Separation of zeros of para-orthogonal rational functions

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Abstract

We generalize a result by L. Golinskii [L. Golinskii. Quadrature formula and zeros of para-orthogonal polynomials on the unit circle. *Acta Math. Hungar.*, 96:169–186, 2002] on separation of the zeros of para-orthogonal polynomials on the unit circle to a similar result for para-orthogonal rational functions.

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Separation of zeros of para-orthogonal rational functions

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Abstract

We generalize a result by L. Golinskii [5] on separation of the zeros of para-orthogonal polynomials on the unit circle to a similar result for para-orthogonal rational functions.

1 Introduction

Every probability measure on the unit circle gives rise to an orthonormal sequence $\{\rho_n\}_{n=0}^{\infty}$ of polynomials, so called Szegő polynomials. See for example [6, 7]. Invariant para-orthogonal polynomials are polynomials of the form $c_n[\rho_n(z) + \tau\rho_n^*(z)]$, $|\tau| = 1$, $c_n \neq 0$, where $\rho_n^*(z) = z^n\overline{\rho_n(1/\overline{z})}$. These polynomials have all their zeros on the unit circle, and they are all simple. The zeros are nodes in a quadrature formula with positive weights which is exact on the space $\text{span}\{1/z^{n-1}, \ldots, 1, \ldots, z^{n-1}\}$. See e.g. [6]. An equivalent representation of the invariant para-orthogonal polynomials is as the class of all polynomials of the form $d_n[\rho_n^*(z)\overline{\rho_n^*(w)} - \rho_n(z)\overline{\rho_n(w)}]$, $|w| = 1$, $d_n \neq 0$. For a given w, the value $z = w$ is a zero of this polynomial. It was shown by Golinskii [5] that the zeros of two consecutive of these polynomials (for a given w) separate each other when the zero $z = w$ is not included among the zeros of the polynomial of highest degree.

The aim of this note is to prove a similar result for orthogonal rational functions on the unit circle. In sections 2 and 3 we give a brief summary of relevant basic properties of such functions. For a more comprehensive treatment, see [3]. In section 4 we give a proof of the indicated result, in the main following the reasoning of Golinskii.

2 Orthogonal rational functions

In the following, $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ denotes the unit circle and $\mathbb{E}$ the exterior of the closed unit disk.

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Let a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of not necessarily distinct points in $\mathbb{D}$ be given. We define

$$\zeta_0 = 1, \quad \zeta_n(z) = z_n \frac{z - \alpha_n}{1 - \overline{\alpha}_n z}, \quad n = 1, 2, \ldots, \tag{2.1}$$

where $z_n = -|\alpha_n|/\alpha_n$ if $\alpha_n \neq 0$ and $z_n = 1$ if $\alpha_n = 0$. Furthermore we define the Blaschke products B_n by

$$B_0 = 1, \quad B_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots. \tag{2.2}$$

The functions $\{B_0, B_1, \ldots, B_n\}$ span the space $\mathcal{L}_n$ consisting of all functions of the form $f(z) = P(z)/\pi(z)$, where $P \in \mathcal{P}_n$ (the space of polynomials of degree at most n) and

$$\pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha}_k z). \tag{2.3}$$

In general we define for any function $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the superstar transform f^* by $f^*(z) = B_n(z) f_*(z)$, where $f_*(z) = \overline{f(1/\overline{z})}$. Note that f^* also belongs to $\mathcal{L}_n$.

Let μ be a probability measure on $\mathbb{T}$, with associated inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_{\mathbb{T}} f(t) \overline{g(t)} d\mu(t). \tag{2.4}$$

We shall use the notation ϕ_n for the elements of the orthonormal basis for $\mathcal{L}_n$ which is ordered such that $\phi_0 = 1$ and $\phi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ for $k = 1, 2, \ldots, n$. We may then write $\phi_n(z) = p_n(z)/\pi_n(z)$, $\phi_n^*(z) = q_n(z)/\pi_n(z)$ where $p_n \in \mathcal{P}_n$, $q_n \in \mathcal{P}_n$.

We note that if $\alpha_n = 0$ for all n, then $B_n(z) = z^n$, $\mathcal{L}_n = \mathcal{P}_n$ and ϕ_n, ϕ_n^* are orthonormal polynomials with respect to μ and their reciprocals. For motivations for studying the rational generalizations of orthogonal polynomials introduced above, we refer to [3, 4].

Let $k_n(z, w)$ denote the reproducing kernel for $\mathcal{L}_n$, i.e.,

$$k_n(z, w) = \sum_{j=0}^{n} \phi_j(z) \overline{\phi_j(w)}. \tag{2.5}$$

The orthonormal functions ϕ_n satisfy

$$\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)} = [1 - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}] k_n(z, w), \tag{2.6}$$

$$\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)} = [1 - \zeta_n(z)\overline{\zeta_n(w)}] k_n(z, w). \tag{2.7}$$

It follows easily from these formulas that

$$\begin{array}{ll} |\phi_n(z)| < |\phi_n^*(z)| & \text{for } z \in \mathbb{D} \\ |\phi_n(z)| = |\phi_n^*(z)| & \text{for } z \in \mathbb{T} \\ |\phi_n(z)| > |\phi_n^*(z)| & \text{for } z \in \mathbb{E}. \end{array} \tag{2.8}$$

(Note that $|\zeta_n(z)| < 1$ for $z \in \mathbb{D}$, $|\zeta_n(z)| = 1$ for $z \in \mathbb{T}$ and $|\zeta_n(z)| > 1$ for $z \in \mathbb{E}$.) Furthermore all the zeros of ϕ_n lie in $\mathbb{D}$. Simple examples (e.g. with μ the normalized Lebesgue measure and $\alpha_n = 0$ for all n, which gives $\phi_n(z) = z^n$) show that the zeros may be multiple.

For more exhaustive treatments, see e.g., [1, 3].

3 Para-orthogonal rational functions

Quadrature formulas with positive weights and nodes on $\mathbb{T}$ have important uses. Of special interest are such formulas which integrate exactly all functions in spaces of the form $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_q, g_* \in \mathcal{L}_p\}$ with as large value of $p+q$ as possible. The zeros of ϕ_n can not be used as nodes, since they lie in $\mathbb{D}$ (and may even be multiple). It turns out that so-called invariant para-orthogonal functions give rise to such quadrature formulas, exact on $\mathcal{L}_{n-1,n-1}$ (while no quadrature formula as specified can be exact on $\mathcal{L}_{n-1,n}$ of on $\mathcal{L}_{n,n-1}$). See [2, 3].

A function Q_n in $\mathcal{L}_n$ is called invariant if $Q_n^*(z) = k_n Q_n(z)$ for some $k_n \neq 0$. It is called para-orthogonal if $\langle Q_n, f\rangle = 0$ for all $f \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, where $\mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}$, while $\langle Q_n, 1\rangle \neq 0$ and $\langle Q_n, B_n\rangle \neq 0$. (These concepts are direct generalizations of corresponding concepts in the polynomial case, ie., when $\alpha_n = 0$ for all n. These were introduced and studied in [6].)

It can be shown that the invariant para-orthogonal rational functions are exactly functions of the form $c_n Q_n(z,\tau)$, $c_n \neq 0$, where

$$Q_n(z,\tau) = [\phi_n(z) + \tau\phi_n^*(z)], \quad \tau \in \mathbb{T}. \tag{3.1}$$

Furthermore, $Q_n(z,\tau)$ has exactly n simple zeros, all of them lying on $\mathbb{T}$. See [3].

Now consider a function $d_n\Omega_n(z,w)$, $d_n \neq 0$, where

$$\Omega_n(z,w) = [\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}]. \tag{3.2}$$

We may write

$$\Omega_n(z,w) = -\overline{\phi_n(w)}[\phi_n(z) + (-\overline{\left[\frac{\phi_n^*(w)}{\phi_n(w)}\right]}\phi_n^*(z)]. \tag{3.3}$$

Because of (2.8) we have for $w \in \mathbb{T}$ that $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} \in \mathbb{T}$. Thus $\Omega_n(z,w)$ is a function of the form $c_n Q_n(z,\tau)$ as in (3.1). On the other hand, for each $\tau \in \mathbb{T}$, there are n values of w in $\mathbb{T}$ such that $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} = \tau$. (Note that for a given τ, $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} = \tau$ may be written as an algebraic equation of degree n in w, and that according to (2.8), $-\overline{[\frac{\phi_n^*(w)}{\phi_n(w)}]} \in \mathbb{T}$ if and only if $w \in \mathbb{T}$. See also [3, Thm. 5.2.1].) Thus the class of functions of the form $c_n Q_n(z,\tau)$, $c_n \neq 0$, $\tau \in \mathbb{T}$ as given in (3.1) is exactly the same as the class of functions $d_n\Omega_n(z,w)$, $d_n \neq 0$, $w \in \mathbb{T}$ as given in (3.2).

4 Separation of zeros

In [5] Golinskii showed that in the polynomial case, i.e., when all α_n equal zero, a certain separation property of the zeros of two consecutive polynomials $\Omega_n(z,w)$ (for fixed w) holds. We shall prove a similar result in the general rational case. The result as well as the proof are rather straightforward generalizations of Golinskii's discussion in the polynomial case.

In the following, w denotes a fixed point on $\mathbb{T}$. We observe that $[1 - \zeta_n(z)\overline{\zeta_n(w)}] = 0$ if and only if $z = w$. It then follows from (2.5)-(2.6) and (3.2) that $z = w$ is a zero of $\Omega_n(z,w)$ for all n, and that the remaining zeros of $\Omega_n(z,w)$ are exactly the zeros of $k_{n-1}(z,w)$.

Now assume that z_0 is a common zero of $\Omega_n(z,w)$ and $\Omega_{n+1}(z,w)$, $z_0 \neq w$. Note that z_0 has to be on $\mathbb{T}$. It follows from (2.6) and the definition (3.2) that $k_n(z_0,w) = 0$ and

$k_{n-1}(z_0, w) = 0$, hence also $\phi_n(z_0)\overline{\phi_n(w)} = 0$. This is impossible since all the zeros of ϕ_n lie in $\mathbb{D}$. Consequently $\Omega_n(z, w)$ and $\Omega_{n+1}(z, w)$ have no common zeros except $z = w$.

Now for each n let $z_{n,k} = e^{i\theta_{n,k}}$, $k = 0, 1, \ldots, n-1$, be the zeros of $\Omega_n(z, w)$, with $z_{n,0} = w$, ordered such that

$$\theta_{n,0} < \theta_{n,1} < \cdots < \theta_{n,n-1} < \theta_{n,0} + 2\pi. \tag{4.1}$$

Theorem 4.1 *The zeros of $\Omega_n(z, w)$ included $z = w$ and the zeros of $\Omega_{n+1}(z, w)$ not included $z = w$ separate each other in the sense that*

$$\theta_{n,0} < \theta_{n+1,1} < \theta_{n,1} < \theta_{n+1,2} < \cdots < \theta_{n,n-1} < \theta_{n+1,n}. \tag{4.2}$$

Proof. Consider the function

$$\Gamma_n(z) = \Gamma_n(z, w) = \frac{k_n(z, w)}{\Omega_n(z, w)}. \tag{4.3}$$

It follows from the foregoing discussion that the zeros of $k_n(z, w)$ are exactly the points $z_{n+1,1}, \ldots, z_{n+1,n}$ while the zeros of $\Omega_n(z, w)$ are the points $z_{n,0}, \ldots, z_{n,n-1}$. Thus $\Gamma_n(z)$ has simple zeros at the points $z_{n+1,1}, \ldots, z_{n+1,n}$ and simple poles at the points $z_{n,0}, \ldots, z_{n,n-1}$. (Recall that $\Omega_n(z, w)$ and $\Omega_{n+1}(z, w)$ have no common zeros except $z = w$. Also note that the terms $\pi_n(z)$ in the numerator and the denominator cancel.) Expressing $k_n(z, w)$ by (2.7) we may write

$$\Gamma_n(z) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{[1 - \zeta_n(z)\overline{\zeta_n(w)}][\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}]}. \tag{4.4}$$

We introduce the function b_n defined by

$$b_n(z) = \frac{\phi_n(z)}{\phi_n^*(z)}. \tag{4.5}$$

We note that b_n is holomorphic in $\mathbb{D} \cup \mathbb{T}$ and maps $\mathbb{D}$ onto $\mathbb{D}$, $\mathbb{T}$ onto $\mathbb{T}$, according to (2.8). In terms of this function, $\Gamma_n(z)$ may be written as

$$\Gamma_n(z) = \frac{1 - \zeta_n(z)\overline{\zeta_n(w)}b_n(z)\overline{b_n(w)}}{[1 - \zeta_n(z)\overline{\zeta_n(w)}][1 - b_n(z)\overline{b_n(w)}]} \tag{4.6}$$

and hence by a simple calculation

$$\Gamma_n(z) = \frac{1}{2}\left[\frac{1 + b_n(z)\overline{b_n(w)}}{1 - b_n(z)\overline{b_n(w)}} + \frac{1 + \zeta_n(z)\overline{\zeta_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}}\right]. \tag{4.7}$$

The Möbius transformation $z \to \frac{1+z}{1-z}$ maps $\mathbb{D}$ onto the open right half-plane $\mathbb{H}$ and $\mathbb{T}$ onto the extended imaginary axis $\hat{\mathbb{I}}$. Taking into account the mapping properties of the function b_n stated above, we find that each of the two terms in (4.7) maps $\mathbb{D}$ onto $\mathbb{H}$ and $\mathbb{T}$ onto $\hat{\mathbb{I}}$. The function $\Gamma_n(z)$ then has the same property. In other words, $\Gamma_n(z)$ is a lossless Carathéodory function.

A rational lossless Carathéodory function has the property that the zeros and poles separate each other. For the sake of completeness, we sketch the proof.

The function $\Gamma_n(z)$ may be written in the from

$$\Gamma_n(z) = ic + \sum_{k=0}^{n-1} \lambda_k \frac{z + z_{n,k}}{z - z_{n,k}}, \tag{4.8}$$

where $\lambda_k > 0$ and c is a real constant. (See e.g. [6].) We find that

$$\lambda_k \frac{e^{i\theta} + e^{i\theta_{n,k}}}{e^{i\theta} - e^{i\theta_{n,k}}} = -2i\lambda_k \frac{\sin(\theta - \theta_{n,k})}{|e^{i\theta} - e^{i\theta_{n,k}}|^2}.$$

It follows that

$$\operatorname{Im}\left(\lambda_k \frac{e^{i\theta} + e^{i\theta_{n,k}}}{e^{i\theta} - e^{i\theta_{n,k}}}\right) > 0 \text{ for } \theta < \theta_{n,k}, \quad \operatorname{Im}\left(\lambda_k \frac{e^{i\theta} + e^{i\theta_{n,k}}}{e^{i\theta} - e^{i\theta_{n,k}}}\right) < 0 \text{ for } \theta > \theta_{n,k}.$$

When $\theta \to \theta_{n,k}$, the term $\lambda_k \frac{z+z_{n,k}}{z-z_{n,k}} = \lambda_k \frac{e^{i\theta}+e^{i\theta_{n,k}}}{e^{i\theta}-e^{i\theta_{n,k}}}$ in (4.8) dominates, hence we may conclude that

$$\lim_{\theta \to \theta_{n,k}^-} \frac{1}{i}\Gamma_n(e^{i\theta}) = +\infty, \quad \lim_{\theta \to \theta_{n,k}^+} \frac{1}{i}\Gamma_n(e^{i\theta}) = -\infty.$$

Thus the image of the arc $\{z = e^{i\theta} : \theta_{n,k} < \theta < \theta_{n,k+1}\}$ by the mapping $z \to \Gamma_n(z)$ is the whole imaginary axis $\mathbb{I}$. Consequently (at least) one of the zeros of $\Gamma_n(z)$ must lie on this arc. Taking into account the ordering for general n indicated in (4.1) and the fact that $\Gamma_n(z)$ has the same number of zeros and poles, we conclude that (4.2) holds.

This completes the proof of the theorem. □

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