let I be approximated by the quadrature formula*

$$I \approx Q[f, g] := Q_F[f, g, h_a] - Q_F[f, g, h_b], \tag{18}$$

where Q_F is evaluated by an n-point Gauss-Laguerre quadrature rule as specified in (17). *Then the quadrature error behaves asymptotically as $O(\omega^{-2n-1})$.*

Proof. A formula for the error of the n-point Gauss-Laguerre quadrature rule applied to the integral $\int_0^\infty f(x)e^{-x}\mathrm{d}x$ is given by [5]

$$E = \frac{(n!)^2}{(2n)!} f^{(2n)}(\zeta), \quad \zeta \in [0, \infty).$$

Using this formula, one can derive an expression for the error $E := F(a) - Q_F[f, g, h_a]$:

$$\begin{aligned} E &= \frac{e^{i\omega g(a)}}{\omega} \frac{(n!)^2}{(2n)!} \left. \frac{\mathrm{d}^{2n}(f(h_a(q/\omega))h_a'(q/\omega))}{\mathrm{d}q^{2n}} \right|_{q=\zeta} \\ &= \frac{e^{i\omega g(a)}}{\omega^{2n+1}} \frac{(n!)^2}{(2n)!} \left. \frac{\mathrm{d}^{2n}(f(h_a(q))h_a'(q))}{\mathrm{d}q^{2n}} \right|_{q=\zeta/\omega} \end{aligned} \tag{19}$$

with $\zeta \in \mathbb{C}$. The error behaves asymptotically as $O(\omega^{-2n-1})$. The absolute error for the approximation to (1) is composed of 2 contributions of the form (19), and, hence, has the same high order of convergence. □

Remark 3.7. The decomposition $I = F(a) - F(b)$ is of a similar type as the decomposition of I in [15] based on asymptotic expansions. There, the terms in the expansions are given by a combination of f, g and their derivatives, evaluated in the points a and b. Yet, the numerical properties of our approach are different: the convergence rate $O(\omega^{-2n-1})$ when using an n-point quadrature rule for both $Q_F[f, g, h_a]$ and $Q_F[f, g, h_b]$, should be compared to the rate $O(\omega^{-n-1})$ when using an n-term asymptotic expansion of I evaluated in a and b.

Table 1: Absolute error of the approximation of I by $Q_F[f, g, h_a] - Q_F[f, g, h_b]$ with n quadrature points for the functions $f(x) = 1/(1 + x)$ and $g(x) = x$ on $[0, 1]$. The last row shows the value of $\log_2(e_{40}/e_{80})$: this value should approximate $2n + 1$.

$\omega \setminus n$	1	2	3	4	5
10	$1.0E-3$	$3.1E-5$	$1.9E-6$	$1.7E-7$	$2.1E-8$
20	$1.2E-4$	$1.1E-6$	$2.3E-8$	$7.5E-10$	$3.2E-11$
40	$1.7E-5$	$3.9E-8$	$2.1E-10$	$2.0E-12$	$2.8E-14$
80	$2.0E-6$	$1.2E-9$	$1.7E-12$	$4.2E-15$	$1.6E-17$
rate	3.1	5.0	6.9	8.9	10.8

Example 3.8. We end this section with a numerical example to illustrate the sharpness of our convergence result. The absolute error for different values of ω and of n is given in Table 1 for the functions $g(x) = x$ and $f(x) = 1/(1 + x)$ on $[0, 1]$. The parameterization for Γ_x is given by $h_x(p) = g(x) + ip$. The behaviour as a function of ω follows the theory until machine precision is reached. The relative error scales only slightly worse, since $|I| = O(\omega^{-1})$.

One should note that decomposition (11) is exact for all positive values of the parameter $\omega > (m+1)\omega_0 > 0$. The conditions from Theorem 3.3 yield the minimal frequency parameter $(m + 1)\omega_0$. The method itself is therefore not asymptotic, only the convergence estimate is. Table 1 shows an absolute error of $2.1E-8$ (relative error $1.4E-7$) for $\omega = 10$ with a number of quadrature points as small as $n = 5$. The corresponding integral is not highly oscillatory at all. In order to achieve the same absolute error with standard Gaussian quadrature on $[0, 1]$, we had to choose a rule with 10 points. Considering the fact that we evaluate both $Q_F[f, g, h_a]$ and $Q_F[f, g, h_b]$ with $n = 5$ points, the amount of work is the same. Thus, even at relatively low frequencies, our approach is competitive with conventional quadrature on the real axis. For higher frequencies, obviously, the new approach may be many orders of magnitude faster.

4 The case of stationary points

4.1 A new decomposition for the oscillatory integral

At a stationary point ξ, the derivative of g vanishes and the integrand $f(x)e^{i\omega g(x)}$ does not oscillate, at least locally. The contribution of the integrand and its derivatives at ξ can therefore not be neglected. The Theorems of Section 3 do not apply, because the inverse of g does not exist uniquely due to the branch point at ξ.

In order to illustrate the problem, consider the following situation. Assume that the equation $g'(x) = 0$ has one solution ξ and $\xi \in [a, b]$. Now define the restrictions

$$g_1 := g|_{[a,\xi]} \quad \text{and} \quad g_2 := g|_{[\xi,b]} \tag{20}$$

of g to the intervals $[a, \xi]$ and $[\xi, b]$ respectively. Then, the unique inverse of g on $[a, b]$ does not exist, but a single-valued branch g_1^{-1} can be found that satisfies $g_1^{-1}(g_1(x)) = x$, $x \in [a, \xi]$. This branch is analytic everywhere except at the point ξ, and along a branch cut that can be chosen arbitrarily but that always originates at ξ. Similarly, a single-valued branch g_2^{-1} exists that satisfies $g_2^{-1}(g_2(x)) = x$, $x \in [\xi, b]$. Both branches satisfy $g(g_i^{-1}(z)) = z$, $i = 1, 2$, in their domain of analyticity. The integrand is small in the region S_1 with points of the form $g_1^{-1}(c + id)$, $d \geq d_0$, or in the region S_2 with points of the form $g_2^{-1}(c + id)$, $d \geq d_0$. It is easy to see that S_1 and S_2 are not connected: applying g on both sides of the equality $g_1^{-1}(y) = g_2^{-1}(z)$ leads to $y = z$, which is only possible if $z = \xi \notin S_1, S_2$. The path (16) that solves $g(h_x(p)) = g(x) + ip$, as suggested in Section 3, leads to a path in S_1 for a, and to a path in S_2 for b.

The solution is therefore to split the integration interval into the two subintervals $[a, \xi]$ and $[\xi, b]$. This procedure can be repeated for any number of stationary points. The analogues of Theorems 3.2 and 3.3 can be stated as follows.

Theorem 4.1. *Assume that the functions f and g are analytic in a bounded and open complex neighbourhood D of $[a, b]$. If the equation $g'(x) = 0$ has only one solution ξ in D and $\xi \in (a, b)$, then there exist functions $F_j(x)$, $j = 1, 2$, such that*

$$\int_s^t f(z) e^{i\omega g(z)} \,\mathrm{d}z = F_1(s) - F_1(\xi) + F_2(\xi) - F_2(t) + O(e^{-\omega d_0}), \quad d_0 > 0,$$

for $s \in [a, \xi]$ and $t \in [\xi, b]$, where $F_j(x)$ is of the form

$$F_j(x) := \int_{\Gamma_{x,j}} f(z) e^{i\omega g(z)} \,\mathrm{d}z \tag{21}$$

with $\Gamma_{x,j}$ a path that starts at x.

Proof. Define $g_2(x)$ as in (20). A decomposition for $\int_\xi^t f(x) e^{i\omega g_2(x)} \,\mathrm{d}x$ can be found using the proof of Theorem 3.2 with two modifications. First, the equation $g(z) = g(x) + id_0$ now has at least two solutions locally around $x = \xi$. We choose the solution that corresponds to the single-valued branch g_2^{-1} of the inverse of g that satisfies $g_2^{-1}(g(x)) = x$, $x \in [\xi, b]$. The branch cut can always be chosen such that it does not prevent from applying Cauchy's Theorem. Secondly, the set S in the proof is now defined as $S := \{z : \Im g(z) \geq d_0 \text{ and } g_2^{-1}(g(z)) = z\} \cap D$, i.e., the set is restricted to one connected part of D where the integrand is small, as opposed to the set of all points where the integrand is small. The latter set would not be connected in this case. With these modifications, the proof shows the existence of F_2 such that

$$\int_\xi^t f(z) e^{i\omega g_2(z)} \,\mathrm{d}z = F_2(\xi) - F_2(t) + O(e^{-\omega d_0}).$$

The same reasoning can be applied in order to find a decomposition on the interval $[a, \xi]$. This leads to the result. □

The next Theorem is the limit case of Theorem 4.1 where the error vanishes. The notation g_1^{-1} denotes a branch of the multi-valued inverse of g that satisfies $g_1^{-1}(g_1(x)) = x$, $x \in [a, \xi]$. The notation g_2^{-1} is similar.

Theorem 4.2. *Assume that the functions f and g are analytic in a simply connected and sufficiently (infinitely) large complex region D containing the interval $[a, b]$. Assume further that the equation $g'(x) = 0$ has only one solution ξ in D and $\xi \in (a, b)$. Define g_1 and g_2 as in (20). If the following conditions hold:*

$$\exists m \in \mathbb{N} : |f(z)| = O(|z|^m),$$
$$\exists \omega_0 \in \mathbb{R} : |g_1^{-1}(z)| = O(e^{\omega_0 |z|}) \text{ and } |g_2^{-1}(z)| = O(e^{\omega_0 |z|}), \quad |z| \to \infty,$$

then there exist functions $F_j(x)$, $j = 1, 2$, of the form (21) such that

$$\int_s^t f(z) e^{i\omega g(z)} \, \mathrm{d}z = F_1(s) - F_1(\xi) + F_2(\xi) - F_2(t), \quad \forall \omega > (m+1)\omega_0, \tag{22}$$

for $s \in [a, \xi]$ and $t \in [\xi, b]$. A parameterization $h_{\xi,j}(p)$, $p \in [0, \infty)$, for $\Gamma_{x,j}$ exists such that the integrand of (21) is $O(e^{-\omega p})$.

Theorems 4.1 and 4.2 are easily extended to the case where $\xi = a$ (or $\xi = b$), by discarding the two terms $F_1(a) - F_1(\xi)$ (or $F_2(\xi) - F_2(b)$).

Example 4.3. We consider the function $g(x) = (x - 1/2)^2$, with a stationary point at $\xi = 1/2$. The inverse of g, i.e., $g^{-1}(y) = 1/2 \pm \sqrt{y}$, is a two-valued function. One branch is valid on the interval $[0, \xi]$, the other on $[\xi, 1]$. The paths suggested by (16) on $[0, \xi]$ that originate at the endpoints 0 and ξ respectively are parameterized by

$$h_{0,1}(p) = 1/2 - \sqrt{1/4 + ip} \quad \text{and} \quad h_{\xi,1}(p) = 1/2 - \sqrt{ip}$$

The paths on $[\xi, 1]$ for the points 1/2 and 1 are parameterized by

$$h_{\xi,2}(p) = 1/2 + \sqrt{ip} \quad \text{and} \quad h_{1,2}(p) = 1/2 + \sqrt{1/4 + ip}$$

These paths correspond to the two inverse functions. We have found the decomposition $I = F_1(a) - F_1(\xi) + F_2(\xi) - F_2(b)$.

Note that the paths $h_{\xi,1}$ and $h_{\xi,2}$ that originate in the point ξ introduce a numerical problem. Their derivatives, that appear in the integrand of the line integral, behave like $1/\sqrt{p}$, $p \to 0$ at ξ. This weak singularity is integrable, but prevents convergence of the Gauss-Laguerre quadrature rules. We will require a new method to evaluate $F_j(\xi)$.

4.2 The evaluation of $F_j(x)$ by generalized Gauss-Laguerre quadrature

The previous example showed a numerical problem for the evaluation of $F_j(x)$ by numerical quadrature: the integrand of $F_j(\xi)$ along the path suggested by (16) becomes weakly singular at the stationary point ξ. A similar singularity occurs if higher order derivatives of $g(\xi)$ also vanish. Assume that $g^{(k)}(\xi) = 0$, $k = 1, \ldots, r$. The Taylor expansion of g is then

$$g(x) = g(\xi) + 0 + \ldots + 0 + g^{(r+l)}(\xi) \frac{(x - \xi)^{r+1}}{(r+1)!} + O((x - \xi)^{l+2}).$$

The path $h_{\xi,j}(p)$ solves the equation $g(h_{\xi,j}(p)) = g(\xi) + ip$. Its behaviour at $p = 0$ is

$$h_{\xi,j}(p) \sim \xi + \sqrt[r+1]{\frac{(r+1)!\,p}{g^{(r+1)}(\xi)}}\,i. \tag{23}$$

The derivative has a singularity of the form $p^{\frac{1}{r+1}-1}$, $p \to 0$.

Fortunately, these types of singularities can be handled efficiently by generalized Gauss-Laguerre quadrature. Generalized Laguerre polynomials are orthogonal with respect to the weight function $x^\alpha e^{-x}$, $\alpha > -1$ [5]. Function $F_j(\xi)$ with optimal path $h_{\xi,j}(p)$ is given by

$$\begin{aligned} F_j(\xi) &= \int_0^\infty f(h_{\xi,j}(p)) e^{i\omega(g(\xi)+ip)} h'_{\xi,j}(p)\,\mathrm{d}p \\ &= \frac{e^{i\omega g(\xi)}}{\omega} \int_0^\infty f(h_{\xi,j}(q/\omega)) h'_{\xi,j}(q/\omega) e^{-q}\,\mathrm{d}q. \end{aligned} \tag{24}$$

Generalized Gauss-Laguerre quadrature will be used with n points x_i and weights w_i that depend on the value of $\alpha = 1/(r+1) - 1 = -r/(r+1)$. The function $F_j(x)$ is then approximated by

$$Q_F^\alpha[f, g, h_{\xi,j}] := \frac{e^{i\omega g(\xi)}}{\omega} \sum_{i=1}^n w_i\, f(h_{\xi,j}(x_i/\omega))\, h'_{\xi,j}(x_i/\omega)\, x_i^{-\alpha}. \tag{25}$$

This expression is similar to (17) but includes the factor $x_i^{-\alpha}$ to regularize the singularity.

Theorem 4.4. *Assume functions f and g satisfy the conditions of Theorem 4.2. Assume that $g^{(k)}(\xi) = 0$, $k = 1, \ldots, r$ and $g^{(r+1)}(\xi) \neq 0$. Let the function $F_j(\xi)$ be approximated by the quadrature formula*

$$F_j(\xi) \approx Q_F^\alpha[f, g, h_{\xi,j}]$$

with $\alpha = -r/(r+1)$. Then the quadrature error behaves asymptotically as $O(\omega^{-2n-1-\alpha})$.

Proof. The error formula for an n-point generalized Gauss-Laguerre quadrature rule is

$$\frac{n!\Gamma(n+\alpha+1)}{(2n)!} f^{(2n)}(\zeta), \quad 0 < \zeta < \infty. \tag{26}$$

We can repeat the arguments of the proof of Theorem 3.6. An expression for the error $e := F_j(\xi) - Q_F^\alpha[f, g, h_{\xi,j}]$ can be derived by using (26). This leads to

$$\begin{aligned} e &= \frac{e^{i\omega g(\xi)}}{\omega} \frac{n!\Gamma(n+\alpha+1)}{(2n)!} \left. \frac{\mathrm{d}^{2n}(f(h_{\xi,j}(q/\omega)) h'_{\xi,j}(q/\omega) q^{-\alpha})}{\mathrm{d}q^{2n}} \right|_{q=\zeta} \\ &= \frac{e^{i\omega g(\xi)}}{\omega^{2n+1}} \frac{n!\Gamma(n+\alpha+1)}{(2n)!} \left. \frac{\mathrm{d}^{2n}(f(h_{\xi,j}(q)) h'_{\xi,j}(q) (\omega q)^{-\alpha})}{\mathrm{d}q^{2n}} \right|_{q=\zeta/\omega} \end{aligned}$$

with $\zeta \in \mathbb{C}$. Hence, the error is asymptotically of the order $O(\omega^{-2n-1-\alpha})$. □

Remark 4.5. Generalized Gauss-Laguerre quadrature converges rapidly only if the function $v(x)$ in an integrand of the form $v(x)x^\alpha e^{-x}$ has polynomial behaviour. Depending on f, the function $f(h_{\xi,j}(p))$ may not resemble a polynomial very well, due to the root in (23) for small

p. An alternative to generalized Gauss-Laguerre quadrature with $\alpha = -1/2$ is to remove the singularity by the transformation $u = \sqrt{p}$ or $p = u^2$. The same transformation also removes the square root behaviour of $h_{\xi,j}(p)$. The integrand after the transformation decays like e^{-u^2}. In that case, variants of the classical Hermite polynomials that are orthogonal w.r.t. e^{-u^2} on the half-range interval $[0, \infty)$ can be used, with corresponding Gaussian quadrature rules as constructed by Gautschi [10]. A similar convergence analysis yields the order $O(\omega^{-n-1})$ in this case.

We can now characterize the approximation of (1) in the presence of several stationary points.

Theorem 4.6. *Assume that f and g are analytic in a sufficiently large region $D \subset \mathbb{C}$, and that the equation $g'(x) = 0$ has l solutions $\xi_i \in (a, b)$. Define $r_i := (\min_{k>1} g^{(k)}(\xi_i) \neq 0) - 1$ and $r := \max_i r_i$. If the conditions of Theorem 4.2 are satisfied on each subinterval $[\xi_i, \xi_{i+1}]$, and on $[a, \xi_1]$ and $[\xi_r, b]$, then (1) can be approximated by*

$$\begin{aligned} I \approx Q[f,g] := & Q_F[f,g,h_{a,0}] - Q_F^{\alpha_1}[f,g,h_{\xi_1,0}] + \sum_{i=1}^{l-1} \left(Q_F^{\alpha_i}[f,g,h_{\xi_i,i}] - Q_F^{\alpha_{i+1}}[f,g,h_{\xi_{i+1},i}] \right) \\ & + Q_F^{\alpha_l}[f,g,h_{\xi_l,l}] - Q_F[f,g,h_{b,l}] \end{aligned} \tag{27}$$

with $\alpha_i = -r_i/(r_i + 1)$, with a quadrature error of the order $O(\omega^{-2n-1/(r+1)})$.

Proof. This follows from a repeated application of the decomposition given by Theorem 4.2, and from the approximation of each term $F_i(x)$ by $Q_F^{\alpha_i}[f, g, h_{x,i}]$ as in Theorem 4.4. □

Theorem 4.6 can easily be extended to the case where $g'(a) = 0$ or $g'(b) = 0$. If, e.g., $g'(a) = 0$, we can set $\xi_1 = a$ and use the general decomposition (27) with the first two terms left out.

Example 4.7. We return to the Example 4.3 of this section in order to illustrate the convergence results. The approximation of (1) for the function $g(x) = (x - 1/2)^2$ is given by

$$I \approx Q[f,g] := Q_F[f,g,h_{0,1}] - Q_F^{-1/2}[f,g,h_{1/2,1}] + Q_F^{-1/2}[f,g,h_{1/2,2}] - Q_F[f,g,h_{1,2}]. \tag{28}$$

Theorem 4.6 predicts an error of the order $O(\omega^{-2n-1/2})$. The sharpness of this estimate can be verified by the results in Table 2.

4.3 The case of complex stationary points

So far, we have required the stationary point $\xi \in [a, b]$ to be real. But even for functions g that are real-valued on the real axis, the equation $g'(x) = 0$ may have complex solutions. The value of $g'(x)$ on $[a, b]$ can become very small, if a complex stationary point ξ lies close to the real axis. We may therefore expect that such a point contributes to the value of the integral (1). Here, we will not pursue the extension of the theory to the case of complex stationary points in any detail. Instead, we will restrict ourselves to a number of remarks that address some of the relevant issues.

A first observation is that Theorem 4.1 can still be applied, if the region D is chosen small enough such that it does not contain ξ. This means that the contribution of ξ to the value of

Table 2: Absolute error of the approximation of I by $Q[f,g]$ using (generalized) Gauss-Laguerre quadrature with $f(x) = 1/(1+x)$ and $g(x) = (x-1/2)^2$ on $[0,1]$. The last row shows the value of $\log_2(e_{80}/e_{160})$: this value should approximate $2n+1/2$.

$\omega \setminus n$	1	2	3	4	5
10	$4.7E-3$	$7.1E-4$	$1.7E-4$	$4.9E-5$	$1.7E-5$
20	$7.8E-4$	$5.6E-5$	$7.2E-6$	$1.3E-6$	$2.7E-7$
40	$1.2E-4$	$2.8E-6$	$1.5E-7$	$1.2E-8$	$1.3E-9$
80	$1.6E-5$	$1.0E-7$	$1.7E-9$	$5.0E-11$	$2.1E-12$
160	$2.3E-6$	$3.4E-9$	$1.6E-11$	$1.3E-13$	$1.6E-15$
rate	2.8	4.9	6.8	8.6	10.4

I, if any, decays exponentially fast as ω increases. Still, for small values of ω, the error may become prohibitively large if ξ lies close to the real axis.

In order to resolve this problem, one must first know which stationary points can contribute to the error of the approximations of Section 4. In general, the question can be answered by inspecting the integration paths. A stationary point contributes if it lies in the interior of the domain bounded by the integration interval on the real axis and the complex integration path (including the limiting connecting part at infinity). In order to obtain an exact decomposition, the integration path should be changed to pass through ξ explicitly. Specifically, the decomposition should include two additional terms for ξ of the form (24).

As a final remark, we note that the integral of the form (24) has a factor $e^{i\omega g(\xi)}$ with $g(\xi) = c + id$ complex. If $d > 0$ then the contribution decays exponentially as $e^{-\omega d}$. We know from Theorem 4.1 that the error introduced by discarding complex stationary points should decay exponentially. Hence, complex stationary points for which $d \leq 0$ cannot contribute to the value of I.

5 The case where the oscillator is not easily invertible

Theorems 3.3 and 4.2 continue to hold for paths different from the one implicitly defined by (16). The value of $F(a)$ does not depend on the path taken, and does not even depend on the limiting endpoint of the path, as long as the imaginary part of $g(x)$ grows infinitely large. We have merely suggested (16) as it yields a non-oscillatory integrand with exponential decay, suitable for Gauss-Laguerre quadrature. Other integration techniques may be applied for other paths with different numerical properties. We will not explore these possibilities in depth here.

We restrict the discussion to an approach that is useful when the inverse function of g is known to exist, but when the suggested path is not easily obtained by analytical means. As ω increases, we see from (17) that $Q_F[f,g,h_a]$ requires function values in a complex region around a of diminishing size. Therefore, it is reasonable to assume that approximating the path defined by (16) locally around $x = a$ is acceptable. Use of the first order Taylor approximation

$$g(x) \approx g(a) + g'(a)(x-a)$$

to replace the left hand side of (16) leads to the path

$$h_a(p) = a + \frac{ip}{g'(a)}. \tag{29}$$

The second order Taylor approximation leads to the path

$$h_a(p) = a - \frac{g'(a) - \sqrt{g'(a)^2 + 2ipg^{(2)}(a)}}{g^{(2)}(a)}.$$

In the case of stationary points the path can be approximated by using (23),

$$h_{\xi,i}(p) = \xi + \sqrt[r+1]{\frac{(r+1)!\,p}{g^{(r+1)}(\xi)}}\,i.$$

The general expression for the integral along the approximate path is given by

$$F(a) = \int_0^\infty f(h_a(p))e^{i\omega g(h_a(p))}h_a'(p)\,\mathrm{d}p.$$

Computing $F(a)$ by Gauss-Laguerre quadrature yields a numerical approximation with an error given by

$$\begin{aligned} E &= \omega^{-1}\frac{(n!)^2}{(2n)!}\left.\frac{\mathrm{d}^{2n}(f(h_a(q/\omega))e^{i\omega g(h_a(q/\omega))}h_a'(q/\omega)e^q)}{\mathrm{d}q^{2n}}\right|_{q=\zeta} \\ &= \omega^{-2n-1}\frac{(n!)^2}{(2n)!}\left.\frac{\mathrm{d}^{2n}(f(h_a(q))h_a'(q)e^{i\omega g(h_a(q))}e^{\omega q})}{\mathrm{d}q^{2n}}\right|_{q=\zeta/\omega}. \end{aligned}$$

The order of convergence is not necessarily $O(\omega^{-2n-1})$ in this case because the derivative still depends on ω. However, the function $e^{i\omega g(h_a(q))}$ is a good approximation to $e^{i\omega g(a)}e^{-\omega q}$ and we can expect the quadrature to converge. This will be illustrated further on.

The results can be improved to preserve the original convergence rate of $O(\omega^{-2n-1})$ at the cost of a little extra work to determine the optimal path. The optimal path depends only on $g(x)$ and on the interval $[a, b]$, and can therefore be reused for different functions f. The extra computations have to be done once for each combination of $g(x)$ and $[a, b]$.

The Taylor approximation of the path can be used to generate suitable starting values for a Newton-Raphson iteration, applied to find the root x of the equation

$$g(x) - g(a) - ip = 0. \tag{30}$$

For the set of n (fixed) values for p that are required by the quadrature rule, the iteration yields the points $x = h_a(p)$ on the path. The values of $h_a'(p)$, i.e., $\frac{\mathrm{d}x}{\mathrm{d}p}$, are found by taking the derivative of (30) with respect to p,

$$g'(x)\frac{\mathrm{d}x}{\mathrm{d}p} = i.$$

With the Newton-Raphson method, the points on the optimal path and the derivatives at these points can be found to high precision. Since the Taylor approximation is already a good approximation for large ω, the required number of iterations is small.

Example 5.1. We consider the second order Taylor approximation of the path for $f(x) = 1/(1+x)$ and $g(x) = (x^2+x+1)^{1/3}$. The absolute error is shown in Table 3. Use of the Newton-Raphson iteration for the same example yields an error of order $O(\omega^{-2n-1})$. This is shown in Table 4. The number of iterations per quadrature point varied between 1 and 4.

Table 3: Absolute error of approximation of $F(a) - F(b)$ by Gauss-Laguerre quadrature with $f(x) = 1/(1+x)$ and $g(x) = (x^2+x+1)^{1/3}$ on $[0,1]$ and second order Taylor approximation of the optimal path. The last row shows the value of $\log_2(e_{160}/e_{320})$.

$\omega \setminus n$	1	2	3	4	5
20	$1.4E-2$	$2.7E-3$	$7.4E-4$	$2.4E-4$	$8.9E-5$
40	$2.5E-3$	$2.6E-4$	$4.6E-5$	$1.0E-5$	$2.5E-6$
80	$3.8E-4$	$1.8E-5$	$1.7E-6$	$2.0E-7$	$2.9E-8$
160	$5.2E-5$	$1.1E-6$	$4.0E-8$	$2.1E-9$	$1.5E-10$
320	$6.7E-6$	$6.8E-8$	$7.7E-10$	$1.6E-11$	$4.4E-13$
rate	3.0	4.0	5.7	7.0	8.4

Table 4: The same example as in Table 3, but using Newton-Raphson iterations to compute the optimal path. The number of iterations per quadrature point varied between 1 and 4. The last row shows the value of $\log_2(e_{320}/e_{640})$: this value should approximate $2n+1$.

$\omega \setminus n$	1	2	3	4	5
20	$1.1E-2$	$2.4E-3$	$7.4E-4$	$2.5E-4$	$7.5E-5$
40	$2.1E-3$	$2.4E-4$	$4.4E-5$	$1.0E-5$	$2.4E-6$
80	$3.3E-4$	$1.5E-5$	$1.2E-6$	$1.5E-7$	$2.3E-8$
160	$4.5E-5$	$6.1E-7$	$1.8E-8$	$8.7E-10$	$6.2E-11$
320	$5.9E-6$	$2.1E-8$	$1.8E-10$	$2.7E-12$	$6.2E-14$
640	$7.2E-7$	$6.7E-10$	$1.5E-12$	$6.3E-15$	$4.3E-17$
rate	3.0	5.0	6.9	8.8	10.5

6 Generalization to a non-analytic function $f(x)$

If $f(x)$ is not analytic in a complex region surrounding $[a,b]$, then the method presented thus far will not work. If $f(x)$ is piecewise analytic (e.g., piecewise polynomial), the integration can be split into the integrals corresponding to the analytic parts of f. More generally however, we need to resort to another approach. For a suitable analytic function $\tilde{f}$ that approximates f, we can expect the integral

$$\tilde{I} := \int_a^b \tilde{f}(x) e^{i\omega g(x)} \, \mathrm{d}x$$

to approximate the value of I.

This leads to a *Filon-type method* that was already mentioned in the introduction. Filon's method was extended by Iserles and Nørsett in [15]. Since the value of I depends on the value of f and its derivatives at $x = a$ and $x = b$, they successfully used Hermite interpolation in the points a and b, in the stationary points and in a few other regular points in the interval $[a, b]$. In [15, Theorem 2.3] it was shown that interpolating $f^{(j)}(x)$ at a and b, $j = 0, \ldots, s-1$, with a polynomial of degree $2s-1$ leads to a quadrature rule with an error of order $O(\omega^{-s-1})$. Since polynomials are analytic, we can also use the Hermite approximation in our approach. This enables the computation of the weights of the Filon-type quadrature rule for general oscillators. (Note that the complex approach also enables the computation of the moments in the *asymptotic method* of [15]). It does not improve the convergence rate of the method. We can improve on the Filon-type method however in a different way. Thanks to the decomposition of (1) as $I = F(a) - F(b)$, it is possible to use different approximations around a and b, and, hence, to approximate $F(a)$ and $F(b)$ independently. Since $F(a)$ only depends on the behaviour of f around a, the approximating Hermite polynomial can have much lower degree. In the theorem below, we show that we can obtain a similar accuracy as in [15, Th.2.3] with two independently constructed polynomials of degree $s-1$ instead of with one polynomial of degree $2s-1$.

Theorem 6.1. *Assume that f is a smooth function, and g is analytic. Let $f_a(x)$ and $f_b(x)$ be the Hermite interpolating polynomials of degree $s-1$ that satisfy*

$$f_a^{(k)}(a) = f^{(k)}(a) \quad \text{and} \quad f_b^{(k)}(b) = f^{(k)}(b), \quad k = 0, \ldots, s-1.$$

Then the approximation of (1) *by*

$$F_{f_a}(a) - F_{f_b}(b) := \int_0^\infty f_a(h_a(p)) e^{i\omega g(h_a(p))} h_a'(p)\, \mathrm{d}p - \int_0^\infty f_b(h_b(p)) e^{i\omega g(h_b(p))} h_b'(p)\, \mathrm{d}p$$

along the paths $h_a(p)$ and $h_b(p)$ that satisfy (16) *has an error of order $O(\omega^{-s-1})$.*

Proof. First we consider the approximation with the Hermite interpolating polynomial $\tilde{f}(x)$ of degree $2s-1$ that satisfies $\tilde{f}^{(k)}(a) = f^{(k)}(a)$ and $\tilde{f}^{(k)}(b) = f^{(k)}(b)$, $k = 0, \ldots, s-1$. Since $\tilde{f}$ is analytic, it can be used to approximate (1) as $I \approx F_{\tilde{f}}(a) - F_{\tilde{f}}(b)$. This approximation has an error of $O(\omega^{-s-1})$ by [15, Theorem 2.3]. Now consider the approximation of $F_{\tilde{f}}(a)$ by $F_{f_a}(a)$. Since $\tilde{f}(x)$ is a polynomial, we can write $F_{\tilde{f}}(a)$ as

$$F_{\tilde{f}}(a) = \sum_{k=0}^{2s-1} \tilde{f}^{(k)}(a) \frac{\mu_k(a)}{k!},$$

where the $\mu_k(a)$ are the moments of the form

$$\mu_k(a) := \int_0^\infty (h_a(p) - a)^k e^{i\omega g(h_a(p))} h_a'(p)\, \mathrm{d}p = \int_0^\infty \frac{e^{i\omega g(a)}}{\omega} (h_a(q/\omega) - a)^k e^{-q} h_a'(q/\omega)\, \mathrm{d}q. \quad (31)$$

Although q goes to infinity, the behaviour for small q/ω dominates due to the factor e^{-q} (this follows from Watson's Lemma [1]). Since $(h_a(q/\omega) - a) \sim \omega^{-1}$, we see that $\mu_k(a) \sim \omega^{-k-1}$. For $F_{f_a}(a)$, we have

$$F_{f_a}(a) = \sum_{k=0}^{s-1} f_a^{(k)}(a) \frac{\mu_k(a)}{k!}. \quad (32)$$

The first discarded moment, $\mu_s(a)$, is of order $O(\omega^{-s-1})$. The approximation of $F_{\tilde{f}}(b)$ by $F_{f_b}(b)$ has an error of the same order. This concludes the proof. □

There are two ways to proceed: either $f_a(x)$ can be evaluated explicitly in the quadrature evaluation of $F_{f_a}(a)$, or the moments (31) can be precomputed with the previous techniques and used in the summation (32). The result is a quadrature rule for integrals of type (1) for fixed g, a and b, and using function values and derivatives of f at a and b. Define $w_{i,1} = \frac{\mu_k(a)}{k!}$ and $w_{i,2} = -\frac{\mu_k(b)}{k!}$. Then

$$I \approx Q_\mu[f] := \sum_{i=0}^{s-1} w_{i,1} f^{(i)}(a) + \sum_{i=0}^{s-1} w_{i,2} f^{(i)}(b) \tag{33}$$

is a quadrature rule with an error of order $O(\omega^{-s-1})$. For a fixed frequency, this *localized Filon-type method* is exact for polynomials up to degree $s-1$, while the regular Filon-type method is exact for polynomials up to degree $2s-1$. Hence, the simplified construction comes at a cost; the order of accuracy as a function of ω is the same, but we can expect the coefficient to be much larger.

We can generalize the result to include stationary points. The same reasoning applies, but we need to interpolate more derivatives in order to achieve a similar convergence rate. That rate depends on the smallest value of r for which $g^{(r+1)}(\xi) \neq 0$, with ξ a stationary point.

Theorem 6.2. *Assume that g is analytic and that $g^{(k)}(\xi) = 0$, $k = 1, \ldots, r$, and $g^{(r+1)}(\xi) \neq 0$. Let f be sufficiently smooth, and let $f_\xi(x)$ be the Hermite interpolating polynomial of degree $s(r+1)-1$ that satisfies*

$$f_\xi^{(k)}(\xi) = f^{(k)}(\xi), \quad j = 0, \ldots, s(r+1)-1.$$

Then the sequence $F_{f_\xi,j}(\xi)$ converges for increasing values of s to a limit with an error of order $O(\omega^{-s-1/(r+1)})$.

Proof. The proof follows essentially the same lines as the proof of Theorem 6.1, and uses the moments $\mu_{k,j}(\xi)$, defined using the path $h_{\xi,j}$,

$$\mu_{k,j}(\xi) := \int_0^\infty \frac{e^{i\omega g(\xi)}}{\omega} (h_{\xi,j}(q/\omega) - \xi)^k e^{-q} h'_{\xi,j}(q/\omega) \, \mathrm{d}q. \tag{34}$$

The derivative of the parameterization $h_{\xi,j}$ in the integrand has an integrable singularity of the form $(q/\omega)^{-r/(r+1)}$ at the stationary point ξ, and leads to a factor $\omega^{r/(r+1)}$. By (23) we have $(h_{\xi,j}(q/\omega) - \xi) \sim \omega^{-1/(r+1)}$. This makes $\mu_{k,j}(\xi) \sim \omega^{r/(r+1)-k/(r+1)-1} = \omega^{(-k-1)/(r+1)}$. The first discarded moment $\mu_{k,j}(\xi)$ in the sum $F_{f_\xi,j}$ of the form (32) has the index $k = s(r+1)$, which leads to the result. □

Theorem 6.2 only shows that the value $F_{f_\xi}(\xi)$ converges with a specific rate if more derivatives of f are interpolated. It does not explicitly state that $F_{f_\xi}(\xi)$ can be used in a decomposition to approximate (1). The existence of an analytic function $\tilde{f}$ that can be used to approximate the value of (1) with an arbitrary accuracy, provided f is smooth enough, was proved in [15, Theorem 3.3] using Hermite interpolation.

Assume there is one stationary point $\xi \in (a, b)$, and $g^{(r+1)}(\xi) \neq 0$. Then we can extend the definition of quadrature rule (33) to

$$I \approx Q_\mu[f] := \sum_{i=0}^{s-1} w_{i,1} f^{(i)}(a) + \sum_{i=0}^{s(r+1)-1} w_{i,2} f^{(i)}(\xi) + \sum_{i=0}^{s-1} w_{i,3} f^{(i)}(b), \tag{35}$$

with $w_{i,1} = \frac{\mu_k(a)}{k!}$, $w_{i,3} = -\frac{\mu_k(b)}{k!}$ and $w_{i,2} = \frac{-\mu_{k,1}(\xi)+\mu_{k,2}(\xi)}{k!}$. This rule has an absolute error of order $O(\omega^{-s-1/(r+1)})$, and a relative error of order $O(\omega^{-s})$.

Example 6.3. We consider the functions $f(x) = 1/(1 + x)$ and $g(x) = (x - 1/3)^2$ on $[0, 1]$. Since f is analytic, we could use the previous techniques. However, here we will only use the values of the first few derivatives of f at 0 and 1 and at the stationary point $\xi = 1/3$. The results are shown in Table 5 for varying degrees of interpolation. The convergence rate is limited to the convergence rate at the stationary point. According to Theorem 6.2, in order to obtain an error of order $O(\omega^{-s-1/(r+1)})$, we need to interpolate up to the derivative of order $m = s(r + 1) - 1$. Hence, solving the latter expression for s, we expect a convergence rate of $(m + 2)/(r + 1)$. The rate is actually higher in the columns with even m, due to the cancellation of the moments at ξ with odd index. For a more general function g there is no exact cancellation, but the difference of the moments at ξ, i.e., $\mu_{k,1}(\xi) - \mu_{k,2}(\xi)$, can have lower order than predicted by Theorem 6.2. This cancellation does not occur if the stationary point ξ is the endpoint of the integration interval.

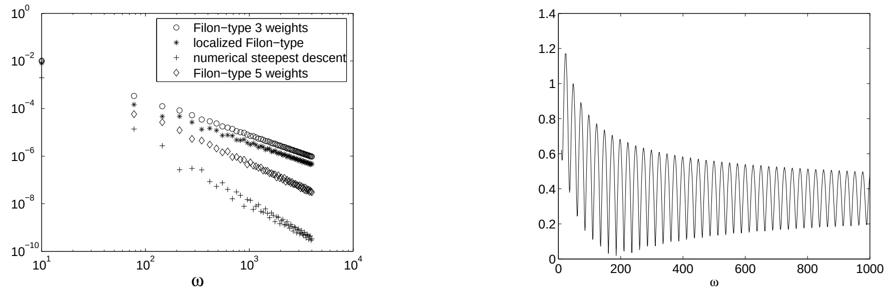
Table 5: Absolute error of the approximation of I for $f(x) = 1/(1+x)$ and $g(x) = (x-1/3)^2$ on $[0, 1]$. We approximate f by interpolating m derivatives. The last row shows the value of $\log_2(e_{1280}/e_{2560})$: this value should approximate $(m+2)/2$ for odd m, and $(m+3)/2$ for even m.

$\omega \setminus m$	0	1	2	3	4
160	$1.0E-4$	$1.8E-4$	$9.5E-7$	$9.7E-7$	$8.6E-9$
320	$6.5E-5$	$6.5E-5$	$1.7E-7$	$1.7E-7$	$7.6E-10$
640	$2.8E-5$	$2.3E-5$	$3.1E-8$	$3.0E-8$	$6.7E-11$
1280	$8.1E-6$	$8.2E-6$	$5.4E-9$	$5.4E-9$	$5.9E-12$
2560	$3.2E-6$	$2.9E-6$	$9.5E-10$	$9.5E-10$	$5.2E-13$
rate	1.4	1.5	2.5	2.5	3.5

Example 6.4. We make a numerical comparison between the regular Filon-type method, the localized Filon-type method and the numerical steepest descent method for $f(x) = 1/(1+x^2)$ and $g(x) = (x - 1/2)^2$ on $[-1, 1]$. Filon-type methods for this integral suffer from Runge's phenomenon: the interpolation error for the function f is large [19]. We choose $s = 1$, i.e., we use only function values of f in $\{-1, 1/2, 1\}$ and no derivatives. The order of the Filon-type methods is then $O(\omega^{-3/2})$. We choose $n = 1$ in Theorem 4.6. The order of the numerical steepest descent method is then $O(\omega^{-5/2})$, using 4 evaluations of f in the complex plane. We also interpolate two additional derivatives at 1/2 for the Filon-type method: this yields a quadrature rule with 5 weights, and order $O(\omega^{-2})$. The results are illustrated in Figure 3.

7 Generalization to a non-analytic function $g(x)$

If $g(x)$ is piecewise analytic, the integration interval can be split into subintervals where the function is analytic. Otherwise we can try to approximate $g(x)$ by an analytic function $\tilde{g}(x)$ on $[a, b]$. We should take care not to introduce new stationary points, and make sure that we



(a) Absolute error for four methods.

(b) The absolute error for numerical steepest descent, scaled by ω^3.

Figure 3: A numerical comparison between the regular and localized Filon-type methods, and the numerical steepest descent method (Example 6.4).

accurately approximate all stationary points of $g(x)$. Alternatively, we can approximate $g(x)$ locally around the special points, possibly by using different functions for each point. This will turn out to be easier and will yield the same convergence rate.

When $g(x)$ is smooth, it can be approximated arbitrarily well by an analytic function $\tilde{g}(x)$ on $[a, b]$. Hence, there exist analytic $\tilde{g}$ such that the integral

$$\tilde{I} := \int_a^b f(x) e^{i\omega\tilde{g}(x)}\, \mathrm{d}x = \tilde{F}(a) - \tilde{F}(b) \tag{36}$$

is arbitrarily close to the value of I. Such function $\tilde{g}$ may be difficult to find however and, if found, intractable for numerical purposes. Hermite interpolation in the points a and b is not a solution in this case, as the resulting polynomial may introduce stationary points that the original function g did not have. However, owing to decomposition (36), it becomes possible to do Hermite interpolation in a and b separately by different polynomials.

Theorem 7.1. *Assume that f and $\tilde{g}$ are analytic. Let $g_a(x)$ be the Hermite interpolating polynomial of degree s that satisfies*

$$g_a^{(k)}(a) = \tilde{g}^{(k)}(a), \quad k = 0, \ldots, s.$$

Then the approximation of $\tilde{F}(a)$ by $F_{g_a}(a)$ has an error of order $O(\omega^{-s-1})$.

Proof. The error $e := \tilde{F}(a) - F_{g_a}(a)$ can be written as

$$\begin{aligned} e &= \int_0^\infty f(h_a(p))(e^{i\omega\tilde{g}(h_a(p))} - e^{i\omega g_a(h_a(p))}) h_a'(p)\, \mathrm{d}p \qquad (37) \\ &= \int_0^\infty f(h_a(p)) e^{i\omega g_a(h_a(p))}(e^{i\omega(\tilde{g}(h_a(p)) - g_a(h_a(p)))} - 1) h_a'(p)\, \mathrm{d}p \\ &= \frac{e^{i\omega g_a(a)}}{\omega} \int_0^\infty f(h_a(q/\omega)) e^{-q}(e^{i\omega(\tilde{g}(h_a(q/\omega)) - g_a(h_a(q/\omega)))} - 1) h_a'(q/\omega)\, \mathrm{d}q \end{aligned}$$

The path $h_a(p)$ was chosen as the solution of (16) with respect to the approximation $g_a(x)$. Using a Taylor approximation around a, we have

$$\tilde{g}(x) - g_a(x) = (\tilde{g}^{(s+1)}(a) - g_a^{(s+1)}(a))\frac{(x-a)^{s+1}}{(s+1)!} + O((x-a)^{s+2}).$$

Because $h_a(q/\omega) - a \sim \omega^{-1}$, we have

$$e^{i\omega(\tilde{g}(h_a(q/\omega)) - g_a(h_a(q/\omega)))} - 1 \sim i\omega(\tilde{g}(h_a(q/\omega)) - g_a(h_a(q/\omega))) \sim \omega^{-s}.$$

The error e is therefore of order $O(\omega^{-s-1})$. □

The value of $\tilde{I}$, defined by (36), is completely determined by the derivatives of $\tilde{g}$ at a and b. If $\tilde{I} - I$ is small, it follows from Theorem 7.1 that $\tilde{g}$ should satisfy $\tilde{g}^{(j)}(a) = g^{(j)}(a)$ and $\tilde{g}^{(j)}(b) = g^{(j)}(b)$, $j = 0, \ldots, s$, for some maximal order s that depends on the smoothness of g. Hence, $\tilde{g}$ need not be explicitly constructed.

At a stationary point ξ, more derivatives are needed. The convergence rate depends on the minimal value of r for which $\tilde{g}^{(r+1)}(\xi) \neq 0$.

Theorem 7.2. *Assume that f and $\tilde{g}$ are analytic and that $\tilde{g}^{(k)}(\xi) = 0$, $k = 1, \ldots, r$, and $\tilde{g}^{(r+1)}(\xi) \neq 0$. Let $g_\xi(x)$ be the Hermite interpolating polynomial of degree $(s+1)(r+1) - 1$ that satisfies*

$$g_\xi^{(k)}(\xi) = \tilde{g}^{(k)}(\xi), \quad k = 0, \ldots, (s+1)(r+1) - 1.$$

Then the approximation of $\tilde{F}_j(\xi)$ by $F_{g_\xi,j}(\xi)$ has an error of order $O(\omega^{-s-1/(r+1)})$.

Proof. The proof follows the same lines as the proof of Theorem 7.1. The difference is that, similar to the situation in the proof of Theorem 6.2, we have $h_{\xi,j}(q/\omega) - \xi \sim \omega^{-1/(r+1)}$ and $h'_{\xi,j}(q/\omega) \sim \omega^{r/(r+1)}$. This leads to

$$e^{i\omega(\tilde{g}(h_{\xi,j}(q/\omega)) - g_\xi(h_{\xi,j}(q/\omega)))} - 1 \sim \omega^{-s}.$$

The error estimate for this case is analogous to (37) in the proof of Theorem 7.1. Adding all contributions, it is of order $O(\omega^{-1-s+r/(r+1)}) = O(\omega^{-s-1/(r+1)})$. □

Example 7.3. We illustrate the convergence with two examples. The function $g(x) = (x - 1/2)^2(x-2)e^{x^2}$ is approximated by a polynomial of degree m in the end points $a = 0$ and $b = 1$, and in the stationary point $\xi = 1/2$. The resulting errors are displayed in Tables 6 and 7. Table 6 shows the error in approximating only $\tilde{F}(a)$. Table 7 shows the error of the approximation of I. The latter error is dominated by the error made at the stationary points but follows the theory. As in the last example for a non-analytic function f, the convergence rate is actually higher for even m, because the difference of the terms at ξ in the decomposition of I can have lower order than predicted by Theorem 7.2. Note that it is not possible to approximate $g(x)$ by a fixed constant since in that case also $e^{i\omega g_a(x)} = e^{i\omega c}$ reduces to a constant. At a stationary point with r vanishing derivatives, the minimal number of derivatives to interpolate is $r + 1$.

Table 6: Absolute error of the approximation of $\tilde{F}(a)$ by $F_a(a)$ for $f(x) = 1/(1+x)$ and $g(x) = (x-1/2)^2(x-2)e^{x^2}$ at $a = 0$. We approximate g by interpolating m derivatives. The last row shows the value of $\log_2(e_{400}/e_{800})$: this value should approximate $m+1$.

$\omega \setminus m$	1	2	3	4
100	$6.1E-5$	$7.6E-7$	$1.3E-8$	$1.9E-10$
200	$1.5E-5$	$9.5E-8$	$8.4E-10$	$6.1E-12$
400	$3.8E-6$	$1.2E-8$	$5.3E-11$	$1.9E-13$
800	$9.6E-7$	$1.5E-9$	$3.3E-12$	$6.0E-15$
rate	2.0	3.0	4.0	5.0

Table 7: Absolute error of the approximation of I by $\tilde{I}$ for $f(x) = 1/(1+x)$ and $g(x) = (x-1/2)^2(x-2)e^{x^2}$ on $[0,1]$. We approximate g by interpolating m derivatives. The last row shows the value of $\log_2(e_{400}/e_{800})$: this value should approximate $m/2$ for odd m, and $(m+1)/2$ for even m.

$\omega \setminus m$	2	3	4
100	$1.6E-4$	$2.7E-4$	$1.8E-7$
200	$5.5E-5$	$9.8E-6$	$3.2E-8$
400	$2.0E-5$	$3.5E-6$	$5.6E-9$
800	$6.9E-6$	$1.2E-6$	$9.9E-10$
rate	1.5	1.5	2.5

8 Concluding remarks

We have presented an approach to compute highly oscillatory integrals of the form (1). The method is quite general, and leads to high order convergence when the frequency increases. The (generalized) Gauss-Laguerre quadrature rules yield the typical Gauss rule convergence exponent of around $2n$, but here as a function of $1/\omega$. This is made possible by transforming the integrand into a numerically well behaved one, i.e., one that is not oscillatory and that has exponential decay which becomes faster with increasing ω.

The approach by Iserles and Nørsett has led us to consider the use of Hermite interpolation for functions $f(x)$ that are not analytic. The resulting polynomial is analytic, and this enables the use of our rapidly converging complex approach. Owing to our decomposition of the integral into a sum of a number of functions that each depend only on one point, this approach could be simplified considerably in our setting. Vice versa, the methods developed in this paper may be used to compute generalized moments of the form $\int_0^1 p(x)e^{i\omega g(x)}\mathrm{d}x$, with $p(x)$ a polynomial of low degree. Such moments are assumed to be available in the approach of Iserles and Nørsett, but an analytical value may not always be available. The details of the latter method can be found in [15].

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