C^1 Hierarchical Riesz Bases of Lagrange Type on Powell–Sabin Triangulations

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Report TW 422, March 2005

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In this paper we construct C^1 continuous piecewise quadratic hierarchical bases on Powell–Sabin triangulations of arbitrary polygonal domains in $\mathbb{R}^2$. Our bases are of Lagrange type instead of the usual Hermite type and we prove that they form strongly stable Riesz bases for the Sobolev spaces $H^s(\Omega)$ with $s \in (1, \frac{5}{2})$. Especially the case $s = 2$ is of interest, because we can use the corresponding hierarchical basis for preconditioning fourth order elliptic equations leading to uniformly well-conditioned stiffness matrices. Compared to the hierarchical Riesz bases by Davydov and Stevenson our construction is simpler.

Keywords : Hierarchical bases, splines, C^1 finite elements, elliptic equations
AMS(MOS) Classification : 41A15, 65D07, 65N30, 65T60

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1 Introduction

The objective of this paper is the construction of a hierarchical Riesz basis on an arbitrary polygonal domain in $\mathbb{R}^2$ that is suitable for preconditioning large linear systems arising from fourth order elliptic boundary value problems. This idea, to use hierarchical bases as preconditioners for stiffness matrices, was first introduced by Yserentant in [21] for the solution of second order elliptic equations. In his paper, Yserentant constructs C^0 hierarchical Riesz bases for the Sobolev spaces $H^s(\Omega)$ with $s \in (1, \frac{3}{2})$, with a suboptimal stability result for the case $s = 1$, leading to logarithmically growing condition numbers of stiffness matrices related to second order elliptic equations. Later, C^1 hierarchical bases for fourth order elliptic equations were constructed in [5, 16]. These bases are Riesz bases for the Sobolev spaces $H^s(\Omega)$ with $s \in (2, \frac{5}{2})$, again leading to logarithmically growing condition numbers of stiffness matrices related to fourth order elliptic equations.

To enlarge the range of stability, wavelet bases can be used, which are in fact stable completions [3] of hierarchical bases. We speak about wavelets when the basis functions at different resolution levels have at least one vanishing moment. Several wavelet constructions on polygonal domains generate Riesz bases on $H^s(\Omega)$ for s in a range around zero, see e.g. [7, 18]. However, for most constructions at least a subset of the wavelets is only continuous, so that the range of stability is restricted to $s < \frac{3}{2}$. The only available wavelet Riesz basis on general polygons for $s \geq \frac{3}{2}$ is the basis constructed in [6].

The previously mentioned C^1 hierarchical bases are constructed using classical C^1 Hermite type finite elements, which restricts the range of stability to values $s > 2$, since, by the Sobolev embedding theorem, the corresponding Hermite interpolant operator is only well defined for functions in $H^s(\Omega)$, $s > 2$. With this in mind, the construction of hierarchical bases based on C^1 finite elements of Lagrange type instead of Hermite type should allow to enlarge the range of stability from $s \in (2, \frac{5}{2})$ to $s \in (1, \frac{5}{2})$. Recently Davydov and Stevenson [8] constructed such hierarchical Riesz bases for $H^s(\Omega), s \in (1, \frac{5}{2})$, based on a Lagrange type interpolation operator for C^1 piecewise cubic polynomials on certain triangulations obtained from checkerboard quadrangulations.

In this paper we construct hierarchical Riesz bases for C^1 piecewise quadratic polynomials on Powell–Sabin (PS) triangulations. A previous construction [12] in this setting, based on a Hermite interpolation operator, yields Riesz bases for $H^s(\Omega), s \in (2, \frac{5}{2})$, which is similar to the results obtained in [5, 16]. Inspired by the work of Davydov and Stevenson we look how we can extend the range of stability from $s \in (2, \frac{5}{2})$ to $s \in (1, \frac{5}{2})$ by using a Lagrange interpolation operator. While the construction of Davydov and Stevenson is rather technical, especially near the domain boundary, our construction is easier and straightforward.

The rest of the paper we will be using the Sobolev spaces $H_0^s(\Omega)$ instead of $H^s(\Omega)$, because, for certain values of s, $\|\cdot\|_{H_0^s(\Omega)}$ appears as the natural energy norm for elliptic boundary value problems. Recall that the Sobolev spaces $H_0^s(\Omega)$ are defined as the closure of the C^∞ continuous functions that vanish on $\partial\Omega$ in $H^s(\Omega)$, and we write $\mathcal{H}^s(\Omega) = H_0^s(\Omega)$. See [1] for a good reference work concerning Sobolev spaces.

In Section 2 we introduce nested spline spaces of C^1 piecewise quadratics. We briefly discuss the construction of a Hermite basis and we review the triadic subdivision scheme that is used to created the nested spline spaces. In Section 3 we construct a Lagrange basis for the space of Powell–Sabin splines with homogeneous boundary conditions, and we prove that this basis is stable and local. Section 4 is devoted to the construction of a $H^s(\Omega)$ stable hierarchical basis for $s \in (1, \frac{5}{2})$, and in Section 5 we use this hierarchical basis to precondition stiffness matrices related to fourth order elliptic boundary value problems.

We close this introduction with a short note about notation. Sometimes we use the notation $A \sim B$ which means that there exist constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$. Similarly $A \lesssim B$ expresses that there exists a constant c_1 such that $A \leq c_1 B$.

2 Nested subspaces of C^1 piecewise quadratics

Let Ω be a domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Suppose we have a conforming triangulation Δ of Ω, constituted of triangles T_j, $j = 1, \ldots, t$ and vertices V_k, $k = 1, \ldots, N$. Then Δ^{PS} is a Powell–Sabin refinement of Δ which divides each triangle T_j into six smaller triangles with a common vertex Z_j as follows, see Figure 1(a):

(a) Choose an interior point Z_j for each triangle T_j, so that if two triangles T_i and T_j have a common edge, the line joining Z_i and Z_j intersects this common edge at a point R_{ij} between its vertices. A good choice for Z_j is the incenter of triangle T_j.

(b) Join the points Z_j to the vertices of T_j.

(c) For each edge of T_j

- which belongs to the boundary $\partial\Omega$, join Z_j to the middle point of the edge.
- which is common to a triangle T_i, join Z_j to R_{ij}.

Now we consider the space of C^1 piecewise quadratic polynomials on Δ^{PS}, the Powell–Sabin splines,

$$S_2^1(\Delta^{PS}) = \{s \in C^1(\overline{\Omega}) \mid s|_T \in \mathcal{P}_2 \text{ for all } T \in \Delta^{PS}\},$$

where $\mathcal{P}_2$ is the space of all bivariate polynomials of total degree at most 2. Each of the $6t$ triangles resulting from the PS-refinement becomes the domain triangle of a quadratic bivariate Bézier polynomial [10]. Powell and Sabin [17] proved that the interpolation problem

$$s(V_k) = f(V_k), \quad D_x s(V_k) = D_x f(V_k), \quad D_y s(V_k) = D_y f(V_k), \quad \forall\, V_k \in \Delta \tag{1}$$

has a unique solution. So, given the function and derivative values at each vertex V_k of Δ, the spline is uniquely defined. Hence the spline space $S_2^1(\Delta^{PS})$ has dimension $3N$.

Dierckx [9] presented a normalized B-spline representation for piecewise polynomials $s(x, y) \in S_2^1(\Delta^{PS})$ based on Hermite interpolation

$$s(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y), \qquad (x, y) \in \Omega \tag{2}$$

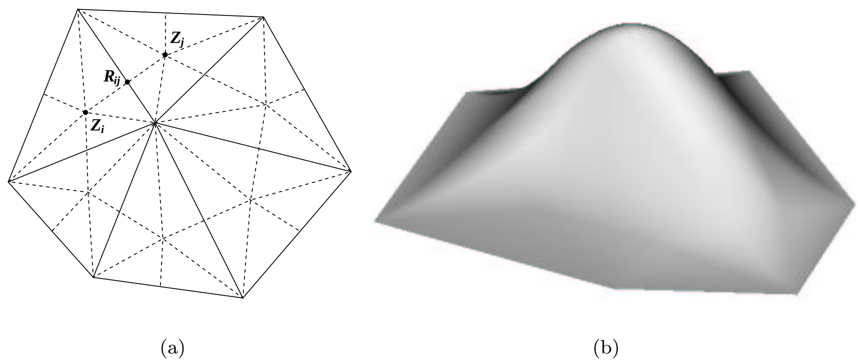


Figure 1: (a) A PS refinement Δ^{PS} of Δ. (b) A B-spline basis function.

where the basis functions form a convex partition of unity

$$B_{ij}(x,y) \geq 0,$$
$$\sum_{i=1}^{N} \sum_{j=1}^{3} B_{ij}(x,y) = 1,$$

and have local support: B_{ij} vanishes outside the union of all triangles $T \in \Delta$ containing V_i. One such basis function is depicted in Figure 1(b). Each basis function B_{ij} is the unique solution of the interpolation problem (1) with

$$B_{ij}(V_k) = \delta_{ik}\alpha_{ij}, \quad D_x B_{ij}(V_k) = \delta_{ik}\beta_{ij}, \quad D_y B_{ij}(V_k) = \delta_{ik}\gamma_{ij}, \quad \forall\ V_k \in \Delta \tag{3}$$

where δ_{ij} is the Kronecker delta and $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, $j = 1, 2, 3$ are three linearly independent triplets of real numbers. The values of these real numbers are determined from the algorithm described in [9].

In [20], Vanraes *et al.* present a subdivision scheme to compute a representation (2) of a PS-spline on a triadic refinement Δ_{l+1} of Δ_l. The subscript l denotes the resolution level. Figure 2 explains the principle of triadic refinement. We place two new vertices on every edge of the current triangulation, each at one side of the intersection with the PS-refinement, and one new vertex is placed on the position of every interior point Z_j in the PS-refinement. This triadic refinement yields nested sequences

$$\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots$$
$$\Delta_0^{PS} \subset \Delta_1^{PS} \subset \Delta_2^{PS} \subset \cdots$$

that are "regular", which means that the minimum angle of any triangle in any Δ_l remains bounded away from zero and that

$$3^{-l} \lesssim \min_{T \in \Delta_l} |T| \leq \max_{T \in \Delta_l} |T| \lesssim 3^{-l}, \qquad l \in \mathbb{N}_0, \tag{4}$$

where $|T|$ is the diameter of triangle T. The same is valid for the triangles in the PS-refinement

$$3^{-l} \lesssim \min_{T \in \Delta_l^{PS}} |T| \leq \max_{T \in \Delta_l^{PS}} |T| \lesssim 3^{-l}, \qquad l \in \mathbb{N}_0.$$

With each triangulation Δ_l we have a corresponding B-spline basis $\{B_{ij,l}\}_{i=1,j=1,2,3}^{N_l}$ for the space $S_2^1(\Delta_l^{PS})$.

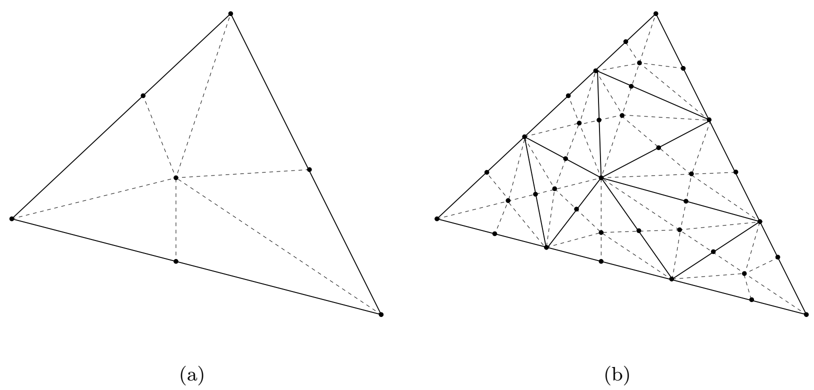


Figure 2: Principle of triadic refinement. We place a new vertex at the position of the interior point in the PS-refinement and two new vertices on the edges each at one side of the intersection with the PS-refinement.

From (2) and (3) we have

$$\begin{bmatrix} s_l(V_i) \\ D_x s_l(V_i) \\ D_y s_l(V_i) \end{bmatrix} = \mathbf{A}_{i,l} \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix} \quad \text{with} \quad \mathbf{A}_{i,l} = \begin{bmatrix} \alpha_{i1,l} & \alpha_{i2,l} & \alpha_{i3,l} \\ \beta_{i1,l} & \beta_{i2,l} & \beta_{i3,l} \\ \gamma_{i1,l} & \gamma_{i2,l} & \gamma_{i3,l} \end{bmatrix}, \ i = 1, \dots, N_l.$$

This gives rise to quasi-interpolant operators $Q_l : C^1(\overline{\Omega}) \to S_2^1(\Delta_l^{PS})$ given by

$$Q_l f(x,y) = \sum_{i=1}^{N_l} \sum_{j=1}^{3} \mu_{ij,l}(f) B_{ij,l}(x,y),$$

where the $\mu_{ij,l}$ are linear functionals of the form

$$\begin{bmatrix} \mu_{i1,l}(f) \\ \mu_{i2,l}(f) \\ \mu_{i3,l}(f) \end{bmatrix} := (\mathbf{A}_{i,l})^{-1} \begin{bmatrix} f(V_i) \\ D_x f(V_i) \\ D_y f(V_i) \end{bmatrix}.$$

It is proven in [13] that these linear functionals can be rewritten as

$$\mu_{ij,l}(f) = f(V_i) + \eta_{ij,l} D_x f(V_i) + \widetilde{\eta}_{ij,l} D_y f(V_i) \tag{5}$$

where the $\eta_{ij,l}$ and $\widetilde{\eta}_{ij,l}$ satisfy

$$|\eta_{ij,l}| \lesssim 3^{-l}, \qquad |\widetilde{\eta}_{ij,l}| \lesssim 3^{-l}. \tag{6}$$

Clearly the operator Q_l satisfies

$$Q_l s_l = s_l, \qquad \forall s_l \in S_2^1(\Delta_l^{PS}).$$

Furthermore we know from the work in [14] that the B-spline functions form a stable basis for the L_∞ norm, i.e.,

$$\left\| \sum_{i=1}^{N_l} \sum_{j=1}^{3} c_{ij} B_{ij,l}(x,y) \right\|_{L_\infty(\Omega)} \sim \|\mathbf{c}\|_\infty. \tag{7}$$

The following lemma is now a trivial result.

Lemma 2.1. *Let T be a triangle in Δ_l with vertices V_1, V_2, V_3. Then for every spline $s_l \in S_2^1(\Delta_l^{PS})$ we have*

$$\|s_l\|_{L_\infty(T)} \lesssim \max\left\{|s_l(V_i)|, 3^{-l}|D_x s_l(V_i)|, 3^{-l}|D_y s_l(V_i)|, i = 1,2,3\right\}.$$

Proof. By construction of Q_l we have

$$\|s_l\|_{L_\infty(T)} = \|Q_l s_l\|_{L_\infty(T)} = \left\|\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_{ij,l}(s_l)B_{ij,l}\right\|_{L_\infty(T)}.$$

Then, by (7), $\|s_l\|_{L_\infty(T)} \lesssim \max_{i,j}|\mu_{ij,l}(s_l)|$, and the lemma follows immediately from (5) and (6). $\square$

3 A stable local Lagrange-type basis for C^1 quadratics

Let Ω be a domain in $\mathbb{R}^2$ with polygonal boundary for which there exists a conforming triangulation Δ_0 such that

(a) $\Omega = \bigcup_{T\in\Delta_0} T$,

(b) the intersection of any two different triangles in Δ_0 is either empty or a common edge or vertex,

(c) all interior vertices have degree six,

(d) all triangles in Δ_0 can be colored black and white in such a way that every interior vertex in Δ_0 belongs to exactly one black triangle.

Such a triangulation can be constructed for any domain with a polygonal boundary. First we create a so-called checkerboard quadrangulation [15] for Ω. This is a quadrangulation consisiting of quadrilaterals with largest interior angle less than π, and the quadrilaterals can be colored black and white in such a way that any two quadrilaterals sharing an edge have the opposite color. If we allow triangles at the boundary, see Figure 3, then such a checkerboard triangulation can always be constructed for a polygonal domain, see [8] for a proof. Note that all interior vertices in the checkerboard quadrangulation are of degree four. Next we construct a triangulation by adding the north-west south-east diagonal to each quadrilateral in the quadrangulation, see Figure 4. Note that all interior vertices have degree six. A suitable coloring can be obtained easily by applying a regular coloring pattern since, by construction, our triangulation has a "regular" pattern, i.e. looking closely at Figure 4 we see a distorted uniform triangulation. So construct a regular coloring pattern satisfying (d) for a uniform triangulation and apply that pattern to the constructed triangulation. Let Δ_1 be the triadic refinement of Δ_0, see Figure 2. It is trivially checked that Δ_1 still satisfies (a)–(d). By induction, each triangulation Δ_l satisfies the properties (a)–(d). We define $\Delta_l^\bullet$ as the subset of Δ_l containing the black triangles, and Δ_l° as $\Delta_l \setminus \Delta_l^\bullet$.

We are interested in the multiscale spaces $S_l \subset S_2^1(\Delta_l^{PS})$ satisfying homogeneous boundary conditions, i.e.

$$S_l = \left\{s \in C^1(\overline{\Omega}) \mid s|_T \in \mathcal{P}_2 \text{ for all } T \in \Delta_l^{PS}, \text{ and } s = D_x s = D_y s = 0 \text{ on } \partial\Omega\right\}.$$

The spaces S_l are nested, i.e.

$$S_0 \subset S_1 \subset \cdots \subset S_l \subset \cdots.$$

The remainder of this section is devoted to the construction of a suitable Lagrange basis for the spaces S_l.

For each interior vertex $V_i \in \Delta_l$ we have a corresponding triangle $T_{V_i} \in \Delta_l^\bullet$. Define $D_l(V_i)$ as the set $\{V_i, V_{ij}, V_{ik}\}$, with V_{ij} and V_{ik} chosen such that the triangle $T(V_i, V_{ij}, V_{ik})$ belongs to the triadic refinement of triangle $T_{V_i} \in \Delta_l^\bullet$, see Figure 5. Define Ξ_l as the union of all sets $D_l(V_i)$, i.e.

$$\Xi_l := \bigcup_{V_i\in\Delta_l\setminus\partial\Omega} D_l(V_i).$$

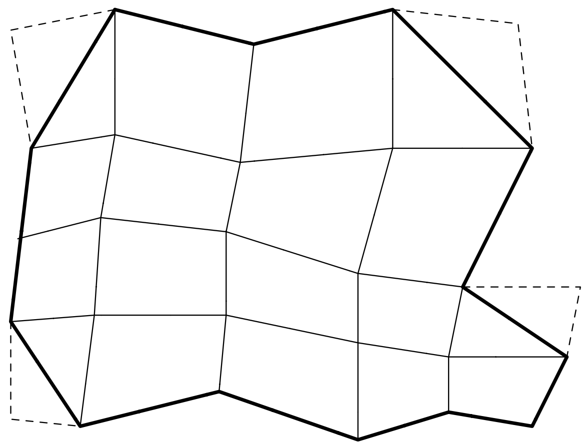


Figure 3: Checkerboard quadrangulation without coloring.

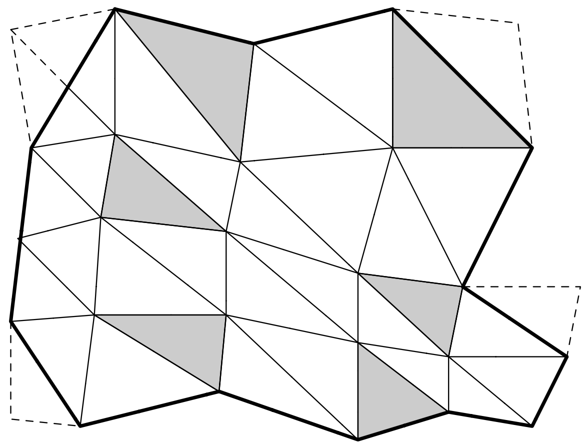


Figure 4: Construction of a suitable triangulation.

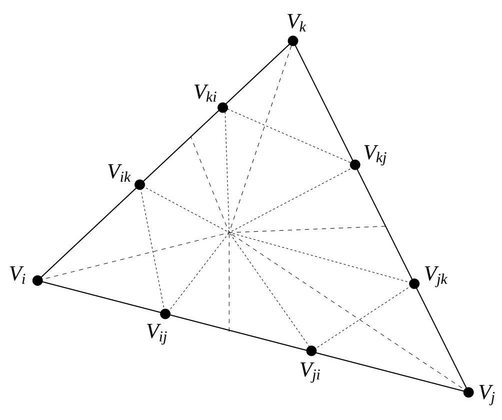


Figure 5: With each interior vertex $V_i \in \Delta_l$ we associate the set $D_l(V_i) := \{V_i, V_{ij}, V_{ik}\}$. Note that the figure also depicts the sets $D_l(V_j) := \{V_j, V_{jk}, V_{ji}\}$ and $D_l(V_k) := \{V_k, V_{ki}, V_{kj}\}$ (assuming that V_j and V_k are interior vertices) .

Theorem 3.1. *The set Ξ_l is a Lagrange interpolation set for S_l, $l \geq 0$, i.e., for any arbitrary function f in $C^0(\Omega)$ there exists a unique spline $s \in S_l$ such that*

$$s(\xi) = f(\xi), \qquad \xi \in \Xi_l.$$

Moreover,

$$\|s\|_{L_\infty(T)} \lesssim \max\left\{|f(\xi)| \mid \xi \in \Xi_l \cap \operatorname{star}(T)\right\}, \qquad T \in \Delta_l, \tag{8}$$

with star(T) *the union of all triangles in Δ_l having at least one vertex in common with T.*

Proof. From the classical work of Powell and Sabin [17] we know that the dimension of the spline space $S_2^1(\Delta_l^{PS})$ equals $3N_l$ with N_l the number of vertices in Δ_l. Let $N_l = N_l^i \cup N_l^b$ with N_l^i the number of interior vertices and with N_l^b the number of boundary vertices. The spline spaces S_l satisfy homogeneous boundary conditions, therefore we find that $\dim S_l = 3N_l^i = \#\Xi_l$. Proving that the set Ξ_l is a Lagrange interpolation set is now equivalent to the statement that $s(\xi) = 0$, $\xi \in \Xi_l$, implies $s = 0$. Therefore, to prove the theorem, it is sufficient to prove (8). We will consider different cases.

Case 1. Suppose that $T(V_1, V_2, V_3) \in \Delta_l^\bullet$ and that $T \cap \partial\Omega = \emptyset$. Then, by construction, we know that $\#(T \cap \Xi_l) = 9$, see Figure 5. In Bernstein–Bézier notation the spline s has a unique representation in function of the 19 Bézier ordinates as depicted in Figure 6. The C^1 continuity conditions in combination with the Lagrange interpolation problem give rise to a system of the form $Ac = b$ that has to be solved for the unknown Bézier ordinates in the vector c. The entries of the vector b corresponding to a continuity constraint are zero, and the entries of b corresponding to an interpolation constraint contain values that can be bounded by a constant times $\max\{|f(\xi)| \mid \xi \in \Xi_l \cap T\}$. It is easy to show that the determinant of the matrix A is nonzero and depends at most on the smallest angle in the triangulation Δ_l. It follows that $\|c\|_\infty \leq c_1\|b\|_\infty$. Indeed, we can take $c_1 := \|A^{-1}\|_\infty$. This implies that

$$\|c\|_\infty \lesssim \max\{|f(\xi)| \mid \xi \in \Xi_l \cap T\}.$$

We immediately deduce that

$$\|s\|_{L_\infty(T)} \lesssim \max\{|f(\xi)| \mid \xi \in \Xi_l \cap T\},$$

because the Bernstein–Bézier polynomials form a partition of unity, i.e. $\sum_{i+j+k=d} B_{ijk}^d = 1$ and $B_{ijk}^d \geq 0$.

Case 2. Suppose that $T(V_1, V_2, V_3) \in \Delta_l^\bullet$ and that $T \cap \partial\Omega = V_1$. Then we have $\#(T \cap \Xi_l) = 6$. The spline space $S_2^1(T)$ has dimension 9. However, because S_l has homogeneous boundary conditions we can add the constraints $s(V_1) = D_x s(V_1) = D_y s(V_1) = 0$. The same reasoning as in the previous case can now be applied.

Case 3. Suppose that $T(V_1, V_2, V_3) \in \Delta_l^\bullet$ and that $T \cap \partial\Omega = [V_1, V_2]$. Then $\#(T \cap \Xi_l) = 3$, and we add the constraints $s(V_1) = D_x s(V_1) = D_y s(V_1) = 0$ and $s(V_2) = D_x s(V_2) = D_y s(V_2) = 0$. The rest is similar to Case 2.

Case 4. Suppose that $T(V_1, V_2, V_3) \in \Delta_l^\circ$. Then, by Lemma 2.1,

$$\|s\|_{L_\infty(T)} \lesssim \max\left\{|s(V_i)|, 3^{-l}|D_x s(V_i)|, 3^{-l}|D_y s(V_i)|\right\},$$

for some vertex V_i, $i = 1, 2, 3$. Because of the homogeneous boundary conditions we find that this vertex V_i is an interior vertex (unless all three vertices V_1, V_2, V_3 are boundary vertices, but that case is trivial). So there exists a triangle $\widetilde{T} \in \Delta_l^\bullet \cap \operatorname{star}(T)$ such that $V_i \in \widetilde{T}$. Now we make use of the Markov inequality for polynomials (see e.g. [11]) and the regularity (4) of the triangulation, and we deduce that $|D_x s(V_i)| \lesssim 3^l \|s\|_{L_\infty(\widetilde{T})}$. A similar result holds for $|D_y s(V_i)|$ and we find $\|s\|_{L_\infty(T)} \lesssim \|s\|_{L_\infty(\widetilde{T})}$. The previous cases now yield

$$\|s\|_{L_\infty(T)} \lesssim \max\{|f(\xi)| \mid \xi \in \Xi_l \cap \operatorname{star}(T)\}.$$

□

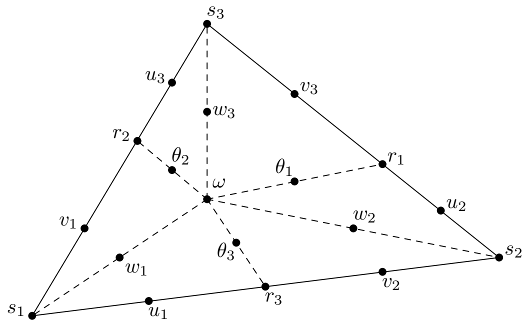


Figure 6: The Powell–Sabin macro-element.

We now define the Lagrange basis functions $B_{\xi,l}$, $\xi \in \Xi_l$, for the space S_l. They are the unique solutions of the interpolation problem

$$B_{\xi,l}(\zeta) = \delta_{\xi,\zeta}, \quad \forall \xi, \zeta \in \Xi_l,$$

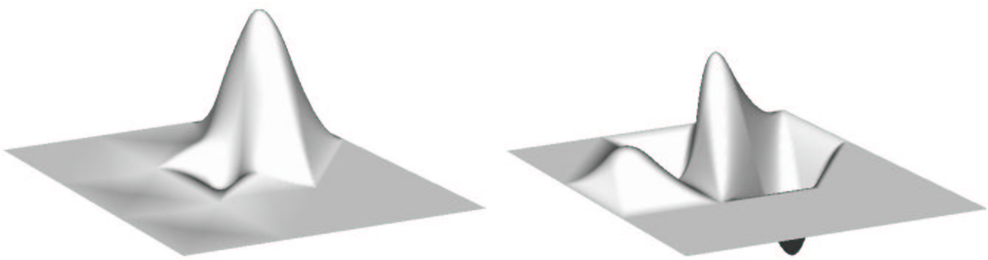
with $\delta_{\xi,\zeta}$ the Kronecker delta. By Theorem 3.1, this interpolation problem is well defined and the basis functions $B_{\xi,l}$ satisfy

$$\operatorname{supp} B_{\xi,l} \subset \operatorname{star}(T), \qquad \xi \in T \subset \Delta_l^{\bullet},$$

and

$$\|B_{\xi,l}\|_{L_\infty(\Omega)} \sim 1.$$

Figure 7 depicts the typical forms of these Lagrange basis functions.



(a) (b)

Figure 7: (a) Basis function $B_{\xi,l}$ with ξ corresponding to a vertex in Δ_l. (b) Basis function $B_{\xi,l}$ with ξ corresponding to a vertex in $\Delta_{l+1} \setminus \Delta_l$.

Theorem 3.2. *If $s_l = \sum_{\xi \in \Xi_l} c_\xi B_{\xi,l}$ is in S_l, then for any $0 < p \leq \infty$ we have*

$$\|s_l\|_{L_p(\Omega)} \sim \left(\sum_{\xi \in \Xi_l} |c_\xi|^p \, 3^{-2l} \right)^{1/p}.$$

Proof. From the properties above we infer immediately that

$$|c_\xi| \lesssim \|s_l\|_{L_\infty(T_\xi)}$$

with $T_\xi \in \Delta_l^{PS}$ such that $\xi \cap T_\xi \neq \emptyset$. By mapping T_ξ to the standard simplex

$$T_s := \{(x,y) \mid 0 \leq x, y \leq 1,\ x+y \leq 1\},$$

and using the fact that all norms on the finite dimensional space of polynomials are equivalent, it is easy to see that

$$\|s_l\|_{L_\infty(T_\xi)} \lesssim 3^{2l/p} \|s_l\|_{L_p(T_\xi)}$$

which implies

$$\left(\sum_{\xi\in\Xi_l} |c_\xi|^p \, 3^{-2l}\right)^{1/p} \lesssim \left(\sum_{\xi\in\Xi_l} \|s_l\|^p_{L_p(T_\xi)}\right)^{1/p} \lesssim \|s_l\|_{L_p(\Omega)}.$$

The other inequality follows from the observation that

$$\left|\sum_{\xi\in\Xi_l} c_\xi B_{\xi,l}(x,y)\right|^p \lesssim \sum_{\xi\in\Xi_l} |c_\xi|^p \, |B_{\xi,l}(x,y)|^p,$$

which holds because at any $(x,y) \in \Omega$ there are at most 27 B-splines nonzero. We find that

$$\begin{aligned}
\|s_l\|^p_{L_p(\Omega)} &= \int_\Omega \left|\sum_{\xi\in\Xi_l} c_\xi B_{\xi,l}(x,y)\right|^p dxdy \\
&\lesssim \sum_{\xi\in\Xi_l} |c_\xi|^p \int_\Omega |B_{\xi,l}(x,y)|^p \, dxdy \\
&\lesssim \sum_{\xi\in\Xi_l} |c_\xi|^p \, 3^{-2l}.
\end{aligned}$$

□

4 C^1 hierarchical Riesz bases for $\mathcal{H}^s(\Omega)$, $s \in (1, \frac{5}{2})$

Let us introduce the interpolant operator $I_l : C^0(\Omega) \to S_l$ that is defined as

$$I_l f := \sum_{\xi\in\Xi_l} f(\xi) B_{\xi,l}. \tag{9}$$

It is obvious that this operator returns the unique spline function in S_l that interpolates the given function f at the positions of the $\xi \in \Xi_l$. We show that this operator is suitable for constructing a multiresolution analysis. A multiresolution analysis consists of

(a) A Banach space $\mathcal{B}$ of functions defined on a bounded subset $\Omega \subset \mathbb{R}^2$ with associated norm $\|\cdot\|_\mathcal{B}$.

(b) A nested sequence of subspaces $S_0 \subset S_1 \subset S_2 \subset \cdots \subset \mathcal{B}$ that are dense in $\mathcal{B}$,

$$\overline{\bigcup_{l=0}^{\infty} S_l} = \mathcal{B}.$$

(c) A collection of uniformly bounded operators

$$I_l : \mathcal{B} \to S_l$$

with the properties

$$\begin{aligned}
I_l I_l &= I_l, \\
I_l I_{l+1} &= I_l, \\
I_l(\mathcal{B}) &= S_l
\end{aligned}$$

for all integers $l \geq 0$.

As Banach space $\mathcal{B}$ we take $C_0^0(\Omega)$, the space of C^0 continuous functions that vanish on $\partial\Omega$, and its associated norm is the L_∞ norm. By Theorem 3.2 for $p = \infty$ and (9), we get $\|I_l f\|_{L_\infty(\Omega)} \lesssim \|f\|_{L_\infty(\Omega)}$, so the operator I_l is uniformly bounded in $C_0^0(\Omega)$. Proving that the spaces S_l are dense in $C_0^0(\Omega)$ is equivalent to proving that $\lim_{l\to\infty} \|f - I_l f\|_{L_\infty(\Omega)} = 0$. Let (x, y) be an arbitrary point in triangle $T \in \Delta_l^\bullet$ with $T \cap \partial\Omega = \emptyset$. Then

$$\begin{aligned} |f(x,y) - I_l f(x,y)| &= |f(x,y) \sum_{\xi\in\Xi_l} B_{\xi,l}(x,y) - \sum_{\xi\in\Xi_l} f(\xi) B_{\xi,l}(x,y)| \\ &= |\sum_{\xi\in\Xi_l} (f(x,y) - f(\xi)) B_{\xi,l}(x,y)|, \end{aligned}$$

and, provided that $B_{\xi,l}(x,y) \neq 0$, $f(x,y) - f(\xi)$ goes to zero as $l \to \infty$ because f is continuous. Similar arguments hold when $T \cap \partial\Omega \neq \emptyset$ or when $T \in \Delta_l^\circ$, so the spaces S_l are dense in $C_0^0(\Omega)$. The last property, $I_l I_{l+1} = I_l$, follows immediately from the fact that the sets Ξ_l are nested, i.e.

$$\Xi_0 \subset \Xi_1 \subset \cdots \subset \Xi_l \subset \cdots .$$

Since for $l \geq 0$, the sets $\{B_{l,\xi} \mid \xi \in \Xi_l \setminus \Xi_{l-1}\}$ are L_∞-stable bases for the spaces $\mathrm{Im}(I_l - I_{l-1})$, we find that $\bigcup_{l=0}^\infty \{B_{l,\xi} \mid \xi \in \Xi_l \setminus \Xi_{l-1}\}$ is a weakly L_∞-stable hierarchical basis for $C_0^0(\Omega)$. The remainder of this section is devoted to the proof of the fact that

$$\bigcup_{l=0}^{\infty} \left\{ 3^{(1-s)l} B_{\xi,l} \mid \xi \in \Xi_l \setminus \Xi_{l-1} \right\}$$

is a strongly stable Riesz basis for $\mathcal{H}^s(\Omega)$ for any $s \in (1, \frac{5}{2})$. The operators I_l will be essential tools. Note that, by the Sobolev embedding theorem, I_l is only well defined on $\mathcal{H}^s(\Omega)$ given that $s > 1$.

Lemma 4.1. *For each $f \in \mathcal{H}^s(\Omega)$, $1 < s$, and arbitrary $p > 0$ we have that*

$$\|f - I_l f\|_{L_p(\Omega)} \to 0 \quad as \quad l \to \infty.$$

Proof. Suppose that $1 < s < 2$. Let (x, y) be some arbitrary point in triangle $T \in \Delta_l$. A simple calculation yields

$$|f(x,y) - I_l f(x,y)| \lesssim \|f\|_{L_\infty(T)} .$$

Then the well-known Bramble–Hilbert lemma [2] and Theorem 7.58 in [1] imply

$$|f(x,y) - I_l f(x,y)| \lesssim 3^{-l(s-1)} |f|_{\mathcal{H}^s(T)} .$$

Using (4), we get

$$\begin{aligned} \|f - I_l f\|_{L_p(T)} &= \left(\int_T |f(x,y) - I_l f(x,y)|^p \, dxdy \right)^{1/p} \\ &\lesssim 3^{-l(s-1)} |f|_{\mathcal{H}^s(T)} 3^{-2l/p}, \end{aligned}$$

and some elementary calculations yield

$$\|f - I_l f\|_{L_p(\Omega)} \lesssim 3^{-l(s-1+2/p)} |f|_{\mathcal{H}^s(\Omega)} .$$

The case $s = 2$ can be proven using the same techniques, and the case $s > 2$ follows from the fact that $\mathcal{H}^s(\Omega) \subset \mathcal{H}^{s-1}(\Omega)$. □

From Lemma 4.1 we know that each function $f \in \mathcal{H}^s(\Omega)$, $s > 1$, can be decomposed as

$$f = \sum_{l=0}^{\infty} g_l, \qquad g_l \in S_l,$$

in the sense of L_p. Moreover, we can use the decomposition

$$f = \sum_{l=0}^{\infty}(I_l - I_{l-1})f,$$

with $I_{-1} := 0$.

We now introduce auxiliary spaces $A^s(\Omega)$. A function $f \in L_2(\Omega)$ belongs to $A^s(\Omega)$ for some fixed $s \geq 0$ if there exists a sequence $g_l \in S_l$, $l = 0, 1, \ldots$ such that $f = \sum_{l=0}^{\infty} g_l$ in the sense of L_2, and $\|\{3^{ls}\|g_l\|_{L_2(\Omega)}\}\|_{l_2} < \infty$. The norm on $A^s(\Omega)$ is defined as

$$\|f\|_{A^s(\Omega)} = \inf \left(\sum_{l=0}^{\infty} \left[3^{ls}\|g_l\|_{L_2(\Omega)}\right]^2 \right)^{1/2}$$

where the infimum must be taken over all admissible representations $\sum_{l=0}^{\infty} g_l$ of f. In order to work with the abstract $A^s(\Omega)$-spaces, we relate them to the more convenient function spaces of Sobolev type. The following fact can be extracted from the results in [16].

Proposition 4.2. *Suppose the nested spaces $\{S_l\}_{l=0}^{\infty}$ satisfy the Jackson estimates*

$$\inf_{g \in S_l} \|f - g\|_{L_2(\Omega)} \lesssim 3^{-3l}\|f\|_{\mathcal{H}^3(\Omega)}, \tag{10}$$

for all $f \in \mathcal{H}^3(\Omega)$, as well as the Bernstein estimate

$$\|g\|_{\mathcal{H}^s(\Omega)} \lesssim 3^{ls}\|g\|_{L_2(\Omega)}, \qquad g \in S_l, \tag{11}$$

for any $s \in [0, \frac{5}{2})$, then

$$\|f\|_{A^s(\Omega)} \sim \|f\|_{\mathcal{H}^s(\Omega)}, \qquad 0 < s < \frac{5}{2}. \tag{12}$$

Jackson estimates of the type (10) and Bernstein estimates of the type (11) hold for the spaces S_l. A precise derivation of these estimates can be found in [12]. Using the norm equivalence (12) we can now prove the following theorem which is inspired by the work in [5] and [8].

Theorem 4.3. *Choose $s \in (1, \frac{5}{2})$. Then it holds that*

$$\|f\|^2_{\mathcal{H}^s(\Omega)} \sim \sum_{l=0}^{\infty} 3^{2ls}\|(I_l - I_{l-1})f\|^2_{L_2(\Omega)}, \qquad f \in \mathcal{H}^s(\Omega).$$

Proof. Because of the norm equivalence (12) it is sufficient to prove that

$$\inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} 3^{2ls}\|g_l\|^2_{L_2(\Omega)} \sim \sum_{l=0}^{\infty} 3^{2ls}\|(I_l - I_{l-1})f\|^2_{L_2(\Omega)}.$$

Since $(I_l - I_{l-1})f \in S_l$ and $\sum_{l=0}^{\infty}(I_l - I_{l-1})f = f$ the inequality "$\lesssim$" is trivial and we will concentrate on the inequality "$\gtrsim$". Let $f = \sum_{l=0}^{\infty} g_l$ with $g_l \in S_l$. Since the operators I_l are projectors and the spaces S_l are nested, we have $(I_l - I_{l-1})S_n = 0$ when $n \leq l - 1$. Moreover the operators I_l also satisfy

$$\|I_l s_n\|_{L_2(\Omega)} \lesssim 3^{n-l}\|s_n\|_{L_2(\Omega)}, \quad s_n \in S_n. \tag{13}$$

Indeed, from the construction of I_l and (4) we easily get

$$\|I_l s_n\|_{L_\infty(\Omega)} \lesssim \|s_n\|_{L_\infty(\Omega)}.$$

Then (13) can be deduced by using (7), Theorem 3.2 and the equivalence of norms on the finite dimensional vector space of polynomials.

From the properties above and Cauchy Schwarz we have

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty} \sum_{l=0}^{\infty} 3^{2ls} \langle (I_l - I_{l-1}) g_n, (I_l - I_{l-1}) g_{n'} \rangle_{L_2(\Omega)} \\
&= \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{2ls} \langle (I_l - I_{l-1}) g_n, (I_l - I_{l-1}) g_{n'} \rangle_{L_2(\Omega)} \\
&\leq \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{2ls} \left(\|I_l g_n\|_{L_2(\Omega)} + \|I_{l-1} g_n\|_{L_2(\Omega)} \right) \\
&\qquad \cdot \left(\|I_l g_{n'}\|_{L_2(\Omega)} + \|I_{l-1} g_{n'}\|_{L_2(\Omega)} \right) \\
&\lesssim \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{2ls} 3^{n+n'-2l} \|g_n\|_{L_2(\Omega)} \|g_{n'}\|_{L_2(\Omega)}.
\end{aligned}$$

The last expression can be rewritten as

$$\sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{(s-1)(2l-n-n')} (3^{ns} \|g_n\|_{L_2(\Omega)}) (3^{n's} \|g_{n'}\|_{L_2(\Omega)}),$$

which is equivalent to

$$\sum_{n,n'=0}^{\infty} 3^{(s-1)(2\min\{n,n'\}-n-n')} (3^{ns} \|g_n\|_{L_2(\Omega)}) (3^{n's} \|g_{n'}\|_{L_2(\Omega)}).$$

The factor $3^{(s-1)(2\min\{n,n'\}-n-n')}$ becomes very small if $|n-n'| \gg 0$. In fact, the infinite matrix $[3^{(s-1)(2\min\{n,n'\}-n-n')}]_{n,n' \in \mathbb{N}}$ defines a bounded mapping on l_2. Therefore

$$\sum_{n,n'=0}^{\infty} 3^{(s-1)(2\min\{n,n'\}-n-n')} (3^{ns} \|g_n\|_{L_2(\Omega)}) (3^{n's} \|g_{n'}\|_{L_2(\Omega)}) \lesssim \sum_{n=0}^{\infty} 3^{2ns} \|g_n\|_{L_2(\Omega)}^2.$$

Since the splitting $f = \sum_{l=0}^{\infty} g_l$ was arbitrary, we have derived that

$$\begin{aligned}
&\inf_{g_l \in S_l : f = \sum_l g_l} \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\infty} 3^{2ls} \langle (I_l - I_{l-1}) g_n, (I_l - I_{l-1}) g_{n'} \rangle_{L_2(\Omega)} \\
&\qquad \lesssim \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{n=0}^{\infty} 3^{2ns} \|g_n\|_{L_2(\Omega)}^2.
\end{aligned}$$

Because $f \in A^s(\Omega)$ (Proposition 4.2) we know that the right expression is bounded. Then from the derivation made above it follows that the left expression is absolutely convergent and we are allowed to write that

$$\begin{aligned}
&\inf_{g_l \in S_l : f = \sum_l g_l} \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\infty} 3^{2ls} \langle (I_l - I_{l-1}) g_n, (I_l - I_{l-1}) g_{n'} \rangle_{L_2(\Omega)} \\
&= \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} \sum_{n,n'=0}^{\infty} 3^{2ls} \langle (I_l - I_{l-1}) g_n, (I_l - I_{l-1}) g_{n'} \rangle_{L_2(\Omega)} \\
&= \sum_{l=0}^{\infty} 3^{2ls} \|(I_l - I_{l-1}) f\|_{L_2(\Omega)}^2.
\end{aligned}$$

We conclude that

$$\sum_{l=0}^{\infty} 3^{2ls} \|(I_l - I_{l-1}) f\|_{L_2(\Omega)}^2 \lesssim \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} 3^{2ls} \|g_l\|_{L_2(\Omega)}^2.$$

□

Since, by Theorem 3.2, the subsets $\left\{3^l B_{\xi,l} \mid \xi \in \Xi_l \setminus \Xi_{l-1}\right\}$ form a L_2-stable Riesz basis for $\mathrm{Im}(I_l - I_{l-1})$, we find from Theorem 4.3 that

$$\begin{aligned}\|f\|^2_{\mathcal{H}^s(\Omega)} &\sim \sum_{l=0}^{\infty} 3^{2ls} \left\| \sum_{\xi\in\Xi_l\setminus\Xi_{l-1}} c_\xi 3^l B_{\xi,l} \right\|^2_{L_2(\Omega)} \\ &\sim \sum_{l=0}^{\infty} 3^{2ls} \sum_{\xi\in\Xi_l\setminus\Xi_{l-1}} |c_\xi|^2.\end{aligned}$$

Hence,

$$\left\| \sum_{l=0}^{\infty} \sum_{\xi\in\Xi_l\setminus\Xi_{l-1}} c_\xi 3^{l(1-s)} B_{\xi,l} \right\|^2_{\mathcal{H}^s(\Omega)} \sim \sum_{l=0}^{\infty} \sum_{\xi\in\Xi_l\setminus\Xi_{l-1}} |c_\xi|^2, \quad s \in (1, \frac{5}{2}). \tag{14}$$

5 The hierarchical basis as preconditioner

We are interested in solving the boundary value problem

$$\Delta^2 u = f, \quad \text{on} \quad \Omega, \qquad u = \frac{\partial u}{\partial n} = 0, \quad \text{on} \quad \partial\Omega, \tag{15}$$

where Δ^2 is the biharmonic operator, and n is the outward normal to $\partial\Omega$. We multiply both sides of (15) by smooth test functions v that vanishes on $\partial\Omega$ and, after integrating by parts, we find that the solution u satisfies

$$\langle \Delta u, \Delta v\rangle_{L_2(\Omega)} = \langle f, v\rangle_{L_2(\Omega)}. \tag{16}$$

It is well-known that the conjugate gradient method is a very efficient solver for large linear systems arising from the Ritz–Galerkin method for problems such as (16). However, because of stability reasons, it is necessary that these systems have been suitably preconditioned. As mentioned in the introduction, one way of preconditioning such stiffness matrices is based on hierarchical bases (see [21, 5, 8, 16]).

So suppose $S \subset \mathcal{H}^2$ is a space of conforming finite elements of dimension N defined on a triangulation with mesh size h. The stiffness matrix

$$A = \Big(\langle \Delta B_i, \Delta B_j\rangle_{L_2(\Omega)}\Big)_{i,j=1}^{N}$$

for a typical nodal basis $\{B_i\}_{i=1}^N$ of S would give rise to condition numbers $\kappa(A) = \mathcal{O}(h^{-4})$. Therefore it is important that we precondition the linear system $Ax = b$ resulting from the Ritz–Galerkin method. This means that we transform the system $Ax = b$ into the equivalent system $C^{1/2}AC^{1/2}y = C^{1/2}b$, $C^{1/2}y = x$, where C is a positive definite matrix such that $\kappa(C^{1/2}AC^{1/2})$ is much smaller than $\kappa(A)$. Such preconditioning can be realized by a change of basis. Let $\{B_i\}_{i=1}^N$ and $\{B_i'\}_{i=1}^N$ be two bases of S. Let g be an arbitrary function in S,

$$g = \sum_{i=1}^{N} c_i B_i = \sum_{i=1}^{N} d_i B_i'$$

and define M as the unique matrix that takes the coefficients c_i into the coefficients d_i. Then the stiffness matrix A' relative to $\{B_i'\}_{i=1}^N$ is given by $A' = M^TAM$, and we solve the equivalent system $M^TAMy = M^Tb$, $My = x$.

Equation (16) defines an energy norm $\|\cdot\|_E$ given by

$$\|\cdot\|_E := \langle \Delta\,\cdot\,, \Delta\,\cdot\,\rangle^{1/2}_{L_2(\Omega)}$$

which satisfies

$$\|\cdot\|_E \sim \|\cdot\|_{\mathcal{H}^2}\,. \tag{17}$$

Suppose that the basis $\{B'_i\}_{i=1}^N$ satisfies

$$c_1 \sum_{i=1}^{N} |d_i|^2 \leq \left\| \sum_{i=1}^{N} d_i B'_i \right\|_{\mathcal{H}^2}^2 \leq c_2 \sum_{i=1}^{N} |d_i|^2, \quad \forall\, d \in \mathbb{R}^N,$$

then by (17) and $\langle g, g \rangle_E = d^T A' d$ we find the well-known fact (see e.g. [5]) that $\kappa(A') = \mathcal{O}\left(c_2/c_1\right)$.

By (14), the properly scaled hierarchical basis $\bigcup_{l=0}^{\infty} \left\{3^{-l} B_{\xi,l} \mid \xi \in \Xi_l\right\}$ leads to uniformly well-conditioned stiffness matrices for the problem (15). However, due to the increasing sizes of the supports of the basis functions on lower resolution levels, the resulting stiffness matrix is not sparse. Fortunately Dahmen *et al.* [7, §4.4] show that a matrix-vector multiplication using this stiffness matrix can be avoided, and they propose an efficient recursive way to solve the system. Therefore, the use of our hierarchical basis of Lagrange type on Powell–Sabin triangulations as preconditioner is extremely well motivated.

Remark 5.1. An other option for preconditioning (15) is to use the bivariate Powell–Sabin spline wavelets from [19]. These wavelets are a stable completion [3] of the hierarchical basis of Hermite type used in Section 2, which is, under a suitable normalization, a Riesz basis for the Sobolev space $H^s(\Omega)$ with $s \in (2, 5/2)$, see [12]. Because the wavelets are designed to be more stable than the hierarchical basis, we expect that the wavelet basis is a strongly stable Riesz basis for the Sobolev spaces $H^s(\Omega)$ for all $s \in (\widetilde{s}, 5/2)$ for a $\widetilde{s} < 2$, though we can not prove this mathematically yet. Under the conjecture that the wavelet basis is a Riesz basis with respect to $H^2(\Omega)$ we find that they give rise to uniformly well-conditioned stiffness matrices for fourth order elliptic problems. Furthermore these wavelets have one vanishing moment which implies that the stiffness matrix is (quasi-)sparse, see [4].

Remark 5.2. The hierarchical basis $\{B_{\xi,l}\}_{l=0}^{\infty}$ can also be used to solve boundary value problems of second order such as, for instance, Poisson's equation

$$-\Delta u = f, \quad \text{on} \quad \Omega, \qquad u = 0, \quad \text{on} \quad \partial\Omega, \tag{18}$$

which appears in many fields of science and engineering. We adapt the proof of Theorem 4.3 slightly to obtain the result

$$n^{-2} \sum_{l=0}^{n} \sum_{\xi \in \Xi_l \setminus \Xi_{l-1}} |c_\xi|^2 \lesssim \left\| \sum_{l=0}^{n} \sum_{\xi \in \Xi_l \setminus \Xi_{l-1}} c_\xi B_{\xi,l} \right\|_{\mathcal{H}^1(\Omega)}^2 \lesssim \sum_{l=0}^{n} \sum_{\xi \in \Xi_l \setminus \Xi_{l-1}} |c_\xi|^2,$$

which yields $\kappa(A) = \mathcal{O}(n^2)$ with A the stiffness matrix arising from the Ritz–Galerkin method for solving (18). In terms of the mesh size h of the underlying triangulation Δ_n this becomes $\kappa(A) = \mathcal{O}(|\log h|^2)$. This is still much better than a typical nodal basis on Δ_n, which would give rise to a condition number $\kappa(A) = \mathcal{O}(h^{-2})$.

Acknowledgements

This work is partially supported by the Flemish Fund for Scientific Research (FWO Vlaanderen) project MISS (G.0211.02), and by the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with the authors.

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