On the stability of Powell–Sabin Wavelets

Jan Maes *Adhemar Bultheel*

Report TW 395, July 2004

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

On the stability of Powell–Sabin Wavelets

Jan Maes *Adhemar Bultheel*

Report TW 395, July 2004

Department of Computer Science, K.U.Leuven

Abstract

Recently we developed multiscale spaces of C^1 piecewise quadratic polynomials relative to arbitrary polygonal domains $\Omega \subset \mathbb{R}^2$. These multiscale bases are weakly stable with respect to the L_2 norm. In this paper we show that these bases form strongly stable Riesz bases for the Sobolev spaces $H^s(\Omega)$ with $s \in (2, \frac{5}{2})$.

Keywords : Wavelets, Powell–Sabin Splines, Hierarchical Bases, Stable Approximation by Splines
AMS(MOS) Classification : 41A15, 65D07, 65T60, 41A63

On the stability of Powell–Sabin Wavelets

Jan Maes Adhemar Bultheel

July 2004

Abstract

Recently we developed multiscale spaces of C^1 piecewise quadratic polynomials relative to arbitrary polygonal domains $\Omega \subset \mathbb{R}^2$. These multiscale bases are weakly stable with respect to the L_2 norm. In this paper we show that these bases form strongly stable Riesz bases for the Sobolev spaces $H^s(\Omega)$ with $s \in (2, \frac{5}{2})$.

keywords: Wavelets, Powell–Sabin Splines, Hierarchical Bases, Stable Approximation by Splines
AMS(MOS) Classification: 41A15, 65D07, 65T60, 41A63

1 Introduction

Powell–Sabin (PS) splines are functions in the bivariate space of C^1 continuous piecewise quadratic polynomials on a PS-refinement of a general triangulation Δ. Working with triangles makes it possible to design surfaces with an arbitary number of edges, which is not possible with the tensor product B-spline representation that is restricted to rectangular domains. In [19] we developed a multiresolution setup for Powell–Sabin splines which is based on a triadic subdivision scheme [20]. Moreover the functions that are added from level to level do have one vanishing moment, i.e. their integral vanishes, so we may refer to the multiscale bases as wavelet bases. In this paper we are concerned with the stability of these PS-wavelet bases.

Stable multiscale bases were constructed first in [21], based on C^0 finite elements. They give rise to Riesz bases for $H^s(\Omega)$, $s \in (1, \frac{3}{2})$, with Ω a bounded polygonal domain in $\mathbb{R}^2$. For $s \geq \frac{3}{2}$, C^1 piecewise polynomials can be used as shown in [7] and [16], where Hermite type finite element bases are used which give rise to Riesz bases for $s \in (2, \frac{5}{2})$. In [8] C^1 bases of Lagrange type are constructed, which allow to enlarge the range of stability from $s \in (2, \frac{5}{2})$ to $s \in (1, \frac{5}{2})$. Since our basis functions are of Hermite type, we will only be able to prove that they form Riesz bases for $s \in (2, \frac{5}{2})$.

The paper is organized as follows. Section 2 is devoted to the idea of multiresolution analysis in the context of Dahmen *et al.* [3, 4, 5]. In Sections 3 and 4 we review the relevant aspects and properties of a multilevel B-spline basis for Powell–Sabin splines. We also introduce quasi-interpolants that are suitable for constructing a multiresolution analysis as seen in Section 2. Finally, in Sections 5 and 6, the theory of multiscale decompositions is applied to show that our multiresolution analysis for Powell–Sabin splines yields stable Riesz bases for $H^s(\Omega)$ with $s \in (2, \frac{5}{2})$.

2 Multiresolution analysis

In this section we briefly review the general idea behind multiresolution analysis and stable bases. We use a general definition of multiresolution analysis in the context of [3, 4, 5].

Definition 2.1. A multiresolution analysis consists of

1. A Hilbert space $\mathcal{H}$ of functions defined on a bounded subset $\Omega \in \mathbb{R}^2$ with associated inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}^2$.

2. A nested sequence of subspaces $S_0 \subset S_1 \subset S_2 \subset \ldots \subset \mathcal{H}$ that are dense in $\mathcal{H}$,
$$\overline{\bigcup_{l=0}^{\infty} S_l} = \mathcal{H}.$$
3. A collection of uniformly bounded operators
$$Q_l : \mathcal{H} \to S_l$$
with the properties
$$\begin{aligned} Q_l Q_l &= Q_l, \\ Q_l Q_{l+1} &= Q_l, \\ Q_l(\mathcal{H}) &= S_l \end{aligned}$$
for all integers $l \geq 0$.

The spaces S_l are spanned by bases $\boldsymbol{\phi}_l = \{\phi_{l,m} : m \in I_l\}$ which are stable, i.e.,
$$\|\mathbf{c}\|_{l_2(I_l)} := \left(\sum_{m \in I_l} |c_m|^2 \right)^{\frac{1}{2}} \sim \left\| \sum_{m \in I_l} c_m \phi_{l,m} \right\|_{\mathcal{H}}. \tag{2.1}$$
Here $A \sim B$ means that both quantities can be bounded by some constant multiple of each other, uniformly with respect to any parameters A and B may depend on. Similarly $A \lesssim B$ is to express that A remains uniformly bounded by a constant multiple of B. We will also refer to the functions $\phi_{l,m}$ as scaling functions.

One is interested in how much an approximation to a given $f \in \mathcal{H}$ from S_l changes when progressing to the next higher resolution S_{l+1}. Therefore we will look for suitable complement spaces W_l such that
$$S_{l+1} = S_l \oplus W_l$$
as well as for stable bases $\boldsymbol{\psi}_l = \{\psi_{l,m} : m \in J_l\}$ of W_l by which one can describe the correction of a current approximation $f_l \in S_l$ in S_{l+1}. With the projectors Q_l given, we can define these complement spaces as
$$W_l = \{s \in S_{l+1} | Q_l s = 0\}.$$
Hence we get a decomposition of $\mathcal{H}$ as the direct sum
$$\mathcal{H} = S_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots .$$
We will refer to the complement spaces W_l as wavelet spaces and the functions $\psi_{l,m} \in W_l$ as wavelets. For now we will neglect the fact that it is common to require from wavelets that they have at least one vanishing moment. Thus, any $f_n \in S_n$ can be written in single scale representation
$$f_n = \sum_{m \in I_n} c_m \phi_{n,m}$$
or in multiscale representation
$$f_n = \sum_{l=-1}^{n-1} \sum_{m \in J_l} d_{l,m} \psi_{l,m},$$
where we have set for simplicity $\boldsymbol{\psi}_{-1} := \boldsymbol{\phi}_0$, $J_{-1} := I_0$.

The following definition introduces the concept of weakly and strongly stable multiscale bases for $\mathcal{H}$.

Definition 2.2. Let $\mathcal{H}$ be a Hilbert space with a multiresolution analysis and corresponding multiscale basis $\boldsymbol{\psi} := \bigcup_{l=-1}^{\infty} \boldsymbol{\psi}_l$. The multiscale basis $\boldsymbol{\psi}$ is said to form a weakly stable basis for $\mathcal{H}$ if for each $n \geq 0$

$$\left\| \sum_{l=-1}^{n-1} \sum_{m \in J_l} d_{l,m} \psi_{l,m} \right\|_{\mathcal{H}} \sim \left(\sum_{l=-1}^{n-1} \sum_{m \in J_l} |d_{l,m}|^2 \right)^{\frac{1}{2}},$$

where the constants of equivalence have at most polynomial growth in n. If the constants of equivalence are independent of n, the basis is said to be strongly stable.

3 Multilevel spaces of C^1 piecewise quadratics

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ for which there is a collection Δ_0 of non-degenerate triangles such that

1. $\Omega = \bigcup_{T \in \Delta_0} T$,
2. The intersection of any two different triangles from Δ_0 is either empty or a common edge or vertex.

This collection Δ_0 of non-degenerate triangles is called a triangulation of Ω. A triangle T_i of Δ_0 is said to be non-conforming if it contains a vertex of another triangle T_j of Δ_0, different from its own three vertices. A triangulation Δ_0 is said to be non-conforming if it contains at least one non-conforming triangle. Otherwise it is said to be conforming. In what follows we will only consider conforming triangulations.

Note that the boundary $\partial\Omega$ of Ω is piecewise linear, and thus is Lipschitz with a constant $L_{\partial\Omega}$ which depends on the size of the angles between the edges of $\partial\Omega$. Because Ω is a bounded domain we have that Ω satisfies the strong local Lipschitz property which implies the uniform cone property [1].

Suppose we have a conforming triangulation Δ_0 of Ω. The Powell–Sabin refinement Δ_0^{PS} of Δ_0 divides each triangle T into six subtriangles with a common vertex. It can be constructed as follows (see Figure 1):

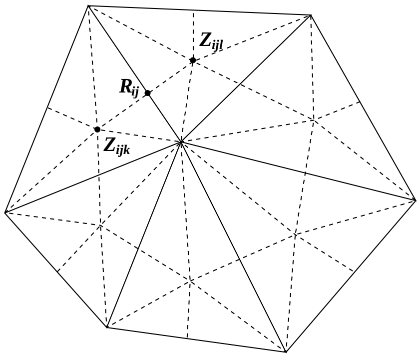


Figure 1: A PS refinement Δ_0^{PS} of Δ_0.

1. Choose an interior point Z_{ijk} for each triangle $T(V_i, V_j, V_k)$ with vertices V_i, V_j, V_k, so that if two triangles $T(V_i, V_j, V_k)$ and $T(V_i, V_j, V_l)$ have a common edge V_iV_j, the line joining Z_{ijk} and Z_{ijl} intersects this common edge V_iV_j at a point R_{ij} between its vertices V_i and V_j. We will choose Z_{ijk} as the incenter of triangle $T(V_i, V_j, V_k)$.
2. Join the points Z_{ijk} to the vertices V_i, V_j and V_k.

3. For each edge of $T(V_i, V_j, V_k)$
   - which belongs to the boundary $\partial\Omega$, join Z_{ijk} to the middle point of the edge.
   - which is common to a triangle $T(V_i, V_j, V_l)$, join Z_{ijk} to R_{ij}.

We discuss a triadic scheme for refining a given triangulation Δ_0 and its associated PS-refinement Δ_0^{PS} to produce nested sequences

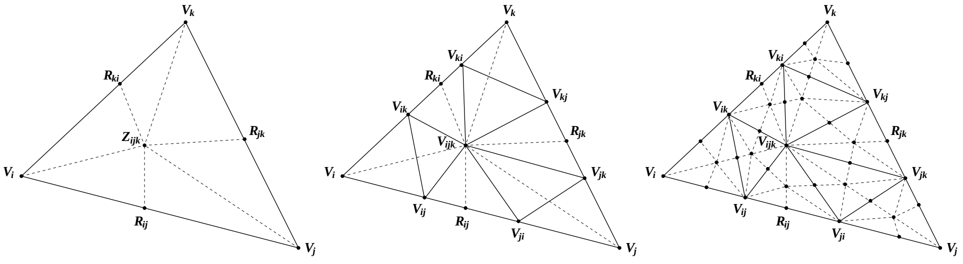
$$\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots \tag{3.1}$$

and

$$\Delta_0^{PS} \subset \Delta_1^{PS} \subset \Delta_2^{PS} \subset \cdots . \tag{3.2}$$

Algorithm 3.1. *Let Δ_0^{PS} be the PS-refinement associated with a triangulation Δ_0. For each triangle T in Δ_0,*

1. *add a new vertex V_{ijk} such that this new vertex coincides with the interior vertex Z_{ijk} of the PS-refinement Δ_0^{PS},*
2. *insert two new vertices on the edges each at one side of the R_{ij} and connect these vertices such that they form a hexagon. These new vertices have to be chosen in such a way that the resulting hexagon contains the interior vertex V_{ijk},*
3. *connect the six new vertices on the edges with the interior vertex V_{ijk},*
4. *for each of the nine new triangles, a new interior point is determined on the line of the old PS-refinement Δ_0^{PS} that crosses the new triangle.*



(a) (b) (c)

Figure 2: Principle of triadic subdivision. We place a new vertex at the position of the interior point Z_{ijk} and two new vertices on the edges each at one side of the R_{ij}.

The steps of the algorithm are illustrated in Figure 2. It is clear that Algorithm 3.1 splits each triangle of Δ_0 into nine subtriangles. This process can be repeated as often as desired to produce the nested sequences in (3.1) and (3.2).

Let $\mathcal{V}_l$, $\mathcal{E}_l$ and $\mathcal{T}_l$ denote the sets of vertices, edges and triangles in the triangulation Δ_l obtained after l steps of Algorithm 3.1 on an initial triangulation Δ_0. Then the following equations can be deduced from Algorithm 3.1:

$$\begin{aligned} \#\mathcal{T}_l &= 9 \cdot \#\mathcal{T}_{l-1}, \\ \#\mathcal{V}_l &= \#\mathcal{V}_{l-1} + 2 \cdot \#\mathcal{E}_{l-1} + \#\mathcal{T}_{l-1}, \end{aligned}$$

and from Euler's formula we get that

$$\#\mathcal{E}_l = \#\mathcal{V}_{l-1} + 2 \cdot \#\mathcal{E}_{l-1} + 10 \cdot \#\mathcal{T}_{l-1} - 1.$$

Proposition 3.2. *For all $l \geq 0$,*

$$\begin{aligned}
\#\mathcal{T}_l &= 9^l \cdot \#\mathcal{T}_0, \\
\#\mathcal{E}_l &= 3^{l-1} \cdot \#\mathcal{V}_0 + 2 \cdot 3^{l-1} \cdot \#\mathcal{E}_0 - 3^{l-1} + \left(\frac{3}{2}9^l - \frac{7}{6}3^l\right) \cdot \#\mathcal{T}_0, \\
\#\mathcal{V}_l &= 3^{l-1} \cdot \#\mathcal{V}_0 + 2 \cdot 3^{l-1} \cdot \#\mathcal{E}_0 - 3^{l-1} + 1 + \left(\frac{9^l}{2} - \frac{7}{6}3^l\right) \cdot \#\mathcal{T}_0.
\end{aligned}$$

Proof. This can easily be proved by induction. □

In the following we will require that the nested sequence of triangulations (3.1) is regular, which means that the minimum angle of any triangle in any Δ_l remains bounded away from zero and that

$$3^{-l} \lesssim \min_{T \in \Delta_l} |T| \leq \max_{T \in \Delta_l} |T| \lesssim 3^{-l}, \qquad l \in \mathbb{N}_0, \tag{3.3}$$

where $|T|$ is the diameter of triangle T. Consequently we also have that

$$3^{-l} \lesssim \min_{T \in \Delta_l^{PS}} |T| \leq \max_{T \in \Delta_l^{PS}} |T| \lesssim 3^{-l}, \qquad l \in \mathbb{N}_0.$$

For each integer $l \geq 0$ we consider the space S_l of C^1 continuous piecewise quadratic polynomials with respect to Δ_l^{PS}, the Powell–Sabin splines, which is defined as

$$S_l := \{s \in C^1(\Omega) : s|_T \in \mathcal{P}_2 \text{ for all } T \in \Delta_l^{PS}\}, \tag{3.4}$$

where $\mathcal{P}_2$ is the space of all bivariate polynomials of total degree at most 2. It is clear that the spaces S_l are nested, i.e.,

$$S_0 \subset S_1 \subset \cdots \subset S_l \subset \cdots . \tag{3.5}$$

Powell and Sabin [17] showed that the following interpolation problem:

$$s(V) = f(V), \quad D_x s(V) = D_x f(V), \quad D_y s(V) = D_y f(V), \quad \forall\ V \in \mathcal{V}_l, \tag{3.6}$$

has a unique solution $s(x, y)$ in S_l. Hence, the dimension of the space S_l equals $3 \cdot \#\mathcal{V}_l$.

4 Stable Powell–Sabin B-splines

For ease of notation we drop the sub/super-script l which denotes the multiscale level and explain the relevant aspects and properties of Powell–Sabin splines in the single scale setting. Dierckx [12] presented a normalized B-spline representation for PS-splines on a triangulation Δ of Ω,

$$s(x, y) = \sum_{i=1}^{\#\mathcal{V}} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y) \quad , \quad (x, y) \in \Omega, \tag{4.1}$$

where the B-splines form a convex partition of unity on Ω, i.e.

$$B_{ij}(x, y) \geq 0 \text{ for all } x, y \in \Omega, \tag{4.2}$$

$$\sum_{i=1}^{\#\mathcal{V}} \sum_{j=1}^{3} B_{ij}(x, y) = 1 \text{ for all } x, y \in \Omega. \tag{4.3}$$

Furthermore these basis functions have local support: $B_{ij}(x, y)$ vanishes outside the so-called molecule M_i of vertex $V_i \in \mathcal{V}$, which is the union of all triangles $T \in \Delta$ containing V_i.

The basis functions $B_{ij}(x, y)$ can be obtained as follows: find three linearly independent triplets $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, $j = 1, 2, 3$, for each vertex $V_i \in \mathcal{V}$. $B_{ij}(x, y)$ is the unique solution of the interpolation problem (3.6) with $(f(V_k), D_x f(V_k), D_y f(V_k)) = (\delta_{ki}\alpha_{ij}, \delta_{ki}\beta_{ij}, \delta_{ki}\gamma_{ij})$, where δ_{ki} is the Kronecker delta.

The triplets $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, $j = 1, 2, 3$, must be determined in such a way that equations (4.2) and (4.3) are satisfied. To find appropriate triplets $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, $j = 1, 2, 3$, we use the algorithm from [12].

Algorithm 4.1.

1. *For each vertex* $V_i \in \mathcal{V}$, *find its PS-triangle points. These are the middle points of the edges in* Δ^{PS} *that contain vertex* V_i, *and vertex* V_i *itself. Figure 3 shows the PS-triangle points* $L, \tilde{L}, L'$ *and* V_1 *for the vertex* V_1 *in the triangle* $T(V_1, V_2, V_3)$.

2. *For each vertex* $V_i \in \mathcal{V}$, *find a triangle* $t_i(Q_{i1}, Q_{i2}, Q_{i3})$ *which contains all the PS-triangle points of* V_i *from all the triangles* T *in the molecule* M_i *of vertex* V_i. *These triangles* t_i, $i = 1, \ldots, \#\mathcal{V}$, *are called PS-triangles and we denote their vertices with* $Q_{ij}(X_{ij}, Y_{ij})$. *Figure 3 shows such a PS-triangle* t_1.

3. *Three linearly independent triplets of real numbers* $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$, $j = 1, 2, 3$, *can be derived from the PS-triangle* t_i *of a vertex* V_i *as follows:*

$$(\alpha_{i1}, \alpha_{i2}, \alpha_{i3}) \text{ are the barycentric coordinates of } V_i \text{ with respect to } t_i, \tag{4.4}$$

$$(\beta_{i1}, \beta_{i2}, \beta_{i3}) = \left(\frac{Y_{i2} - Y_{i3}}{e}, \frac{Y_{i3} - Y_{i1}}{e}, \frac{Y_{i1} - Y_{i2}}{e} \right), \tag{4.5}$$

$$(\gamma_{i1}, \gamma_{i2}, \gamma_{i3}) = \left(\frac{X_{i3} - X_{i2}}{e}, \frac{X_{i1} - X_{i3}}{e}, \frac{X_{i2} - X_{i1}}{e} \right), \tag{4.6}$$

where

$$e = \begin{vmatrix} X_{i1} & Y_{i1} & 1 \\ X_{i2} & Y_{i2} & 1 \\ X_{i3} & Y_{i3} & 1 \end{vmatrix}.$$

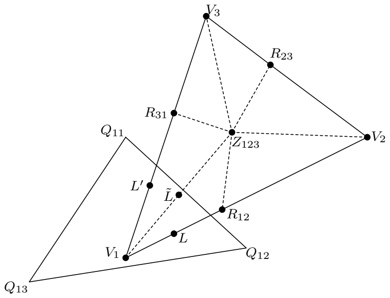


Figure 3: PS-triangle points and PS-triangle.

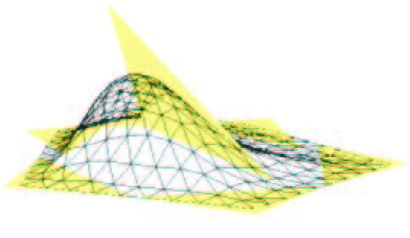


Figure 4: B-spline with control triangles.

This allows to define the useful notion of control triangles. First, we define with the notation introduced above the PS-control points as

$$C_{ij}(X_{ij}, Y_{ij}, c_{ij}). \tag{4.7}$$

For fixed i, they constitute a triangle $\hat{T}_i(C_{i1}, C_{i2}, C_{i3})$ that is tangent to the surface at $(V_i, s(V_i))$, see Figure 4. The projection of the control triangles $\hat{T}_i$ in the (x, y) plane are the PS-triangles t_i.

The following theorem states that the B-spline functions constructed by Algorithm 4.1 form a stable basis with respect to the L_∞ norm.

Theorem 4.2. *The B-spline basis is a stable basis for the $L_\infty(\Omega)$ norm, i.e,*

$$\left\| \sum_{i=1}^{\#\mathcal{V}} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y) \right\|_{L_\infty(\Omega)} \sim \|\mathbf{c}\|_\infty.$$

Proof. For the proof we refer to the work of Maes *et al.* [15]. ☐

Let us now return to the multiscale setting. For each level $l \geq 0$ we have a corresponding triangulation Δ_l of Ω. With each triangulation Δ_l we have a corresponding B-spline basis $\left\{B_{ij}^l\right\}$ for the space S_l. From Theorem 4.2 we get that

$$\|B_{ij}^l\|_{L_\infty(\Omega)} \sim 1. \tag{4.8}$$

If we combine this result with the local support of the basis functions and the regularity assumption (3.3) on the triangulation Δ_l, then we can deduce that [19]

$$\|B_{ij}^l\|_{L_2(\Omega)} \sim 3^{-l}.$$

These properties immediately imply that the bases $\{3^l B_{ij}^l\}$, $l \geq 0$, are uniformly L_2-stable, i.e.,

$$\left\| \sum_{i=1}^{\#\mathcal{V}_l} \sum_{j=1}^{3} c_{ij} B_{ij}^l(x, y) \right\|_{L_2(\Omega)} \sim 3^{-l} \|\mathbf{c}\|_{l_2}, \tag{4.9}$$

for arbitrary coefficients c_{ij}, and the constants of equivalence are independent of l. From the Markov inequality for polynomials in $\mathcal{P}_2$ (see e.g. [13]), the regularity assumption (3.3), and equation (4.8) it is not so difficult to prove that

$$\|D_x^\alpha D_y^\beta B_{ij}^l\|_{L_\infty(\Omega)} \lesssim 3^{l(\alpha+\beta)}, \qquad 0 \leq \alpha + \beta \leq 2. \tag{4.10}$$

From (4.1) and the construction of the B-spline basis we have

$$\begin{bmatrix} s_l(V_i) \\ D_x s_l(V_i) \\ D_y s_l(V_i) \end{bmatrix} = \mathbf{A}_i^l \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix} \quad \text{with} \quad \mathbf{A}_i^l = \begin{bmatrix} \alpha_{i1}^l & \alpha_{i2}^l & \alpha_{i3}^l \\ \beta_{i1}^l & \beta_{i2}^l & \beta_{i3}^l \\ \gamma_{i1}^l & \gamma_{i2}^l & \gamma_{i3}^l \end{bmatrix}, \qquad i = 1, \ldots, \#\mathcal{V}_l.$$

With Definition 2.1 in mind we will now introduce the following quasi-interpolants $Q_l : \mathcal{H} \to S_l$ given by

$$Q_l f = \sum_{i=1}^{\#\mathcal{V}_l} \sum_{j=1}^{3} \mu_{ij}^l(f) B_{ij}^l(x, y), \tag{4.11}$$

where the μ_{ij}^l are linear functionals of the form

$$\begin{bmatrix} \mu_{i1}^l(f) \\ \mu_{i2}^l(f) \\ \mu_{i3}^l(f) \end{bmatrix} := (\mathbf{A}_i^l)^{-1} \begin{bmatrix} f(V_i) \\ D_x f(V_i) \\ D_y f(V_i) \end{bmatrix}.$$

Clearly the operator Q_l satisfies

$$Q_l s_l = s_l, \qquad \forall s_l \in S_l,$$

and

$$Q_l f(V_i) = f(V_i), \qquad \nabla Q_l f(V_i) = \nabla f(V_i), \qquad i = 1, \ldots, \#\mathcal{V}_l.$$

The following proposition shows that the operators Q_l are suitable for constructing a multiresolution analysis.

Proposition 4.3. *For each $l \geq 0$ we have*

$$Q_l Q_{l+1} = Q_l.$$

Proof. From the construction of Q_l (4.11) we know that

$$Q_{l+1} f(V) = f(V), \qquad \nabla Q_{l+1} f(V) = \nabla f(V), \qquad \forall\, V \in \mathcal{V}_{l+1}.$$

Then it is also obvious that

$$Q_l Q_{l+1} f(V) = f(V), \qquad \nabla Q_l Q_{l+1} f(V) = \nabla f(V), \qquad \forall\, V \in \mathcal{V}_l \subset \mathcal{V}_{l+1},$$

and

$$Q_l f(V) = f(V), \qquad \nabla Q_l f(V) = \nabla f(V), \qquad \forall\, V \in \mathcal{V}_l \subset \mathcal{V}_{l+1}.$$

From the uniqueness of the interpolation problem (3.6) we conclude that $Q_l Q_{l+1} = Q_l$. □

The operators Q_l satisfy the following approximation property.

Proposition 4.4. *Let $n \geq l \geq 0$, then*

$$\|Q_l s_n\|_{L_2(\Omega)} \lesssim 3^{2(n-l)} \|s_n\|_{L_2(\Omega)}, \qquad s_n \in S_n.$$

Proof. Fix (x, y) in a triangle $T \in \Delta_l^{PS}$. From the work in [14] we can extract that

$$|Q_l f(x, y)| \lesssim \|f\|_{L_\infty(T)} + |T| (\|D_x f\|_{L_\infty(T)} + \|D_y f\|_{L_\infty(T)}).$$

If we apply this inequality to s_n in combination with the regularity assumption (3.3) and the Markov inequality for polynomials [13] we find that

$$\|Q_l s_n\|_{L_\infty(\Omega)} \lesssim \|s_n\|_{L_\infty(\Omega)} + 3^{-l} \cdot 3^n \|s_n\|_{L_\infty(\Omega)} \lesssim 3^{n-l} \|s_n\|_{L_\infty(\Omega)}.$$

If we use Theorem 4.2 and equation (4.9) then we find that

$$\begin{aligned} \|Q_l s_n\|_{L_2(\Omega)} &\lesssim 3^{-l} \cdot 3^{n-l} \|s_n\|_{L_\infty(\Omega)} \\ &\lesssim 3^{n-2l} \|\mathbf{c}\|_{L_\infty(\Omega)} \\ &\lesssim 3^{2n-2l} \|s_n\|_{L_2(\Omega)}. \end{aligned}$$

□

Until now we did not specify a Hilbert space $\mathcal{H}$. Note that the operator Q_l is not bounded on the space $L_2(\Omega)$. In the next section we will prove that certain Sobolev spaces are suitable for a multiresolution analysis.

5 Approximation properties

Many applications of multiresolution rely on the approximation properties of the spaces S_l. The main theorem of this section (Theorem 5.7) is build on these approximation properties and it states the existence of Hilbert spaces $H^s(\Omega)$ such that

$$\|f\|^2_{H^s(\Omega)} \sim \sum_{l=0}^{\infty} 3^{2ls} \|(Q_l - Q_{l-1}) f\|^2_{L_2(\Omega)}, \qquad f \in H^s(\Omega).$$

We mention that many of the proofs below are inspired by the work in [10, 16, 7, 8].

First we will introduce some appropriate function spaces. By $W_p^m(\Omega)$, $m \in \mathbb{N}$, $1 \leq p \leq \infty$, we mean the usual Sobolev space, i.e. the set of all functions in $L_p(\Omega)$ whose distributional derivatives of order less than or equal to m are in $L_p(\Omega)$. We can define the following norm for these Banach spaces

$$\|f\|^p_{W^m_p(\Omega)} = \sum_{\alpha+\beta\leq m} \|D^\alpha_x D^\beta_y f\|^p_{L_p(\Omega)},$$

with α and β positive integers. We will also use the semi-norm

$$|f|^p_{W^m_p(\Omega)} = \sum_{\alpha+\beta=m} \|D^\alpha_x D^\beta_y f\|^p_{L_p(\Omega)}.$$

For the special case $p = 2$ we will use the notation $H^m(\Omega) \equiv W^m_2(\Omega)$. These spaces $H^m(\Omega)$ are Hilbert spaces with inner product

$$\langle f, g\rangle_{H^m(\Omega)} = \sum_{\alpha+\beta\leq m} \langle D^\alpha_x D^\beta_y f, D^\alpha_x D^\beta_y g\rangle_{L_2(\Omega)}.$$

We also define spaces $W^s_p(\Omega)$ for arbitrary real values of $s \geq 0$ and $1 < p < \infty$. These spaces coincide for integer values of s with the spaces $W^m_p(\Omega)$. If s is not an integer, we write $s = m + \sigma$ where m is an integer and $0 < \sigma < 1$. Then $W^s_p(\Omega)$ is a Banach space with respect to the norm

$$\|f\|^p_{W^s_p(\Omega)} = \|f\|^p_{W^m_p(\Omega)} + \sum_{\alpha+\beta=m} \int_\Omega\int_\Omega \frac{|D^\alpha_x D^\beta_y f(\vec{u}) - D^\alpha_x D^\beta_y f(\vec{v})|^p}{|\vec{u} - \vec{v}|^{2+\sigma p}} d\vec{u}d\vec{v}.$$

Again for the special case $p = 2$ we write $H^s(\Omega) \equiv W^s_2(\Omega)$ and the spaces $H^s(\Omega)$ are Hilbert spaces for arbitrary real values of $s \geq 0$ with inner product

$$\langle f, g\rangle_{H^s(\Omega)} = \langle f, g\rangle_{H^m(\Omega)} + \sum_{\alpha+\beta=m} \int_\Omega\int_\Omega \frac{(D^\alpha_x D^\beta_y (f(\vec{u}) - f(\vec{v})))(D^\alpha_x D^\beta_y (g(\vec{u}) - g(\vec{v})))}{|\vec{u} - \vec{v}|^{2(1+\sigma)}} d\vec{u}d\vec{v}.$$

Proposition 5.1. *The operator Q_l is bounded on $H^s(\Omega)$ with $s \in (2, 3]$.*

Proof. We will prove that $H^s(\Omega) \subset C^1(\overline{\Omega})$ where $C^1(\overline{\Omega})$ is a Banach space with norm

$$\|f\|_{C^1(\overline{\Omega})} = \max_{0\leq\alpha+\beta\leq1} \sup_{(x,y)\in\Omega} |D^\alpha_x D^\beta_y f(x, y)|.$$

Then the boundedness of Q_l follows from the Uniform Boundedness Principle. We know that Ω satisfies the strong local Lipschitz property and the uniform cone property. The special case $s = 3$ follows immediately from the Sobolev imbedding theorem (cf. Theorem 5.4 in [1]). Now suppose that $2 < s < 3$. Define $q = \frac{2}{3-s}$, then $2 < q < \infty$. By the direct imbedding theorem for fractional order Sobolev spaces (cf. Theorem 7.58 in [1]) we have that

$$H^s(\Omega) \equiv W^s_2(\Omega) \subset W^2_q(\Omega).$$

The Sobolev imbedding theorem (cf. Theorem 5.4 in [1]) yields that for $q > 2$

$$W^2_q(\Omega) \subset C^{1,\lambda}(\overline{\Omega}), \qquad 0 < \lambda \leq 1 - \frac{2}{q} = s - 2,$$

where the space $C^{1,\lambda}(\overline{\Omega})$ is defined as the subspace of $C^1(\overline{\Omega})$ consisting of those functions f for which $D_x f$ and $D_y f$ satisfy in Ω a Hölder condition of exponent λ. This yields $H^s(\Omega) \subset C^1(\overline{\Omega})$. □

The following theorem states that the quasi-interpolation operator Q_l maps $H^s(\Omega)$, $2 < s \leq 3$, into S_l such that $Q_l f$ approximates f and its derivatives to optimal order.

Theorem 5.2. *For each $f \in H^3(\Omega)$ we have that*

$$\|D^\alpha_x D^\beta_y (f - Q_l f)\|_{L_2(\Omega)} \lesssim 3^{-(3-\alpha-\beta)l} |f|_{H^3(\Omega)}$$

for all $0 \leq \alpha + \beta \leq 3$, and for each $f \in H^s(\Omega)$, $2 < s < 3$, we have that

$$\|D^\alpha_x D^\beta_y (f - Q_l f)\|_{L_2(\Omega)} \lesssim 3^{-(2-\alpha-\beta)l} |f|_{H^2(\Omega)}$$

for all $0 \leq \alpha + \beta \leq 2$.

Proof. The proof of this theorem is based on the well-known Bramble–Hilbert lemma [2]. For a detailed proof we refer to the work of Maes *et al.* [14]. ☐

From Theorem 5.2 it is clear that

$$\lim_{l\to\infty} \|f - Q_l f\|_{L_2(\Omega)} = 0$$

for all functions $f \in H^s(\Omega)$, $2 < s \le 3$. From this we obtain that such a function $f \in H^s(\Omega)$ can be decomposed as

$$f = \sum_{l=0}^{\infty} g_l, \qquad g_l \in S_l,$$

in the sense of L_2. Moreover, we can also use the decomposition

$$f = \sum_{l=0}^{\infty} (Q_l - Q_{l-1}) f,$$

with $Q_{-1} := 0$.

We introduce the difference operator

$$(\Delta^r_{\vec{h}} f)(x) := \sum_{j=0}^{r} \left(\begin{array}{c} r \\ j \end{array} \right) (-1)^{r-j} f((x,y) + j\vec{h}), \qquad (x,y) \in \mathbb{R}^2,$$

and define the r-th order L_2-modulus of smoothness of $f \in L_2(\Omega)$ (see e.g. [9])

$$\omega_r(f,t) := \sup_{|\vec{h}|\le t} \|\Delta^r_{\vec{h}} f\|_{L_2(\Omega(r\vec{h}))}, \tag{5.1}$$

where $\Omega(r\vec{h}) := \{(x,y) \in \Omega : (x,y) + j\vec{h} \in \Omega,\ j = 0,\dots,r\}$. The r-th order L_2-modulus of smoothness has the following properties

$$\begin{array}{rcl} \omega_r(f,t) & \le & 2^r \|f\|_{L_2(\Omega)}, \\ \lim_{t\to 0_+} \omega_r(f,t) & = & 0, \\ \omega_r(f+g,t) & \le & \omega_r(f,t) + \omega_r(g,t). \end{array} \tag{5.2}$$

A deeper property of moduli of smoothness is the equivalence with certain K-functionals. Given $f \in L_2(\Omega)$ define

$$K_r(f,t) := \inf_{g\in H^r(\Omega)} \left\{ \|f-g\|_{L_2(\Omega)} + t^r |g|_{H^r(\Omega)} \right\}. \tag{5.3}$$

Given that Ω is a bounded polygonal domain in $\mathbb{R}^2$ one can obtain that

$$K_r(f,t) \sim \omega_r(f,t). \tag{5.4}$$

See [6] for the result and some references.

For a function $f \in L_2(\Omega)$, let

$$E_l(f) := \inf_{g\in S_l} \|f-g\|_{L_2(\Omega)}, \qquad l \ge 0$$

be the error in approximating f by the elements of S_l. We will now prove the classical Jackson and Bernstein inequalities for the space S_l of Powell–Sabin splines on a triangulation Δ_l.

Theorem 5.3 (Jackson estimate). *For each $f \in H^3(\Omega)$, we have the Jackson inequality*

$$E_l(f) \lesssim \omega_3(f, 3^{-l}). \tag{5.5}$$

Proof. From Theorem 5.2 we know that

$$\|g - Q_l g\|_{L_2(\Omega)} \lesssim 3^{-3l} |g|_{H^3(\Omega)}, \qquad g \in H^3(\Omega).$$

Then

$$\begin{aligned} \|f - Q_l g\|_{L_2(\Omega)} &\le \|f - g\|_{L_2(\Omega)} + \|g - Q_l g\|_{L_2(\Omega)} \\ &\lesssim \|f - g\|_{L_2(\Omega)} + 3^{-3l} |g|_{H^3(\Omega)} \end{aligned}$$

From the equations (5.3) and (5.4) we get

$$\inf_{g \in S_l} \|f - Q_l g\|_{L_2(\Omega)} \lesssim \omega_3(f, 3^{-l}).$$

□

Theorem 5.4 (Bernstein estimate). *For each $l \ge 0$, and each $r = 1, 2, 3$, we have for $\lambda := r - \frac{1}{2}$ the Bernstein inequality*

$$\omega_r(g_l, t) \lesssim \left(\min\{1, 3^l t\}\right)^{\lambda} \|g_l\|_{L_2(\Omega)}, \qquad g_l \in S_l. \tag{5.6}$$

Proof. Since for $t \ge 3^{-l}$ this inequality reduces to

$$\omega_r(g_l, t) \lesssim \|g_l\|_{L_2(\Omega)},$$

which directly follows from (5.2), we concentrate on $t < 3^{-l}$. From the definition of the operator Q_l (4.11) we can write

$$g_l(x, y) = Q_l g_l(x, y) = \sum_{i=1}^{\#\mathcal{V}_l} \sum_{j=1}^{3} \mu_{ij}^l(g_l) B_{ij}^l(x, y),$$

and also

$$(\Delta_{\vec{h}}^r g_l)(x, y) = \sum_{i=1}^{\#\mathcal{V}_l} \sum_{j=1}^{3} \mu_{ij}^l(g_l) (\Delta_{\vec{h}}^r B_{ij}^l)(x, y).$$

For any $(x, y) \in \Omega$ at most 9 B-splines are nonzero at (x, y), hence

$$|(\Delta_{\vec{h}}^r g_l)(x, y)|^2 \le 9 \sum_{i=1}^{\#\mathcal{V}_l} \sum_{j=1}^{3} |\mu_{ij}^l(g_l)|^2 |(\Delta_{\vec{h}}^r B_{ij}^l)(x, y)|^2. \tag{5.7}$$

We shall give two estimates for $|(\Delta_{\vec{h}}^r B_{ij}^l)(x, y)|$. First define Γ_{ij}^l as the set of all (x, y) such that (x, y) and $(x, y) + r\vec{h}$ are in the same triangle $T \in \Delta_l^{PS}$ and $\operatorname{supp} B_{ij}^l \cap T \neq 0$. Then B_{ij}^l is a polynomial on T whose r-th order derivatives can be bounded by the Markov inequality for polynomials (4.10) and we find that

$$|(\Delta_{\vec{h}}^r B_{ij}^l)(x, y)| \lesssim (3^l |\vec{h}|)^r, \qquad (x, y) \in \Gamma_{ij}^l. \tag{5.8}$$

The second estimate is for the set $\tilde{\Gamma}_{ij}^l$ which consists of all (x, y) such that (x, y) and $(x, y) + r\vec{h}$ are in different triangles from Δ^{PS} and B_{ij}^l does not vanish identically on both of these triangles. It is easy to see that $B_{ij}^l \in W_\infty^{r-1}(\Omega)$. Hence B_{ij}^l has $(r-1)$-th order derivatives whose $L_\infty(\Omega)$ norms do not exceed $c3^{l(r-1)}$ (4.10). We find that

$$|(\Delta_{\vec{h}}^r B_{ij}^l)(x, y)| \lesssim (3^l |\vec{h}|)^{r-1}, \qquad (x, y) \in \tilde{\Gamma}_{ij}^l. \tag{5.9}$$

The set Γ_{ij}^l has measure $\lesssim (3^{-l})^2$ because the support of B_{ij}^l has measure $\lesssim (3^{-l})^2$, and a similar argument shows that $\tilde{\Gamma}_{ij}^l$ has measure $\lesssim |\vec{h}| 3^{-l}$.

If we combine the estimates (5.8) and (5.9) with the estimates for the measures of Γ^l_{ij} and $\tilde{\Gamma}^l_{ij}$ we obtain

$$\begin{aligned}\int_{\Omega(r\vec{h})} |(\Delta^r_{\vec{h}} B^l_{ij})(x,y)|^2 &\lesssim (3^l|\vec{h}|)^{2r}(3^{-l})^2 + (3^l|\vec{h}|)^{2(r-1)}|\vec{h}|3^{-l} \\ &\lesssim |\vec{h}|^{2\lambda}3^{2l\lambda}3^{-2l} \qquad (5.10)\end{aligned}$$

where we have used that $|\vec{h}| \le t < 3^{-l}$.

We integrate (5.7) and use (5.10) to find

$$\|\Delta^r_{\vec{h}} g_l\|^2_{L_2(\Omega(r\vec{h}))} \lesssim \sum_{i=1}^{\#\mathcal{V}_l}\sum_{j=1}^{3} |\mu^l_{ij}(g_l)|^2|\vec{h}|^{2\lambda}3^{2l\lambda}3^{-2l},$$

and because $\sum_{i=1}^{\#\mathcal{V}_l}\sum_{j=1}^{3} 3^{-2l}|\mu^l_{ij}(g_l)|^2 \sim \|g_l\|^2_{L_2(\Omega)}$ (4.9) we get

$$\|\Delta^r_{\vec{h}} g_l\|^2_{L_2(\Omega(r\vec{h}))} \lesssim (|\vec{h}|3^l)^{2\lambda}\|g_l\|^2_{L_2(\Omega)}.$$

□

We now define the scale of auxiliary spaces $A^s_{p,q}(\{S_l\})$ as follows:

Definition 5.5. A function $f \in L_p(\Omega)$ belongs to $A^s_{p,q} \equiv A^s_{p,q}(\{S_l\})$ for some fixed $s \ge 0$, $1 \le p, q \le \infty$ if there exists a sequence $g_l \in S_l$, $l = 0, 1, \dots$ such that $f = \sum_{l=0}^{\infty} g_l$ in the sense of L_p, and $\|\{3^{ls}\|g_l\|_{L_p(\Omega)}\}\|_{l_q} < \infty$. The norm on $A^s_{p,q}$ is defined as

$$\|f\|_{A^s_{p,q}} = \inf \left\|3^{ls}\|g_l\|_{L_p(\Omega)}\right\|_{l_q}$$

where the infimum must be taken with respect to all admissible representations $\sum_{l=0}^{\infty} g_l$ of f.

In order to work with the abstract $A^s_{p,q}$-spaces in applications, we must relate them to the more convenient function spaces of Besov–Sobolev type. Let $B^s_{p,q}(\Omega)$, $s > 0$, $1 \le p, q \le \infty$, denote the Besov spaces, i.e., the Banach spaces of functions $f \in L_p(\Omega)$ with the norm

$$\|f\|_{B^s_{p,q}(\Omega)} := \|f\|_{L_p(\Omega)} + \|\{3^{ls}\omega_r(f, 3^{-l})\}\|_{l_q} < \infty,$$

where the choice of the integer $r > s$ is arbitrary. It is well known that

$$H^s(\Omega) \equiv W^s_2(\Omega) \cong B^s_{2,2}(\Omega), \qquad s > 0. \qquad (5.11)$$

The following fact can be extracted from the results in [16].

Proposition 5.6. *Suppose the nested spaces $\{S_l\}$ satisfy Jackson estimates (5.5) for all $f \in H^3(\Omega)$, as well as Bernstein estimates (5.6) for $r = 3$, then*

$$A^s_{2,2} \cong B^s_{2,2}(\Omega), \qquad 0 < s < \frac{5}{2}.$$

Equation (5.11) and Proposition 5.6 yield

$$\|f\|^2_{H^s(\Omega)} \sim \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} 3^{2ls}\|g_l\|^2_{L_2(\Omega)}, \qquad 0 < s < \frac{5}{2}. \qquad (5.12)$$

We can now prove the main theorem of this section.

Theorem 5.7. *Choose $s \in (2, \frac{5}{2})$. Then it holds that*

$$\|f\|^2_{H^s(\Omega)} \sim \sum_{l=0}^{\infty} 3^{2ls}\|(Q_l - Q_{l-1})f\|^2_{L_2(\Omega)}, \qquad f \in H^s(\Omega). \qquad (5.13)$$

Proof. First we mention that from Proposition 5.1 it follows that $(Q_l - Q_{l-1})f$ is well defined for all $f \in H^s(\Omega)$ and $s \in (2, \frac{5}{2})$.

From (5.12) it is sufficient to prove that

$$\inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} 3^{2ls} \|g_l\|_{L_2(\Omega)}^2 \sim \sum_{l=0}^{\infty} 3^{2ls} \|(Q_l - Q_{l-1})f\|_{L_2(\Omega)}^2.$$

Since $(Q_l - Q_{l-1})f \in S_l$ and $\sum_{l=0}^{\infty}(Q_l - Q_{l-1})f = f$ the inequality "$\lesssim$" is trivial and we will concentrate on the inequality "$\gtrsim$". Let $f = \sum_{l=0}^{\infty} g_l$ with $g_l \in S_l$. Since the quasi-interpolants Q_l are projectors and the spaces S_l are nested, we have $(Q_l - Q_{l-1})S_n = 0$ when $n \le l-1$. From this, Cauchy Schwartz and Proposition 4.4 we have

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty} \sum_{l=0}^{\infty} 3^{2ls} \langle (Q_l - Q_{l-1})g_n, (Q_l - Q_{l-1})g_{n'} \rangle_{L_2(\Omega)} \\
&= \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{2ls} \langle (Q_l - Q_{l-1})g_n, (Q_l - Q_{l-1})g_{n'} \rangle_{L_2(\Omega)} \\
&\le \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{2ls} \left(\|Q_l g_n\|_{L_2(\Omega)} + \|Q_{l-1} g_n\|_{L_2(\Omega)} \right) \left(\|Q_l g_{n'}\|_{L_2(\Omega)} + \|Q_{l-1} g_{n'}\|_{L_2(\Omega)} \right) \\
&\lesssim \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{2ls} 3^{2(n+n')-4l} \|g_n\|_{L_2(\Omega)} \|g_{n'}\|_{L_2(\Omega)}.
\end{aligned}$$

The last expression can be rewritten as

$$\sum_{n,n'=0}^{\infty} \sum_{l=0}^{\min\{n,n'\}} 3^{(s-2)(2l-n-n')} (3^{ns} \|g_n\|_{L_2(\Omega)}) (3^{n's} \|g_{n'}\|_{L_2(\Omega)}),$$

which is equivalent to

$$\sum_{n,n'=0}^{\infty} 3^{(s-2)(2\min\{n,n'\}-n-n')} (3^{ns} \|g_n\|_{L_2(\Omega)}) (3^{n's} \|g_{n'}\|_{L_2(\Omega)}).$$

The factor $3^{(s-2)(2\min\{n,n'\}-n-n')}$ becomes very small if $|n - n'| \gg 0$. In fact, the infinite matrix $[3^{(s-2)(2\min\{n,n'\}-n-n')}]_{n,n' \in \mathbb{N}}$ defines a bounded mapping on l_2. Therefore

$$\sum_{n,n'=0}^{\infty} 3^{(s-2)(2\min\{n,n'\}-n-n')} (3^{ns} \|g_n\|_{L_2(\Omega)}) (3^{n's} \|g_{n'}\|_{L_2(\Omega)}) \lesssim \sum_{n=0}^{\infty} 3^{2ns} \|g_n\|_{L_2(\Omega)}^2.$$

Since the splitting $f = \sum_{l=0}^{\infty} g_l$ was arbitrary, we have derived that

$$\inf_{g_l \in S_l : f = \sum_l g_l} \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\infty} 3^{2ls} \langle (Q_l - Q_{l-1})g_n, (Q_l - Q_{l-1})g_{n'} \rangle_{L_2(\Omega)} \lesssim \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{n=0}^{\infty} 3^{2ns} \|g_n\|_{L_2(\Omega)}^2.$$

Because $f \in A_{2,2}^s$ (Proposition 5.6) we know that the right expression is bounded. Then from the derivation made above it follows that the left expression is absolutely convergent and we are allowed to write that

$$\begin{aligned}
&\inf_{g_l \in S_l : f = \sum_l g_l} \sum_{n,n'=0}^{\infty} \sum_{l=0}^{\infty} 3^{2ls} \langle (Q_l - Q_{l-1})g_n, (Q_l - Q_{l-1})g_{n'} \rangle_{L_2(\Omega)} \\
&= \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} \sum_{n,n'=0}^{\infty} 3^{2ls} \langle (Q_l - Q_{l-1})g_n, (Q_l - Q_{l-1})g_{n'} \rangle_{L_2(\Omega)} \\
&= \sum_{l=0}^{\infty} 3^{2ls} \|(Q_l - Q_{l-1})f\|_{L_2(\Omega)}^2.
\end{aligned}$$

We conclude that

$$\sum_{l=0}^{\infty} 3^{2ls} \|(Q_l - Q_{l-1})f\|^2_{L_2(\Omega)} \lesssim \inf_{g_l \in S_l : f = \sum_l g_l} \sum_{l=0}^{\infty} 3^{2ls} \|g_l\|^2_{L_2(\Omega)}.$$

□

As a consequence of (5.12) we can show that the nested sequence of spaces S_l is dense in $H^s(\Omega)$, $2 < s < \frac{5}{2}$.

Proposition 5.8. *The spaces* $\{S_l\}_{l=0}^{\infty}$ *are dense in* $H^s(\Omega)$*,* $2 < s < \frac{5}{2}$*.*

Proof. Let $f \in H^s(\Omega)$ and write $f = \sum_l h_l$ with $h_l \in S_l$ in the sense of L_2. It is sufficient to prove that

$$\lim_{n \to \infty} \inf_{g \in S_n} \|f - g\|_{H^s(\Omega)} = 0.$$

From (5.12) we get that

$$\begin{aligned} \lim_{n \to \infty} \inf_{g \in S_n} \|f - g\|^2_{H^s(\Omega)} &= \lim_{n \to \infty} \inf_{g_l \in S_l} \|f - \sum_{l=0}^{n} g_l\|^2_{H^s(\Omega)} \\ &\leq \inf_{g_l \in S_l} \|f - \lim_{n \to \infty} \sum_{l=0}^{n} g_l\|^2_{H^s(\Omega)} \\ &\leq c \inf_{g_l, h_l \in S_l} \sum_{l=0}^{\infty} 3^{2ls} \|h_l - g_l\|^2_{L_2(\Omega)} \equiv 0. \end{aligned}$$

□

6 Stable Powell–Sabin wavelets

In [20], Vanraes *et al.* present a subdivision scheme to compute a representation (4.1) of a PS-spline on a triadic refinement Δ_{l+1} of Δ_l. This subdivision scheme is used as the prediction step in the Lifting Scheme [18] to create second generation Powell–Sabin wavelets in [19].

It is convenient to write the following in matrix form. To indicate scaling and wavelet functions we will use the same notation as in Section 2. From (4.9) we know that the bases $\{3^l B^l_{ij}\}$, $l \geq 0$, are uniformly L_2-stable. Therefore we define new basisfunctions $\boldsymbol{\phi}_l := 3^l \mathbf{B}_l$ for S_l such that

$$s_l(x, y) = \boldsymbol{\phi}_l \mathbf{c}_l \qquad \text{and} \qquad \|\boldsymbol{\phi}_l \mathbf{c}_l\|_{L_2(\Omega)} \sim \|\mathbf{c}_l\|_{l_2}.$$

We can do the same for the complement spaces W_l and define $\boldsymbol{\psi}_l$ as a set of basis functions for W_l. A spline s_{l+1} in $S_{l+1} = S_l \oplus W_l$ can now be written as

$$s_{l+1}(x, y) = \boldsymbol{\phi}_l \mathbf{c}_l + \boldsymbol{\psi}_l \mathbf{d}_l.$$

From the construction in [19] we know the existence of matrices $\mathbf{P}_l$ and $\mathbf{Q}_l$ such that

$$[\boldsymbol{\phi}_l \quad \boldsymbol{\psi}_l] = \boldsymbol{\phi}_{l+1} [\mathbf{P}_l \quad \mathbf{Q}_l],$$

and such that the wavelets $\boldsymbol{\psi}_l$ (see Figure 5) have one vanishing moment, i.e. they satisfy

$$\langle 1, \psi_{l,m} \rangle_{L_2(\Omega)} = 0, \qquad \forall m \in J_l.$$

Denote $\mathbf{M}_l = [\mathbf{P}_l \quad \mathbf{Q}_l]$, then we can find a multiscale representation for $\boldsymbol{\phi}_{l+1}$ as

$$[\boldsymbol{\phi}_0 \quad \boldsymbol{\psi}_0 \quad \boldsymbol{\psi}_1 \cdots \boldsymbol{\psi}_{l-1} \quad \boldsymbol{\psi}_l] = \boldsymbol{\phi}_{l+1} \mathbf{T}_l,$$

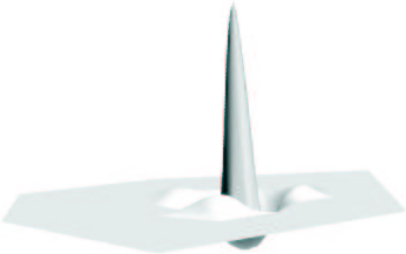


Figure 5: The Powell–Sabin wavelet $\psi_{l,m}$.

$$\mathbf{T}_l = \begin{bmatrix} \mathbf{M}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{l-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

When doing the wavelet transform there should be no significant loss of accuracy in the data, or in other words the condition numbers should remain uniformly bounded

$$\|\mathbf{T}_l\|, \|(\mathbf{T}_l)^{-1}\| = \mathcal{O}(1), \qquad l \to \infty. \tag{6.1}$$

From Dahmen [4, 5] we know that the wavelet transform $\mathbf{T}_l$ is uniformly stable, i.e. (6.1) holds, if the multiscale basis $\boldsymbol{\psi} := \bigcup_{l=-1}^{\infty} \boldsymbol{\psi}_l$ is a Riesz basis. A necessary condition is that each basis $\boldsymbol{\psi}_l$ is a uniformly stable Riesz basis or alternatively that the condition numbers of the one level transforms $\mathbf{M}_l$ are uniformly bounded in l, i.e.

$$\|\mathbf{M}_l\|, \|(\mathbf{M}_l)^{-1}\| = \mathcal{O}(1). \tag{6.2}$$

Equation (6.2) was already proven in [19] and the wavelets $\boldsymbol{\psi}_l$ form a L_2-stable Riesz basis for W_l. Evidently the basis $\boldsymbol{\phi}_l \cup \boldsymbol{\psi}_l$ is also a L_2-stable Riesz basis for S_{l+1}. Thus, according to Definition 2.2, the multiscale basis $\boldsymbol{\psi}$ forms a weakly stable basis for $L_2(\Omega)$.

As a consequence of the Riesz Representation Theorem, (6.2) is equivalent to the existence of another L_2-stable Riesz basis $\tilde{\boldsymbol{\phi}}_l \cup \tilde{\boldsymbol{\psi}}_l = \{\tilde{\phi}_{l,m} : m \in I_l\} \cup \{\tilde{\psi}_{l,m} : m \in J_l\}$ which is biorthogonal to $\boldsymbol{\phi}_l \cup \boldsymbol{\psi}_l$, i.e.

$$\begin{aligned}
\langle \phi_{l,m}, \tilde{\phi}_{l,n} \rangle_{L_2(\Omega)} &= \delta_{m,n}, \qquad m,n \in I_l, \\
\langle \psi_{l,m}, \tilde{\psi}_{l,n} \rangle_{L_2(\Omega)} &= \delta_{m,n}, \qquad m,n \in J_l, \\
\langle \phi_{l,m}, \tilde{\psi}_{l,n} \rangle_{L_2(\Omega)} &= 0, \qquad m \in I_l, n \in J_l, \\
\langle \tilde{\phi}_{l,m}, \psi_{l,n} \rangle_{L_2(\Omega)} &= 0, \qquad m \in I_l, n \in J_l.
\end{aligned}$$

The dual scaling functions $\tilde{\boldsymbol{\phi}}_l$ span a space $\tilde{S}_l$ and the dual wavelet functions $\tilde{\boldsymbol{\psi}}_l$ span a space $\tilde{W}_l$. Let $f \in S_{l+1}$ and $\tilde{f} \in \tilde{S}_{l+1}$, then we have the following expansions,

$$f = \sum_{m \in I_l} \langle f, \tilde{\phi}_{l,m} \rangle_{L_2(\Omega)} \phi_{l,m} + \sum_{m \in J_l} \langle f, \tilde{\psi}_{l,m} \rangle_{L_2(\Omega)} \psi_{l,m}, \tag{6.3}$$

$$\tilde{f} = \sum_{m \in I_l} \langle \tilde{f}, \phi_{l,m} \rangle_{L_2(\Omega)} \tilde{\phi}_{l,m} + \sum_{m \in J_l} \langle \tilde{f}, \psi_{l,m} \rangle_{L_2(\Omega)} \tilde{\psi}_{l,m}. \tag{6.4}$$

Proposition 6.1. *The dual wavelets $\tilde{\boldsymbol{\psi}}_l$ have three vanishing moments, i.e.*

$$\langle \pi, \tilde{\psi}_{l,m} \rangle_{L_2(\Omega)} = 0, \qquad \forall \pi \in \mathcal{P}_2, \quad l \geq 0, \quad m \in J_l,$$

where $\mathcal{P}_2$ is the space of all bivariate polynomials of total degree at most 2.

Proof. Because $\pi \in S_l$ we get from (6.3) that $\sum_{m \in J_l} \langle \pi, \tilde{\psi}_{l,m} \rangle_{L_2(\Omega)} \psi_{l,m} = 0$. The wavelet functions $\{\psi_{l,m} : m \in J_l\}$ form a basis for W_l, thus $\langle \pi, \tilde{\psi}_{l,m} \rangle_{L_2(\Omega)} = 0$. □

The dual basis $\tilde{\boldsymbol{\phi}}_l \cup \tilde{\boldsymbol{\psi}}_l$ is not unique. We will construct a dual basis such that

$$\begin{aligned} \text{diam}(\text{supp } \tilde{\phi}_{l,m}) &\sim \text{diam}(\text{supp } \phi_{l,m}), & (6.5)\\ \text{diam}(\text{supp } \tilde{\psi}_{l,m}) &\sim \text{diam}(\text{supp } \psi_{l,m}). & (6.6)\end{aligned}$$

Consider all the scaling functions $\boldsymbol{\phi}_{M_i^l} := \{\phi_{l,m}\}$ whose support vanishes not identically on the molecule M_i^l of vertex $V_i \in \mathcal{V}_l$, which is the union of triangles in $\mathcal{T}_l$ that contain this vertex $V_i \in \mathcal{V}_l$. By construction we know that there are (at least) 3 scaling functions in the set $\boldsymbol{\phi}_{M_i^l}$ whose support coincides with the molecule M_i^l. It is also clear that the set $(\boldsymbol{\phi}_{M_i^l})|_{M_i^l}$ forms a stable basis for $S_l|_{M_i^l}$. Define $\mathbf{G}$ as the Gram matrix of the functions in $\boldsymbol{\phi}_{M_i^l}$, restricted to the domain M_i^l. Thus after reordering some indices we can write that $\mathbf{G}_{p,q} = \langle \phi_{l,p}, \phi_{l,q}\rangle_{L_2(M_i^l)}$ with $\phi_{l,p}$ and $\phi_{l,q}$ members of the set $\boldsymbol{\phi}_{M_i^l}$ and $p, q \in \{1, \ldots, \#\boldsymbol{\phi}_{M_i^l}\}$. The matrix $\mathbf{G}$ is invertible and we write $\mathbf{G}^{-1}_{p,q} = b_{p,q}$. Now define $\tilde{\phi}_{l,q} := \sum_p b_{p,q}\phi_{l,p}|_{M_i^l}$, then for an arbitrary function $\phi_{l,r} \in \boldsymbol{\phi}_{M_i^l}$,

$$\langle \tilde{\phi}_{l,q}, \phi_{l,r}\rangle_{L_2(\Omega)} = \sum_p b_{p,q}\langle \phi_{l,p}, \phi_{l,r}\rangle_{L_2(M_i^j)} = \sum_p b_{p,q}\mathbf{G}_{p,r} = \delta_{q,r},$$

thus the set $\tilde{\boldsymbol{\phi}}_{M_i^l} := \{\tilde{\phi}_{l,q}\}$ is biorthogonal with respect to $\boldsymbol{\phi}_{M_i^l}$ and the supports of the dual functions $\tilde{\phi}_{l,q}$ satisfy diam(supp $\tilde{\phi}_{l,q}$) $\sim$ diam(M_i^l). Let $\phi_{l,m} \in \boldsymbol{\phi}_{M_i^l}$ be one of the scaling functions with supp $\phi_{l,m} = M_i^l$ and let $\tilde{\phi}_{l,m} \in \tilde{\boldsymbol{\phi}}_{M_i^l}$ be its corresponding dual function. Then we have that $\langle \tilde{\phi}_{l,m}, \phi_{l,n}\rangle_{L_2(\Omega)} = \delta_{m,n}$ for all functions $\phi_{l,n} \in \boldsymbol{\phi}_{M_i^l}$. Moreover we have that $\langle \tilde{\phi}_{l,m}, \phi_{l,n}\rangle_{L_2(\Omega)} = \delta_{m,n}$ for all functions $\phi_{l,n}$ for all $n \in I_l$. This way we construct a dual basis $\tilde{\boldsymbol{\phi}}_l$ for $\boldsymbol{\phi}_l$ with diam(supp $\tilde{\phi}_{l,m}$) $\sim$ diam(supp $\phi_{l,m}$). The construction of a dual basis $\tilde{\boldsymbol{\psi}}_l$ for $\boldsymbol{\psi}_l$ with diam(supp $\tilde{\psi}_{l,m}$) $\sim$ diam(supp $\psi_{l,m}$) is analogous. And of course there will also exist a dual basis $\tilde{\boldsymbol{\phi}}_l \cup \tilde{\boldsymbol{\psi}}_l$ for $\boldsymbol{\phi}_l \cup \boldsymbol{\psi}_l$ such that both (6.5) and (6.6) are satisfied.

The following proposition summarizes the key properties of our wavelets and their dual counterparts.

Proposition 6.2. *The elements of $\boldsymbol{\psi}_l$ and $\tilde{\boldsymbol{\psi}}_l$ all have compact support that scales properly, i.e.*

$$\textit{diam(supp } \psi_{l,m}) \sim \textit{diam(supp } \tilde{\psi}_{l,m}) \sim 3^{-l}. \tag{6.7}$$

The wavelets are normalized in L_2, i.e.

$$\|\psi_{l,m}\|_{L_2(\Omega)} \sim \|\tilde{\psi}_{l,m}\|_{L_2(\Omega)} \sim 1. \tag{6.8}$$

The wavelets satisfy the following cancellation properties,

$$\begin{aligned} |\langle f, \psi_{l,m}\rangle_{L_2(\Omega)}| &\lesssim 3^{-l}|f|_{H^1(\Omega)}, & (6.9)\\ |\langle f, \tilde{\psi}_{l,m}\rangle_{L_2(\Omega)}| &\lesssim 3^{-3l}|f|_{H^3(\Omega)}. & (6.10)\end{aligned}$$

Proof. (6.7) follows immediately from the construction of $\boldsymbol{\psi}_l$ and $\tilde{\boldsymbol{\psi}}_l$, and (6.8) is a trivial consequence from the fact that both $\boldsymbol{\psi}_l$ and $\tilde{\boldsymbol{\psi}}_l$ form a L_2-stable Riesz basis for W_l respectively $\tilde{W}_l$. The vanishing moments of $\psi_{l,m}$ and $\tilde{\psi}_{l,m}$ imply that

$$\begin{aligned} |\langle f, \psi_{l,m}\rangle_{L_2(\Omega)}| &= \inf_{c\in\mathbb{R}} |\langle f-c, \psi_{l,m}\rangle_{L_2(\Omega)}|\\ &\leq \inf_{c\in\mathbb{R}} \|f-c\|_{L_2(\Omega)}\|\psi_{l,m}\|_{L_2(\Omega)},\\ |\langle f, \tilde{\psi}_{l,m}\rangle_{L_2(\Omega)}| &= \inf_{\pi\in\mathcal{P}_2} |\langle f-\pi, \tilde{\psi}_{l,m}\rangle_{L_2(\Omega)}|\\ &\leq \inf_{\pi\in\mathcal{P}_2} \|f-\pi\|_{L_2(\Omega)}\|\tilde{\psi}_{l,m}\|_{L_2(\Omega)}.\end{aligned}$$

Standard estimates on local polynomial approximation (see e.g. [10, 11]) yield

$$\inf_{\pi\in\mathcal{P}_k} \|f-\pi\|_{L_2(G)} \lesssim (\text{diam } G)^{k+1}|f|_{H^{k+1}(G)}.$$

Together with (6.7) and (6.8) we find (6.9) and (6.10). ☐

As a consequence of Theorem 5.7 we can also prove that the basis $\bigcup_{l=0}^{\infty}\{3^{-sl}\boldsymbol{\psi}_{l-1}\}$ is a strongly stable basis for $H^s(\Omega)$, $2 < s < \frac{5}{2}$.

Theorem 6.3. *The multiscale basis* $\bigcup_{l=0}^{\infty}\{3^{-sl}\boldsymbol{\psi}_{l-1}\}$ *is a strongly stable basis for* $H^s(\Omega)$*,* $2 < s < \frac{5}{2}$*, in the context of Definition 2.2.*

Proof. Since the wavelets $\boldsymbol{\psi}_l$ form a L_2-stable Riesz basis for W_l we find from Theorem 5.7 that

$$\begin{aligned}\|f\|^2_{H^s(\Omega)} &\sim \sum_{l=-1}^{\infty} 3^{2(l+1)s} \| \sum_{m\in J_l} d_{l,m}\psi_{l,m}\|^2_{L_2(\Omega)} \\ &\sim \sum_{l=-1}^{\infty} 3^{2(l+1)s} \sum_{m\in J_l} |d_{l,m}|^2.\end{aligned}$$

Hence,

$$\| \sum_{l=-1}^{\infty} \sum_{m\in J_l} d_{l,m} 3^{-s(l+1)}\psi_{l,m}\|^2_{H^s(\Omega)} \sim \sum_{l=-1}^{\infty} \sum_{m\in J_l} |d_{l,m}|^2.$$

□

Note that a similar proof yields that $\bigcup_{l=0}^{\infty}\{3^{(1-s)l}B^l_{ij} : B^l_{ij} \in W_l\}$ is a strongly stable basis for $H^s(\Omega)$, $2 < s < \frac{5}{2}$.

Acknowledgements

This work is partially supported by the Flemish Fund for Scientific Research (FWO Vlaanderen) project MISS (G.0211.02), and by the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with the authors.

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[2] J. H. Bramble and S. R. Hilbert. Bounds for a class of linear functionals with applications to hermite interpolation. *Numer. Math.*, 16:362–369, 1971.

[3] J. M. Carnicer, W. Dahmen, and J. M. Peña. Local decomposition of refinable spaces. *Appl. Comp. Harm. Anal.*, 3:127–153, 1996.

[4] W. Dahmen. Some remarks on multiscale transformations, stability and biorthogonality. In P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, editors, *Curves and surfaces.* Academic Press, 1994.

[5] W. Dahmen. Stability of multiscale transformations. *J. Fourier Anal. Appl.*, 4:341–362, 1996.

[6] W. Dahmen, R. DeVore, and K. Scherer. Multidimensional spline approximation. *SIAM J. Numer. Anal.*, 17:380–402, 1980.

[7] W. Dahmen, P. Oswald, and X.-Q. Shi. C^1-hierarchical bases. *J. Comput. Appl. Math.*, 51:37–56, 1994.

[8] O. Davydov and R. Stevenson. Hierarchical Riesz bases for $H^s(\Omega)$, $1 < s < 5/2$. Preprint 1281, Department of Mathematics, Utrecht University, 2003.

[9] R. A. Devore and G. G. Lorentz. *Constructive Approximation.* Springer-Verlag, Berlin, 1993.

[10] R. A. Devore and V. A. Popov. Interpolation of Besov spaces. *Trans. Amer. Math. Soc.*, 305(1):397–414, 1988.

[11] R. A. Devore and R. C. Sharpley. Besov spaces on domains in $\mathbb{R}^d$. *Trans. Amer. Math. Soc.*, 335:843–864, 1993.

[12] P. Dierckx. On calculating normalized Powell–Sabin B-splines. *Comput. Aided Geom. Design*, 15(3):61–78, 1997.

[13] M. Lai and L. Schumaker. On the approximation power of bivariate splines. *Advances in Comp. Math.*, 9:251–279, 1998.

[14] J. Maes and A. Bultheel. Surface compression with hierarchical Powell–Sabin B-splines. TW Report 391, Department of Computer Science, Katholieke Universiteit Leuven, Belgium, May 2004.

[15] J. Maes, E. Vanraes, P. Dierckx, and A. Bultheel. On the stability of normalized Powell–Sabin B-splines. *J. Comput. Appl. Math.*, 170(1):181–196, 2004.

[16] P. Oswald. Hierarchical conforming finite element methods for the biharmonic equation. *SIAM J. Numer. Anal.*, 29:1610–1625, 1992.

[17] M. J. D. Powell and M. A. Sabin. Piecewise quadratic approximations on triangles. *ACM Transactions on Mathematical Software*, 3:316–325, 1977.

[18] W. Sweldens. The lifting scheme: A construction of second generation wavelets. *SIAM J. Math. Anal.*, 29(2), 1997.

[19] E. Vanraes, J. Maes, and A. Bultheel. Powell–Sabin spline wavelets. *Intl. Journal of Wavelets, Multiresolution and Information Processing*, 2(1):23–42, 2004.

[20] E. Vanraes, J. Windmolders, A. Bultheel, and P. Dierckx. Automatic construction of control triangles for subdivided Powell–Sabin splines. *Computer Aided Geometric Design*, 2004. Accepted.

[21] H. Yserentant. On the multi-level splitting of finite element spaces. *Numer. Math.*, 49:379–412, 1986.