General Linear Methods for Volterra Integro-Differential Equations with Memory

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GENERAL LINEAR METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH MEMORY

CHENGJIAN ZHANG* AND STEFAN VANDEWALLE†

Abstract. A new class of numerical methods for Volterra integro-differential equations with memory is developed. The methods are based on the combination of general linear methods with compound quadrature rules. Sufficient conditions that guarantee global and asymptotic stability of the solution of the differential equation and its numerical approximation are established. Numerical examples illustrate the convergence and effectiveness of the numerical methods.

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1. Introduction.

Volterra integro-differential equations (VIDEs) arise widely in the mathematical modelling of physical and biological phenomena. Significant advances in the theoretical analysis of such equations, and in the numerical analysis and implementation of time-integration techniques for these problems have been made in the last few decades. For a survey of early results we refer to the books [7, 24]. More recently, one has found that VDIDEs with memory (MVIDEs), also called Volterra delay-integro-differential equations (VDIDEs), can be even more effective than standard VIDEs for the modelling of real-life problems, see, e.g., [6, 19]. This fact has led researchers to develop a theory and numerical analysis for MVIDEs. For example, numerical time-integration techniques of one-step collocation and Runge-Kutta (RK) type were investigated in [8, 14, 9, 20, 27, 18]. Linear multistep (LM) based methods were studied in [1, 2, 3, 25, 26].

So far, only very few papers have been concerned with the *nonlinear* stability of MVIDEs. In the papers [5, 26, 27], the authors study the nonlinear stability of continuous RK methods, discrete RK methods and BDF methods for MVIDEs, respectively. Here, we will study the use of general linear (GL) methods. For regular ODEs, it is well-known that the class of general linear methods covers most of the common methods (cf. [10, 12, 13]).

The paper is organized as follows. In §2, a fairly general class of MVIDEs is defined. This class of problems contains the problems considered in [26, 27]. We present a stability criterion for such problems, which generalizes the criteria in the above references. In §3, a class of extended GL methods is derived for solving MVIDEs. They are obtained by combining classical GL methods with compound quadrature rules. In §4, some technical lemmas are derived. These lemmas will play a key role in the derivation of the numerical stability results. In §5 and §6, we study the global and asymptotic stability of the extended methods. In §7 we present some numerical examples in order to illustrate the convergence of the extended GL methods. These numerical results show that the new methods are quite effective.

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2. A class of MVIDEs and its stability. We consider a complex N-dimensional system of MVIDEs with constant delay $\tau > 0$ of the form,

$$\begin{cases} y'(t) &= f(t, y(t), y(t-\tau), \int_{t-\tau}^{t} g(t, v, y(v))dv), \quad t \in [t_0, +\infty), \\ y(t) &= \varphi(t), \qquad\qquad\qquad\qquad\qquad\quad t \in [t_0 - \tau, t_0], \end{cases} \tag{2.1}$$

where the mappings f, g and φ are smooth enough, such that system (2.1) has a unique solution $y(t)$. We assume in particular that the following conditions are satisfied,

$$\Re\langle f(t,x,y,z) - f(t,\tilde{x},\tilde{y},\tilde{z}), x - \tilde{x}\rangle \le \alpha\|x-\tilde{x}\|^2 + \beta\|y-\tilde{y}\|^2 + \sigma\|z-\tilde{z}\|^2, \tag{2.2}$$

$$\|g(t,v,x) - g(t,v,\tilde{x})\| \le \gamma\|x-\tilde{x}\|, \quad (t,v) \in \mathbb{D}, \tag{2.3}$$

for $t \in [t_0, +\infty)$, $\mathbb{D} = \{(t,v) : t \in [t_0,+\infty), v \in [t-\tau, t]\}$, and $x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{C}^N$. The notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote a given inner product in $\mathbb{C}^N$ and its induced norm. The constants $(-\alpha), \beta, \sigma$, and γ are nonnegative. Further on, problems of type (2.1) with (2.2)-(2.3) will be called *problems of class* $\mathbb{GRI}(\alpha,\beta,\sigma,\gamma)$. Some concrete examples will be given in §7. Note that this class includes the $\mathbb{DI}$ and $\mathbb{RI}$ problem classes that were investigated in [26, 27]. The relation among these classes is as follows,

$$\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma) \subset \mathbb{RI}(\alpha,\beta\sigma_1,\beta\sigma_2,\gamma) \subset \mathbb{GRI}(\alpha+\beta(\sigma_1+\sigma_2)/2, \beta\sigma_1/2, \beta\sigma_2/2, \gamma).$$

Hence, the present research in the framework of class $\mathbb{GRI}$ will be more general and will extend our earlier results.

In [26, 27], sufficient conditions are derived for system (2.1) belonging to $\mathbb{DI}$ or $\mathbb{RI}$ to satisfy the following global and asymptotic stability properties:

$$\|y(t) - \tilde{y}(t)\| \le \max_{\theta\in[t_0-\tau,\ t_0]} \|\varphi(\theta) - \psi(\theta)\|, \quad \forall t \ge t_0 \tag{2.4}$$

$$\lim_{t\to+\infty} \|y(t) - \tilde{y}(t)\| = 0, \tag{2.5}$$

where $\tilde{y}(t)$ is the solution of a system similar to (2.1) but with $\varphi(t)$ replaced by a different function $\psi(t)$. Here, we derive analogous results for the $\mathbb{GRI}$ class.

Theorem 2.1. *Assume that system* (2.1) *belongs to class* $\mathbb{GRI}(\alpha,\beta,\sigma,\gamma)$ *with* $\beta + \sigma\gamma^2\tau^2 < -\alpha$. *Then the stability properties* (2.4) *and* (2.5) *hold.*

Proof. Denote by $\Delta y(t)$ the difference $y(t) - \tilde{y}(t)$. Using (2.2) and (2.3), we find

$$\begin{aligned} \frac{d}{dt}(\|\Delta y(t)\|^2) &= 2\Re\langle \Delta y(t), (\Delta y(t))'\rangle \\ &\le \alpha\|\Delta y(t)\|^2 + \beta\|\Delta y(t-\tau)\|^2 + \sigma\left\|\int_{t-\tau}^{t}[g(t,v,y(v)) - g(t,v,\tilde{y}(v))]dt\right\|^2 \\ &\le \alpha\|\Delta y(t)\|^2 + \beta\|\Delta y(t-\tau)\|^2 + \sigma\gamma^2\tau^2 \max_{t-\tau\le v\le t}\|\Delta y(v)\|^2 \\ &\le \alpha\|\Delta y(t)\|^2 + (\beta + \sigma\gamma^2\tau^2)\max_{t-\tau\le v\le t}\|\Delta y(v)\|^2. \end{aligned}$$

Application of the generalized Halanay inequality from [4], leads to

$$\|\Delta y(t)\|^2 \le \max_{\theta\in[t_0-\tau,\ t_0]}\|\varphi(\theta) - \psi(\theta)\|^2, \quad \forall t \ge t_0, \quad \text{and} \quad \lim_{t\to+\infty}\|\Delta y(t)\|^2 = 0.$$

Thus, (2.4) and (2.5) are satisfied. □

3. The extended GL methods. The numerical methods we suggest for (2.1) will be based on GL methods for ODEs of the form $y'(t) = f(t, y(t))$ with $y(t_0) = y_0$, see [10, 12, 13]. A GL method can be formulated as follows

$$\left\{\begin{array}{lcll} Y_i^{(n)} & = & h\sum\limits_{j=1}^{s} c_{ij}^{(11)} f(t_n + c_j h, Y_j^{(n)}) + \sum\limits_{j=1}^{r} c_{ij}^{(12)} y_j^{(n-1)}, & i = 1, 2, \ldots, s, \\ y_i^{(n)} & = & h\sum\limits_{j=1}^{s} c_{ij}^{(21)} f(t_n + c_j h, Y_j^{(n)}) + \sum\limits_{j=1}^{r} c_{ij}^{(22)} y_j^{(n-1)}, & i = 1, 2, \ldots, r, \end{array}\right. \tag{3.1}$$

with real coefficients $c_{ij}^{(IJ)}$ for $I, J = 1, 2$, with a stepsize $h > 0$, and with $t_n = t_0 + nh$ for $n \geq 0$. The vector $Y_i^{(n)}$ approximates $y(t_n + c_i h)$. The precise nature of vector $y_i^{(n)}$ differs from method to method; generally, it contains the necessary information for doing the next time-integration step. For use in the subsequent analysis, we introduce the coefficient matrices $C_{IJ} = (c_{ij}^{(IJ)})$ and the abscissa vector $c = (c_1, c_2, \ldots, c_s)^T$.

An important subclass of the GL methods, that will be considered further on, are the multistep Runge-Kutta (MRK) methods, see e.g. [11, 23],

$$\left\{\begin{array}{lcll} Y_i^{(n)} & = & h\sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j^{(n)}) + \sum\limits_{j=1}^{r} \hat{a}_{ij} y_{n+j-1}, & i = 1, 2, \ldots, s, \\ y_{n+r} & = & h\sum\limits_{j=1}^{s} b_j f(t_n + c_j h, Y_j^{(n)}) + \sum\limits_{j=1}^{r} \hat{b}_j y_{n+j-1} & \end{array}\right. \tag{3.2}$$

These methods can be written in the form (3.1) when setting $y_{n+i} = y_i^{(n)}$. With $b = (b_1, b_2, \ldots, b_s)^T$ and $\hat{b} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r)^T$, we can identify the coefficient matrices:

$$C_{11} = (a_{ij}), \quad C_{12} = (\hat{a}_{ij}), \quad C_{21} = \begin{bmatrix} 0 \\ b^T \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 0 & I_{r-1} \\ & \hat{b}^T \end{bmatrix},$$

For the formulation of other numerical methods as a GL method we refer to [13].

First, we recall a number of elementary concepts of GL methods that will be important for our stability analysis.

Definition 3.1. *Let* $R(z) = C_{22} + zC_{21}(I_s - zC_{11})^{-1}C_{12}$, *where* $z \in \mathbb{C}$ *and where* I_s *denotes the* $s \times s$ *identity matrix. A GL method is called strictly stable at infinity if* $R(\infty) := \lim\limits_{z\to\infty} R(z)$ *exists and if its spectral radius satisfies* $\rho(R(\infty)) < 1$.

Definition 3.2. *(cf. [10]) A GL method is called* $(k, l)-$*algebraically stable if there exist real constants* k *and* l*, and a symmetric positive definite matrix* $G = (g_{ij}) \in \mathbb{R}^{r\times r}$ *and a nonnegative diagonal matrix* $D = diag(d_1, d_2, \ldots, d_s) \in \mathbb{R}^{s\times s}$ *such that matrix* $M = (m_{ij}) \in \mathbb{R}^{(r+s)\times(r+s)}$ *is nonnegative definite, with*

$$M = \begin{bmatrix} kG - C_{22}^T G C_{22} - 2lC_{12}^T G C_{12} & C_{12}^T D - C_{22}^T G C_{21} - 2lC_{12}^T D C_{11} \\ DC_{12} - C_{21}^T G C_{22} - 2lC_{11}^T D C_{12} & DC_{11} + C_{11}^T D - C_{21}^T G C_{21} - 2lC_{11}^T D C_{11} \end{bmatrix}.$$

In particular, a (1,0)-algebraically stable method is called algebraically stable.

Adapting method (3.1) to MVIDE (2.1), and considering the case where the stepsize $h = \tau/m$ for m a given positive integer, yields the following numerical scheme

$$\left\{\begin{array}{l} Y_i^{(n)} = h\sum\limits_{j=1}^{s} c_{ij}^{(11)} f(t_n + c_j h, Y_j^{(n)}, Y_j^{(n-m)}, Z_j^{(n)}) + \sum\limits_{j=1}^{r} c_{ij}^{(12)} y_j^{(n-1)}, \ i = 1, 2, \ldots, s, \\ y_i^{(n)} = h\sum\limits_{j=1}^{s} c_{ij}^{(21)} f(t_n + c_j h, Y_j^{(n)}, Y_j^{(n-m)}, Z_j^{(n)}) + \sum\limits_{j=1}^{r} c_{ij}^{(22)} y_j^{(n-1)}, \ i = 1, 2, \ldots, r. \end{array}\right. \tag{3.3}$$

TABLE 3.1

Value for the bound ν in (3.5) *for different compound quadrature rules.*

	CT rule	CG rule	CS rule	CN rule
ν	2τ	2τ	$\sqrt{20}\,\tau/3$	$2\sqrt{1194}\,\tau/45$

$Z_j^{(n)}$ is an approximation to

$$Z(t_n+c_jh) := \int_{t_{n-m}+c_jh}^{t_n+c_jh} g(t_n+c_jh, v, y(v))dv$$

and is computed by a convergent, compound quadrature formula

$$Z_j^{(n)} = h\sum_{q=0}^{m} \nu_q g(t_n+c_jh, t_{n-q}+c_jh, Y_j^{(n-q)}), \quad j=1,2,\dots,s. \tag{3.4}$$

Such a quadrature formula can be derived from a uniform repeated rule (cf. [3, 7, 26]). For our stability analysis we need the rule to satisfy the following condition:

$$h\sqrt{(m+1)\sum_{q=0}^{m}|\nu_q|^2} < \nu \quad \text{with } \ mh=\tau \ \text{ and a positive constant } \ \nu. \tag{3.5}$$

This condition holds for many of the common quadrature rules. For example, the compound Gregory (CG) rule of order three (cf. [7])

$$\int_0^\tau \Phi(s)ds \cong \frac{h}{12}[5\Phi(0)+13\Phi(h)+12\sum_{q=2}^{m-2}\Phi((m-q)h)+13\Phi((m-1)h))+5\Phi(mh)],$$

with $mh=\tau$, satisfies

$$h\sqrt{(m+1)\sum_{q=0}^{m}|\nu_q|^2} = h\sqrt{(m+1)\left[\frac{5^2+13^2+(m-3)12^2+13^2+5^2}{12^2}\right]}$$

$$= h\sqrt{(m+1)(m-\frac{11}{36})} \le h(m+1) \le 2mh = 2\tau.$$

A value for ν for the compound trapezoidal (CT) rule of second order, the compound Simpson (CS) rule of fourth order and the compound Newton-Cotes (CN) rule of sixth order has been derived in [26]. We recall those values in Table 3.1.

We end this section with some more notational conventions. When the extended GL method is applied to system (2.1) with initial function $\varphi(t)$ replaced by $\psi(t)$, the resulting numerical approximations will be denoted by $\tilde{Y}_j^{(n)}, \tilde{Z}_j^{(n)}$ and $\tilde{y}_i^{(n)}$. Also, when a time point t_n or t_n+c_jh falls in the initial interval $[t_0-\tau, t_0]$, we set the approximation at that point equal to the corresponding (known) true solution.

4. Some elementary lemmas.

In this section, we will derive some technical lemmas, which are important for the derivation of the main results in the subsequent sections. First, we introduce some more notations:

$$\Delta y_j{}^{(n)} = y_j^{(n)} - \tilde{y}_j^{(n)}, \quad \Delta Y_j^{(n)} = Y_j^{(n)} - \tilde{Y}_j^{(n)}, \quad \Delta Z_j^{(n)} = Z_j^{(n)} - \tilde{Z}_j^{(n)},$$

$$\Delta f_j^{(n)} = f(t_j^{(n)}, Y_j^{(n)}, Y_j^{(n-m)}, Z_j^{(n)}) - f(t_j^{(n)}, \tilde{Y}_j^{(n)}, \tilde{Y}_j^{(n-m)}, \tilde{Z}_j^{(n)}),$$

$$\Delta y^{(n)} = \begin{pmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \vdots \\ \Delta y_r^{(n)} \end{pmatrix}, \quad \Delta Y^{(n)} = \begin{pmatrix} \Delta Y_1^{(n)} \\ \Delta Y_2^{(n)} \\ \vdots \\ \Delta Y_s^{(n)} \end{pmatrix}, \quad \Delta f^{(n)} = \begin{pmatrix} \Delta f_1^{(n)} \\ \Delta f_2^{(n)} \\ \vdots \\ \Delta f_s^{(n)} \end{pmatrix}.$$

With (3.3) we can write the relation between the above quantities compactly as

$$\begin{cases} \Delta Y^{(n)} &= h(C_{11} \otimes I_N)\Delta f^{(n)} + (C_{12} \otimes I_N)\Delta y^{(n-1)}, \\ \Delta y^{(n)} &= h(C_{21} \otimes I_N)\Delta f^{(n)} + (C_{22} \otimes I_N)\Delta y^{(n-1)}, \end{cases} \tag{4.1}$$

where symbol $\otimes$ denotes the Kronecker product. We will use the following inner product and norms on $\mathbb{C}^{Ns}$:

$$\langle U, V\rangle = \sum_{i=1}^{s} \langle u_i, v_i\rangle, \quad \|U\| = \sqrt{\langle U, U\rangle}, \quad \|U\|_G = \sqrt{\langle U, GU\rangle}$$

where vectors $U = \left(u_1^T, u_2^T, \ldots, u_s^T\right)^T, V = \left(v_1^T, v_2^T, \ldots, v_s^T\right)^T \in \mathbb{C}^{Ns}$ and $u_i, v_i \in \mathbb{C}^N$.

Lemma 4.1. *Suppose that GL method* (3.1) *is* (k,l)*-algebraically stable with* $0 < k \le \theta \le 1$. *Suppose that quadrature formula* (3.4) *satisfies* (3.5) *and that conditions* (2.2) *and* (2.3) *hold. Set* $d = \sum_{j=1}^{s} d_j$. *Then, method* (3.3) *satisfies*

$$\begin{aligned} \|\Delta y^{(n)}\|_G^2 \le \theta^n \|\Delta y^{(0)}\|_G^2 &+ 2\beta\tau\theta^{n-m} d \max_{-m\le i\le -1} \max_{1\le j\le s} \{\|\Delta Y_j^{(i)}\|^2\} \\ &+ 2[(h\alpha - l)\theta^m + h\beta] \sum_{i=0}^{n-1} \theta^{n-m-i-1} \sum_{j=1}^{s} d_j \|\Delta Y_j^{(i)}\|^2 \\ &+ \frac{2h\sigma(\nu\gamma)^2}{m+1} \sum_{i=0}^{n-1} \theta^{n-i-1} \sum_{j=1}^{s}\sum_{q=0}^{m} d_j \|\Delta Y_j^{(i-q)}\|^2, \quad n \ge 1. \end{aligned} \tag{4.2}$$

Proof. It follows from a fairly straightforward (but tedious) computation and $(k,l)-$algebraic stability that (compare also [10, 17])

$$\|\Delta y^{(n)}\|_G^2 - k\|\Delta y^{(n-1)}\|_G^2 - 2\sum_{j=1}^{s} d_j \Re\langle \Delta Y_j^{(n)}, h\Delta f_j^{(n)} - l\Delta Y_j^{(n)}\rangle = -\sum_{i=1}^{r+s}\sum_{i=1}^{r+s} m_{ij}\langle \omega_i, \omega_j\rangle \le 0$$

where $M = (m_{ij})$, as defined in Definition 3.2, and $\omega_i = \begin{cases} y_i^{(n-1)}, & 1 \le i \le r, \\ h\Delta f_i^{(n)}, & r+1 \le i \le r+s. \end{cases}$

This, together with the condition $0 < k \le \theta$, implies

$$\|\Delta y^{(n)}\|_G^2 \le \theta\|\Delta y^{(n-1)}\|_G^2 + 2\sum_{j=1}^{s} d_j \Re\langle \Delta Y_j^{(n)}, h\Delta f_j^{(n)} - l\Delta Y_j^{(n)}\rangle. \tag{4.3}$$

By condition (2.2) one has

$$\sum_{j=1}^{s} d_j \Re\langle \Delta Y_j^{(n)}, h\Delta f_j^{(n)}\rangle \le h\sum_{j=1}^{s} d_j[\alpha\|\Delta Y_j^{(n)}\|^2 + \beta\|\Delta Y_j^{(n-m)}\|^2 + \sigma\|\Delta Z_j^{(n)}\|^2].$$

When this bound is inserted into (4.3), one finds

$$\begin{aligned}\|\Delta y^{(n)}\|_G^2 \le \theta\|\Delta y^{(n-1)}\|_G^2 + 2(h\alpha - l)\sum_{j=1}^{s} d_j\|\Delta Y_j^{(n)}\|^2 \\ +\ 2h\beta\sum_{j=1}^{s} d_j\|\Delta Y_j^{(n-m)}\|^2 + 2h\sigma\sum_{j=1}^{s} d_j\|\Delta Z_j^{(n)}\|^2.\end{aligned} \tag{4.4}$$

An induction argument applied to (4.4) yields

$$\begin{aligned}\|\Delta y^{(n)}\|_G^2 \le \theta^n\|\Delta y^{(0)}\|_G^2 + 2(h\alpha - l)\sum_{i=0}^{n-1}\theta^{n-i-1}\sum_{j=1}^{s} d_j\|\Delta Y_j^{(i)}\|^2 \\ + 2h\beta\sum_{i=0}^{n-1}\theta^{n-i-1}\sum_{j=1}^{s} d_j\|\Delta Y_j^{(i-m)}\|^2 + 2h\sigma\sum_{i=0}^{n-1}\theta^{n-i-1}\sum_{j=1}^{s} d_j\|\Delta Z_j^{(i)}\|^2.\end{aligned} \tag{4.5}$$

With (2.3), (3.5) and the Cauchy inequality, we have

$$\begin{aligned}\|\Delta Z_j^{(i)}\|^2 &\le (h\gamma)^2\left(\sum_{q=0}^{m}|\nu_q|\|\Delta Y_j^{(i-q)}\|\right)^2 \le (h\gamma)^2\left(\sum_{q=0}^{m}|\nu_q|^2\right)\left(\sum_{q=0}^{m}\|\Delta Y_j^{(i-q)}\|^2\right) \\ &\le \frac{(\gamma\nu)^2}{m+1}\sum_{q=0}^{m}\|\Delta Y_j^{(i-q)}\|^2.\end{aligned} \tag{4.6}$$

Moreover, we have

$$\begin{aligned}&h\sum_{i=0}^{n-1}\theta^{n-i-1}\sum_{j=1}^{s} d_j\|\Delta Y_j^{(i-m)}\|^2 \\ &\quad= h\sum_{i=0}^{n-m-1}\theta^{n-m-i-1}\sum_{j=1}^{s} d_j\|\Delta Y_j^{(i)}\|^2 + h\sum_{i=-m}^{-1}\theta^{n-m-i-1}\sum_{j=1}^{s} d_j\|\Delta Y_j^{(i)}\|^2 \\ &\quad\le h\sum_{i=0}^{n-1}\theta^{n-m-i-1}\sum_{j=1}^{s} d_j\|\Delta Y_j^{(i)}\|^2 + \tau\theta^{n-m} d\max_{-m\le i\le -1}\max_{1\le j\le s}\{\|\Delta Y_j^{(i)}\|^2\}.\end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) results in (4.2). ◻

The second lemma gives an existence condition for the stability function $R(z)$ that is referred to in Definition 3.1.

LEMMA 4.2. *Suppose that a GL method is* $(k,l)-$*algebraically stable for a symmetric positive-definite matrix* $G \in \mathbb{R}^{r\times r}$ *and a positive diagonal matrix* $D \in \mathbb{R}^{s\times s}$. *Then the limit* $R(\infty) := \lim_{z\to\infty} R(z)$ *exists.*

Proof. It suffices to prove the existence of $C_0 := \lim_{\varepsilon\to 0} C_{21}(C_{11} + \varepsilon I_s)^{-1}$. This can be shown in a similar way as Lemma 3.8 is proven in [17] (compare also [15]), i.e., by considering the nonnegative-definiteness of the bottom right-hand principal minor $DC_{11} + C_{11}^T D - C_{21}^T G C_{21} - 2lC_{11}^T D C_{11}$ of matrix M. ◻

REMARK 4.3. *In papers [15, 17] the positivity of the diagonal matrix* D *is guaranteed by requiring the underlying methods to be irreducible and algebraically stable. Here, we loosen these conditions, by requesting positivity for* D *explicitly. This allows us to consider more general* $(k,l)-$*algebraically stable GL methods.*

REMARK 4.4. *When the conditions in Lemma 4.2 are satisfied, it holds that*

$$R(\infty)=\left\{\begin{array}{ll} C_{22}-C_{21}C_{11}^{-1}C_{12}, & \text{if } C_{11} \text{ is invertible,}\\ C_{22}-C_0C_{12}, & \text{if } C_{11} \text{ is singular.}\end{array}\right.$$

Two inequalities from [27] will also play important roles in the subsequent sections.

LEMMA 4.5. *Suppose that* $\{A_i\}_{i=0}^n$ *and* $\{B_i\}_{i=-m}^n$ *are two arbitrary nonnegative real sequences. Then the following two inequalities hold, for all* $n, m \geq 0$*:*

$$\sum_{i=0}^{n}(A_i\sum_{j=0}^{m}B_{i-j}) \leq \sum_{j=0}^{m}\sum_{i=0}^{n}A_{i+j}B_i + (\sum_{j=1}^{m}\sum_{i=1}^{j}A_{j-i}) \max_{-m\leq q\leq -1}\{B_q\} \tag{4.8}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i-j} \leq (m+1)\sum_{i=0}^{n}B_i + \frac{m(m+1)}{2}\max_{-m\leq q\leq -1}\{B_q\}. \tag{4.9}$$

Finally, we mention an asymptotic property of vector difference equations that can be derived easily from a combination of Theorems 105B and 123D in [13].

LEMMA 4.6. *Given a matrix* $\mathcal{A} \in \mathbb{C}^{Q\times Q}$ *and a sequence* $V_n \in \mathbb{C}^Q$*. Then the solution sequence* $\{X_n\} \subseteq \mathbb{C}^Q$ *of the linear difference equation* $X_n = \mathcal{A}X_{n-1} + V_n$ *satisfies* $\lim_{n\to\infty}\|X_n\| = 0$ *if and only if* $\rho(\mathcal{A}) < 1$ *and* $\lim\limits_{n\to\infty}\|V_n\| = 0$*.*

5. Global stability of the extended GL methods.

Numerical stability is an important feature of an effective numerical method. An unstable numerical method may be consistent of high order, yet arbitrarily small perturbations will eventually cause large deviations from the true solution. In this section, we will focus on the global stability of the extended GL methods.

DEFINITION 5.1. *The extended GL method* (3.3)-(3.4) *is called globally stable for problems of class* $\mathbb{GRI}(\alpha,\beta,\sigma,\gamma)$ *if there exists a constant* $\mathcal{H} > 0$*, that depends only on* $\alpha,\beta,\sigma,\gamma,\tau,\nu$ *and the method, such that*

$$\|\Delta y^{(n)}\| \leq \mathcal{H}\max\left\{\|\Delta y^{(0)}\|, \max_{-m\leq i\leq -1}\max_{1\leq j\leq s}\{\|\Delta Y_j^{(i)}\|\}\right\}, \quad \forall n\geq 1. \tag{5.1}$$

REMARK 5.2. *For the extended one-step RK methods, we have* $\Delta y^{(n)} = \Delta y_n := y_{n+1} - \tilde{y}_{n+1}$*. Also, under the usual assumption that* $0 \leq c_i \leq 1$*, it holds that*

$$\max\left\{\|\Delta y_0\|, \max_{-m\leq i\leq -1}\max_{1\leq j\leq s}\{\|\Delta Y_j^{(i)}\|\}\right\} \leq \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|.$$

Hence, we have for (5.1) *the following equivalent formulation:*

$$\|\Delta y_n\| \leq \mathcal{H}\max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|.$$

Thus, the concept of global stability in Def. 5.1 is slightly stronger than that in [27].

THEOREM 5.3. *Suppose the GL method* (3.1) *is* (k,l)*-algebraically stable for a nonnegative diagonal matrix* $D = diag(d_1, d_2, \ldots, d_s) \in \mathbb{R}^{s\times s}$ *and a symmetric positive-definite matrix* $G = (g_{ij}) \in \mathbb{R}^{r\times r}$*, where* $0 < k \leq 1$*, and suppose the quadrature formula* (3.4) *satisfies condition* (3.5)*. Then the extended GL method* (3.3)-(3.4) *is globally stable for class* $\mathbb{GRI}(\alpha,\beta,\sigma,\gamma)$ *with stability constant*

$$\mathcal{H} = \sqrt{\frac{\lambda_{max}^G + d\tau(2\beta+\sigma\gamma^2\nu^2)}{\lambda_{min}^G}}, \tag{5.2}$$

where λ^G_{min} *and* λ^G_{max} *denote the minimum and maximum eigenvalues of matrix* G *and* $d = \sum_{j=1}^s d_j$, *whenever the following condition holds:*

$$h(\alpha + \beta + \sigma\gamma^2\nu^2) \le l. \tag{5.3}$$

Proof. Setting $\theta = 1$ in (4.2) gives

$$\begin{aligned} \|\Delta y^{(n)}\|_G^2 \le{}& \|\Delta y^{(0)}\|_G^2 + 2\beta\tau d \max_{-m\le i\le -1} \max_{1\le j\le s} \{\|\Delta Y_j^{(i)}\|^2\} \\ &+ 2[h(\alpha+\beta) - l] \sum_{i=0}^{n-1}\sum_{j=1}^{s} d_j \|\Delta Y_j^{(i)}\|^2 + \frac{2h\sigma(\nu\gamma)^2}{m+1} \sum_{i=0}^{n-1}\sum_{j=1}^{s}\sum_{q=0}^{m} d_j \|\Delta Y_j^{(i-q)}\|^2. \end{aligned} \tag{5.4}$$

Also, by virtue of inequality (4.9), we have for $n \ge 1$

$$\sum_{i=0}^{n-1}\sum_{q=0}^{m} \|\Delta Y_j^{(i-q)}\|^2 \le (m+1)\sum_{i=0}^{n-1} \|\Delta Y_j^{(i)}\|^2 + \frac{m(m+1)}{2} \max_{-m\le i\le -1} \{\|\Delta Y_j^{(i)}\|^2\}. \tag{5.5}$$

Inserting (5.5) into (5.4) and using the condition that $mh = \tau$ yields

$$\begin{aligned} \|\Delta y^{(n)}\|_G^2 \le{}& \|\Delta y^{(0)}\|_G^2 + d\tau(2\beta + \sigma\gamma^2\nu^2) \max_{-m\le i\le -1} \max_{1\le j\le s} \{\|\Delta Y_j^{(i)}\|^2\} \\ &+2[h(\alpha+\beta+\sigma\gamma^2\nu^2) - l] \sum_{i=0}^{n-1}\sum_{j=1}^{s} d_j \|\Delta Y_j^{(i)}\|^2. \end{aligned} \tag{5.6}$$

With condition (5.3) this leads to

$$\|\Delta y^{(n)}\|_G^2 \le \|\Delta y^{(0)}\|_G^2 + d\tau(2\beta + \sigma\gamma^2\nu^2) \max_{-m\le i\le -1} \max_{1\le j\le s} \{\|\Delta Y_j^{(i)}\|^2\}.$$

Considering that $\lambda^G_{min}\|\Delta y^{(n)}\|^2 \le \|\Delta y^{(n)}\|_G^2$ and that $\|\Delta y^{(0)}\|_G^2 \le \lambda^G_{max}\|\Delta y^{(0)}\|^2$, we arrive at stability inequality (5.1) with stability constant (5.2). □

Example 5.1. As an application of Theorem 5.3, we consider an extended GL method with $s = 1$ and $r = 2$, induced by the combination of the two-step BDF method and the compound trapezoidal rule:

$$\left\{ \begin{array}{rcl} Y^{(n)} &=& \frac{2}{3}hF^{(n)} + \left[\left(\frac{4}{3}, -\frac{1}{3}\right) \otimes I_N\right] y^{(n-1)}, \\ y^{(n)} &=& h\left[\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \otimes I_N\right] F^{(n)} + \left[\begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{pmatrix} \otimes I_N\right] y^{(n-1)}, \end{array} \right. \tag{5.7}$$

where $Y^{(n)} = y_{n+2}$, $y_n \cong y(t_n)$, $y^{(n)} = \left(y_{n+2}^T, y_{n+1}^T\right)^T$ and

$$\begin{aligned} F^{(n)} &= f_{n+2} := f\left(t_{n+2}, Y^{(n)}, Y^{(n-m)}, Z^{(n)}\right), \\ Z^{(n)} &= h{\sum_{q=0}^{m}}'' \, g\left(t_{n+2}, t_{n+2-q}, Y^{(n-q)}\right), \end{aligned} \tag{5.8}$$

where the special summation symbol stands for a sum with the first and last term halved. By Table 3.1 we know that (5.8) satisfies (3.5) with $\nu = 2\tau$. In [12], it was

TABLE 5.1

l-value minimizing (5.9) *and corresponding stability constant $\mathcal{H}$ of the extended two-step BDF method for the MVIDEs in* (7.1) *and* (7.2).

	MVIDE (7.1)		MVIDE (7.2)	
m	l	$\mathcal{H}$	l	$\mathcal{H}$
8	-0.2224	6.4650	-0.0554	7.1616
16	-0.1112	6.8497	-0.0277	7.2997
32	-0.0556	7.0912	-0.0138	7.3733
64	-0.0278	7.2279	-0.0069	7.4116

shown that the two-step BDF method is $(1/(1-2l), l)-$algebraically stable with $l \le 0$ for $D = 1$ and the 2×2 real symmetric-positive-definite matrix

$$G := \begin{pmatrix} (5-2l)/2 & -1 \\ -1 & 1/2 \end{pmatrix}.$$

Note that $0 < k \le 1$ whenever $l \le 0$, and that there always exists an l such that (5.3) holds whenever $\beta + \sigma\gamma^2\nu^2 \le -\alpha$. Hence, we may conclude that the method is globally stable for class $\mathbb{GRI}(\alpha, \beta, \sigma, \gamma)$ whenever $\beta + \sigma\gamma^2\nu^2 \le -\alpha$. For the parameters in the formula of the stability constant, the following values are used:

$$d = 1, \;\; \nu = 2\tau, \;\; \lambda^G_{min} = \frac{3 - l - \sqrt{(l-2)^2+4}}{2}, \;\; \lambda^G_{max} = \frac{3 - l + \sqrt{(l-2)^2+4}}{2}.$$

The range of l is given by $h(\alpha+\beta+\sigma\gamma^2\nu^2) \le l \le 0$. In particular, when the extended two-step BDF method with a fixed stepsize h is applied to a concrete problem of class $\mathbb{GRI}(\alpha, \beta, \sigma, \gamma)$, one should look for the value of l that minimizes the stability constant $\mathcal{H}$. This leads on to consider the following constrained minimization problem:

$$\begin{cases} \text{minimize} \quad \mathcal{H} := \sqrt{\frac{3-l+\sqrt{(l-2)^2+4}+4\tau(\beta+2\sigma\gamma^2\tau^2)}{3-l-\sqrt{(l-2)^2+4}}} \\ \text{subject to} \quad h(\alpha+\beta+\sigma\gamma^2\nu^2) \le l \le 0. \end{cases} \tag{5.9}$$

In §7 results of numerical experiments will be reported for two concrete MVIDEs. The first of those, specified in (7.1), belongs to class $\mathbb{GRI}(-4, \frac{1}{2}, \frac{1}{2}, 1)$; the second one, specified in (7.2), belongs to class $\mathbb{GRI}(-4 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, 2)$. We have numerically solved problem (5.9) for these two examples. The l-value at which the minimum is obtained and the corresponding $\mathcal{H}$-value are given in Table 5.1, as a function of the parameter m, with $h = \tau/m$.

The method in the above example is also algebraically stable. For such methods, a general result can be derived immediately from Theorem 5.3.

COROLLARY 5.4. *Suppose GL method* (3.1) *is algebraically stable for a non-negative diagonal matrix $D = diag(d_1, d_2, \dots, d_s) \in \mathbb{R}^{s\times s}$ and a symmetric positive-definite matrix $G = (g_{ij}) \in \mathbb{R}^{r\times r}$, and suppose quadrature formula* (3.4) *satisfies condition* (3.5)*. Then the extended GL method* (3.3)-(3.4) *is globally stable for class $\mathbb{GRI}(\alpha, \beta, \sigma, \gamma)$ with stability constant* (5.2) *whenever the following condition holds*

$$\beta + \sigma\gamma^2\nu^2 \le -\alpha. \tag{5.10}$$

In [11], Burrage has derived a class of s-stage MRK methods of order $2s$ of the form (3.2) that satisfy

$$\begin{cases} B(2s), \quad C(s), \quad E(s); \quad c_i \neq c_j \quad \text{whenever} \quad i \neq j; \\ \sum\limits_{j=1}^{r} \hat{b}_j = 1, \quad i = 1, 2, \ldots, s; \quad \hat{b}_1 > 0, \quad \hat{b}_j \geq 0, \quad j = 2, 3, \ldots, r, \end{cases} \tag{5.11}$$

where $B(2s)$, $C(s)$ and $E(s)$ denote the usual order conditions. It was shown that these methods are algebraically stable for the matrices

$$G := \operatorname{diag}(\hat{b}_1, \hat{b}_1 + \hat{b}_2, \ldots, \sum_{j=1}^{r} \hat{b}_j), \quad D := \operatorname{diag}(b_1, b_2, \ldots, b_s). \tag{5.12}$$

Obviously, those are positive-definite. Hence, (5.10) is sufficient as a condition for global stability of the corresponding extended GL method.

Example 5.2. By combining the fourth order MRK method from [11] with the compound Simpson quadrature rule of order four, we can obtain an extended GL method with $r = 2$ and $s = 2$. It is written schematically below; for the values of the coefficients we refer to Table 5.2.

$$\begin{cases} Y^{(n)} &= h \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes I_N \right] F^{(n)} + \left[\begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{pmatrix} \otimes I_N \right] y^{(n-1)}, \\ y^{(n)} &= h \left[\begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} \otimes I_N \right] F^{(n)} + \left[\begin{pmatrix} 0 & 1 \\ \hat{b}_1 & \hat{b}_2 \end{pmatrix} \otimes I_N \right] y^{(n-1)}, \end{cases} \tag{5.13}$$

where $Y^{(n)} = (Y_1^{(n)^T}, Y_2^{(n)^T})^T$, and $y^{(n)} = \left(y_{n+1}{}^T, y_{n+2}{}^T\right)^T$, with $Y_i^{(n)} \cong y(t_n + c_i h)$, and $y_n \cong y(t_n)$, and

$$F^{(n)} = \left(f\left(t_n + c_1 h, Y_1^{(n)}, Y_1^{(n-m)}, Z_1^{(n)}\right)^T, f\left(t_n + c_2 h, Y_2^{(n)}, Y_2^{(n-m)}, Z_2^{(n)}\right)^T \right)^T.$$

$Z_j^{(n)}$ is computed by the compound Simpson rule with an even integer $m \geq 4$:

$$\begin{aligned} Z_j^{(n)} = \frac{h}{3} \Bigg[& g\left(t_{n,j}, t_{n,j}, Y_j^{(n)}\right) + 4 \sum_{q=1}^{m/2} g\left(t_{nj}, t_{n-2q+1,j}, Y_j^{(n-2q+1)}\right) \\ & + 2 \sum_{q=1}^{(m-2)/2} g\left(t_{n,j}, t_{n-2q,j}, Y_j^{(n-2q)}\right) + g\left(t_{n,j}, t_{n-m,j}, Y_j^{(n-m)}\right) \Bigg]. \end{aligned} \tag{5.14}$$

Here the notation $t_{n,j}$ is used as a shorthand for $t_n + c_j h$. This rule satisfies (3.5) for $\nu = \frac{\sqrt{20}}{3}\tau$. Under condition (5.10) this extended MRK method is globally stable. The stability constant (5.2) can be computed using $d = 1.7460$, $\nu = \frac{\sqrt{20}}{3}\tau$, $\lambda_{min}^G = 2\sqrt{15} - 7$, and $\lambda_{max}^G = 1$.

In [23], the MRK methods that satisfy (5.11) are called *MRK methods of the first class*. Five more classes of MRK methods are identified by Li in the above reference.

Table 5.2
Coefficients of the fourth order MRK method form Example 5.2

a_{11}	a_{12}	a_{21}	a_{22}
0.47790690818421	0.87165188291653	−0.08663699023763	0.50361252124048
$\hat{a}_{11}$	$\hat{a}_{12}$	$\hat{a}_{21}$	$\hat{a}_{22}$
0.75576439912123	0.24423560087877	0.97380878183171	0.02619121816829
b_1	b_2	$\hat{b}_1$	$\hat{b}_2$
0.95532987568936	0.79063681672548	$2\sqrt{15}-7$	$8-2\sqrt{15}$
	c_1	c_2	
	1.59379439197950	0.44316674917114	

Table 5.3
Coefficients of the third order MRK method from Example 5.3

a_{11}	a_{12}	a_{21}	a_{22}
0.41623635782678	0.43372361014760	−0.22963576074350	0.62694215804990
$\hat{a}_{11}$	$\hat{a}_{12}$	$\hat{a}_{21}$	$\hat{a}_{22}$
0.04995996797438	0.95004003202562	0.48484848484848	0.51515151515152
b_1	b_2	$\hat{b}_1$	$\hat{b}_2$
0.59227996965099	0.55772003034901	0.15	0.85
	c_1	c_2	
	1.80000000000000	0.91245791245791	

All of those are proven to be algebraically stable for the matrices (5.12). Hence, Corollary 5.4 is immediately applicable to the corresponding extended GL methods.

Example 5.3. We present an example of the second MRK class. Using formulae (3.11)-(3.16) in [23], we compute the coefficients for a two-stage third order MRK method. Its combination with the compound Gregory rule of order three (cf. [7]), yields another extended GL method of the form (5.13). The coefficients are given in Table 5.3. The $Z_j^{(n)}$ values are computed as follows:

$$Z_j^{(n)} = \frac{h}{12}\Big[5\,g(t_{n,j},t_{n,j},Y_j^{(n)}) + 13\,g(t_{n,j},t_{n-1,j},Y_j^{(n-1)}) + 12\sum_{q=2}^{m-2} g(t_{n,j},t_{n-q,j},Y_j^{(n-q)})$$
$$+ 13\,g(t_{n,j},t_{n-m+1,j},Y_j^{(n-m+1)}) + 5\,g(t_{n,j},t_{n-m,j},Y_j^{(n-m)})\Big]. \tag{5.15}$$

Since the corresponding underlying method is algebraically stable, and since the compound Gregory rule satisfies (3.5) with $\nu = 2\tau$, we immediately have global stability of the extended GL method under condition (5.11). The stability constant (5.2) can be computed using $d = 1.1500$, $\nu = 2\tau$, $\lambda^G_{min} = 0.15$, and $\lambda^G_{max} = 1$.

6. Asymptotic stability of the extended GL methods. In this section, we focus on the concept of asymptotic stability.

Definition 6.1. *The extended GL method* (3.3)-(3.4) *is called asymptotically stable for problems of class* $\mathbb{GRI}(\alpha,\beta,\sigma,\gamma)$ *if* $\lim_{n\to\infty}\|\Delta y^{(n)}\| = 0$.

THEOREM 6.2. *Suppose GL method* (3.1) *is* (k,l)*-algebraically stable for a non-negative diagonal matrix* $D = diag(d_1, d_2, \dots, d_s) \in \mathbb{R}^{s\times s}$ *and a real symmetric-positive-definite matrix* $G = (g_{ij}) \in \mathbb{R}^{r\times r}$*, where* $0 < k < 1$*, and suppose quadrature formula* (3.4) *satisfies* (3.5). *Then the extended GL method* (3.3)-(3.4) *is asymptotically stable for class* $\mathbb{GRI}(\alpha,\beta,\sigma,\gamma)$ *whenever the following condition holds*

$$h(\alpha+\beta+\sigma\gamma^2\nu^2) < l. \tag{6.1}$$

Proof. Define the quantity $\theta = \max\left\{k,\ \left[\frac{h(\beta+\sigma\gamma^2\nu^2)}{l-h\alpha}\right]^{\frac{1}{m}}\right\}$. From $0 < k < 1$ and (6.1) it follows that $0 < k \le \theta < 1$. Hence, bound (4.2) from Lemma 4.1 holds. We will simplify its right hand-side. By using (4.8), we can derive the following bound:

$$\begin{aligned}
&h\sum_{i=0}^{n-1}\theta^{-i}\sum_{q=0}^{m}\|\Delta Y_j^{(i-q)}\|^2 \\
&\quad\le h\left[\sum_{q=0}^{m}\sum_{i=0}^{n-1}\theta^{-(i+q)}\|\Delta Y_j^{(i)}\|^2 + \left(\sum_{q=1}^{m}\sum_{i=1}^{q}\theta^{-(q-i)}\right)\max_{-m\le\hat{q}\le-1}\{\|\Delta Y_j^{(\hat{q})}\|^2\}\right] \\
&\quad\le (m+1)h\sum_{i=0}^{n-1}\theta^{-(i+m)}\|\Delta Y_j^{(i)}\|^2 + m\tau\theta^{1-m}\max_{-m\le i\le-1}\{\|\Delta Y_j^{(i)}\|^2\}.
\end{aligned} \tag{6.2}$$

Substituting (6.2) into the last term of the right hand-side of (4.2) yields

$$\begin{aligned}
\|\Delta y^{(n)}\|_G^2 \le &\theta^n\|\Delta y^{(0)}\|_G^2 + 2\tau\theta^{n-m}d\left[\beta + \frac{m\sigma\gamma^2\nu^2}{m+1}\right]\max_{-m\le i\le -1}\max_{1\le j\le s}\{\|\Delta Y_j^{(i)}\|^2\} \\
&+ 2[(h\alpha-l)\theta^m + h(\beta+\sigma\gamma^2\nu^2)]\sum_{i=0}^{n-1}\theta^{n-m-i-1}\sum_{j=1}^{s}d_j\|\Delta Y_j^{(i)}\|^2.
\end{aligned} \tag{6.3}$$

By the definition of θ and with the inequalities $h\alpha \le h(\alpha+\beta+\sigma\gamma^2\nu^2) < l$, one finds

$$(h\alpha - l)\theta^m + h(\beta+\sigma\gamma^2\nu^2) \le 0.$$

This inequality, together with the knowledge that $0 < \theta < 1$ and (6.3), leads to $\lim_{n\to\infty}\|\Delta y^{(n)}\|_G = 0$, or $\lim_{n\to\infty}\|\Delta y^{(n)}\| = 0$. □

REMARK 6.3. *The major difference between Theorem 6.2 and Theorem 5.3 lies in the strict inequalities present in both* $k < 1$ *and* (6.1).

EXAMPLE 6.1. As an illustration we consider method (5.7)-(5.8) again. Recall that (5.8) satisfies (3.5) with $\nu = 2\tau$ and that the underlying GL method (5.7) is $(1/(1-2l), l)$-algebraically stable for any $l \le 0$. Thus, we have $0 < k < 1$ whenever $l < 0$. Moreover, there always exists an $l(< 0)$ such that $h(\alpha+\beta+\sigma\gamma^2\nu^2) < l$ whenever $\beta+\sigma\gamma^2\nu^2 < -\alpha$. Hence, the latter is sufficient as a condition for asymptotic stability.

From Theorem 6.2, it is not immediately possible to derive a corollary that applies specifically for algebraically stable methods, because of the condition $k < 1$. To remedy this situation, we present alternative approach for proving asymptotic stability, where the parameter k will be allowed to take the value one.

Theorem 6.4. *Suppose GL method* (3.1) *is strictly stable at infinity and* (k, l)-*algebraically stable for a positive diagonal matrix* $D = diag(d_1, d_2, \dots, d_s) \in \mathbb{R}^{s\times s}$ *and a real symmetric-positive-definite matrix* $G = (g_{ij}) \in \mathbb{R}^{r\times r}$, *where* $0 < k \leq 1$. *Suppose quadrature formula* (3.4) *satisfies* (3.5). *Then the extended GL method* (3.3)-(3.4) *is asymptotically stable for class* $\mathbb{GRI}(\alpha, \beta, \sigma, \gamma)$ *whenever* (6.1) *holds.*

Proof. Inequality (5.6) can be written as

$$\|\Delta y^{(n)}\|_G^2 + 2[l - h(\alpha+\beta+\sigma\gamma^2\nu^2)] \sum_{i=0}^{n-1}\sum_{j=1}^{s} d_j \|\Delta Y_j^{(i)}\|^2 \\ \leq \|\Delta y^{(0)}\|_G^2 + d\tau(2\beta+\sigma\gamma^2\nu^2) \max_{-m\leq i\leq -1} \max_{1\leq j\leq s} \{\|\Delta Y_j^{(i)}\|^2\}. \tag{6.4}$$

Since D is a positive diagonal matrix and since (6.1) holds, we may conclude from (6.4) that $\lim_{n\to\infty} \|\Delta Y_j^{(n)}\|^2 = 0$ for $j = 1, 2, \dots, s$, and thus

$$\lim_{n\to\infty} \|\Delta Y^{(n)}\| = \lim_{n\to\infty} \sqrt{\sum_{j=1}^{s} \|\Delta Y_j^{(n)}\|^2} = 0. \tag{6.5}$$

The remainder of the proof is divided into two parts. First, we consider the case where matrix C_{11} is invertible. Then, by (4.1), we have

$$\Delta y^{(n)} = [R(\infty)\otimes I_N]\Delta y^{(n-1)} + [(C_{21}C_{11}^{-1}\otimes I_N]\Delta Y^{(n)}, \tag{6.6}$$

where $R(\infty) = C_{22} - C_{21}C_{11}^{-1}C_{12}$. Because of the strictly stability at infinity, one has

$$\rho[R(\infty)\otimes I_N] = \rho[R(\infty)] < 1. \tag{6.7}$$

Therefore, applying Lemma 4.6 to (6.6) yields $\lim_{n\to\infty} \|\Delta y^{(n)}\| = 0$.

Next, we consider the case that matrix C_{11} is singular. We replace it in (6.6) by the invertible matrix $(C_{11} + \varepsilon I_s)$ and let $\varepsilon \to 0$. This gives

$$\Delta y^{(n)} = [R(\infty)\otimes I_N]\Delta y^{(n-1)} + (C_0\otimes I_N)\Delta Y^{(n)}, \tag{6.8}$$

where $R(\infty) = C_{22} - C_0C_{12}$ and $C_0 = \lim_{\varepsilon\to 0} C_{21}(C_{11} + \varepsilon I_s)^{-1}$, whose existence is ensured by Lemma 4.2. Another application of Lemma 4.6 completes the proof. □

When the GL method satisfies algebraic stability, Theorem 6.4 can be simplified.

Corollary 6.5. *Suppose that GL method* (3.1) *is strictly stable at infinity and algebraically stable for a positive diagonal matrix* $D = diag(d_1, d_2, \dots, d_s) \in \mathbb{R}^{s\times s}$ *and a real symmetric-positive-definite matrix* $G = (g_{ij}) \in \mathbb{R}^{r\times r}$, *and suppose quadrature formula* (3.4) *satisfies* (3.5). *Then the extended GL method* (3.3)-(3.4) *is asymptotically stable for the class* $\mathbb{GRI}(\alpha, \beta, \sigma, \gamma)$ *whenever the following condition holds*

$$\beta + \sigma\gamma^2\nu^2 < -\alpha. \tag{6.9}$$

Example 6.2. Consider the extended GL method from Example 5.2. A computation shows that $\rho[R(\infty)] \cong 0.9816 < 1$, which implies strict stability. So, the method is asymptotically stable for problem class $\mathbb{GRI}(\alpha, \beta, \sigma, \gamma)$ under condition (6.9).

Example 6.3. Consider the extended GL method from Example 5.3. Now, we find that $\rho[R(\infty)] \cong 0.6158 < 1$. Hence, also for this method, (6.9) is sufficient a condition to guarantee asymptotic stability.

7. Numerical experiments and convergence. In the previous sections we have proven that the extended GL methods possess an excellent stability behavior under a set of suitable conditions. Naturally, one also wishes to find out the actual computational performance, and their accuracy in particular. Therefore, we will apply the methods from Examples 5.1, 5.2 and 5.3 to two MVIDE systems. In order to measure the quality of the numerical methods, we introduce the following:

$$err := \|y(t_n) - y_n\|_\infty, \qquad \varepsilon := \max_{0\le n\le N_f} \|y(t_n) - y_n\|_\infty, \qquad p := \ln(\varepsilon)/\ln(h),$$

where N_f is a positive integer such that $N_f h = T$ with T being the length of the solution interval. The symbols err, ε and p denote the error at time point t_n, the maximum error over the interval, and the convergence order of the method.

We will pay particular attention to the estimate of the convergence order. Our numerical results will seem to indicate that the convergence order of the extended GL method equals the minimum of the orders of the underlying GL method and the quadrature rule. This results is not entirely unexpected, and will be verified through a number of examples. In the first two examples, the methods of Examples 5.1, 5.2, and 5.3 will be used. Those are characterized by a correct match between the orders of the GL method and the quadrature rule (order 2 for Example 5.1, order 3 for Example 5.3 and order 4 for Example 5.2). Further on, we will also consider methods where the order of the GL method is different from that of the quadrature rule.

Example 7.1. Consider the linear system with partially variable coefficients:

$$\begin{cases} y'(t) = -(6+\sin t)y(t) + y(t-\frac{\pi}{4}) - \int_{t-\frac{\pi}{4}}^{t} \sin(v)y(v)dv + 5\exp(\cos t), t \ge 0 \\ y(t) = \exp(\cos t), \quad -\frac{\pi}{4} \le t \le 0. \end{cases} \tag{7.1}$$

One may check that this system has the solution $y(t) = \exp(\cos t)$ and that it belongs to class $\mathbb{GRI}(-4, \frac{1}{2}, \frac{1}{2}, 1)$. With Theorem 2.1 we conclude that system (7.1) satisfies stability properties (2.4)-(2.5). Moreover, based on our earlier discussions, we deduce that the numerical methods from Examples 5.1, 5.2, and 5.3 retain the stability properties of the analytical solution; for this problem they are also globally and asymptotically stable.

In order to check the convergence behavior and accuracy of the extended GL methods, they will be applied using a sequence of step lengths characterized by the parameter m, on the time interval $[0, 9\pi]$. The accuracy of the obtained numerical solutions is displayed in Figs. 7.1-7.3. The numerically estimated convergence orders are given in Table 7.1. The numerical results show that the methods are quite effective, and preserve the inherent order of accuracy of the underlying components.

Example 7.2. Consider the following two-dimensional nonlinear system

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = -4\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ \cos t & 0 \end{pmatrix}\begin{pmatrix} y_1(t-\frac{\pi}{5}) \\ y_2(t-\frac{\pi}{5}) \end{pmatrix} \\ \qquad + \frac{1}{\sqrt{2}} \int_{t-\frac{\pi}{5}}^{t} \begin{pmatrix} \frac{(1+\sin^2 v)y_1^2(v)}{1+y_1^2(v)} \\ \frac{(1+\cos^2 v)y_2^2(v)}{1+y_2^2(v)} \end{pmatrix} dv + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \quad 0 \le t \\ \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad -\pi/5 \le t \le 0. \end{cases} \tag{7.2}$$

TABLE 7.1

Convergence order of extended GL methods for system (7.1). *Example 5.1: 2nd order BDF + 2nd order quadrature; Example 5.3: 3rd order MRK + 3rd order quadrature; Example 5.2: 4th order MRK + 4th order quadrature.*

m	Example 5.1	Example 5.3	Example 5.2
8	2.5666	3.7821	4.0022
16	2.4327	3.5499	3.9982
32	2.3510	3.4254	3.9964
64	2.2956	3.3491	3.9960

TABLE 7.2

Convergence order of extended GL methods for system (7.2). *Example 5.1: 2nd order BDF + 2nd order quadrature; Example 5.3: 3rd order MRK + 3rd order quadrature; Example 5.2: 4th order MRK + 4th order quadrature.*

m	Example 5.1	Example 5.3	Example 5.2
8	2.9132	4.1053	4.7479
16	2.7183	3.8449	4.5847
32	2.5918	3.6858	4.4804
64	2.5029	3.5787	4.4078

Functions $f(t)$ and $g(t)$ are chosen in such a way that the solution of the system equals $y(t) = (\sin t, \cos t)^T$. This problem belongs to class $\mathbb{GRI}(-4 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, 2)$. It follows that the system is satisfies stability properties (2.4)-(2.5), and that our three GL methods are globally and asymptotically stable. We consider the numerical results obtained when these methods are applied to (7.2). Note that a nonlinear implicit equation is to be solved in every time-step. To speed up the computation, we adopted a Newton-Raphson technique, with initial iteration values obtained by a Newton backward interpolation formula

$$\hat{Y}_i^{(n)} = y_{n+1} + \sum_{j=1}^{Q} \frac{(c_i - 1)c_i(c_i + 1)(c_i + 2)\dots(c_i + j - 2)}{j!} \nabla^j y_{n+1}, \quad i = 1, 2,$$

where $\hat{Y}_i^{(n)} \cong y(t_n + c_i h)$ $(= y(t_{n+1} + (c_i - 1)h))$, $0 < c_i \leq 2$. More precisely, in order to get a starting value with accuracy order matching the estimated order of the extended GL methods, we take $Q = 2, 3, 4$ for the methods of Example 5.1, 5.3 and 5.2, respectively. This proved to provide a great savings in computational cost. The accuracy of the numerical solutions are shown in Figs. 7.4-7.6, and the estimated convergence orders are given in Table 7.2. As in the previous example, the methods again seem to preserve the inherent accuracy of the underlying components.

EXAMPLE 7.3. Naturally, one also wishes to know what happens to an extended GL method when the order of the underlying method differs from that of the quadrature formula. In order to gain some insight, we considered the third order MRK method from Example 5.3 combined with the first order quadrature formula:

$$Z_j^{(n)} = h \sum_{q=0}^{m} g(t_{n,j}, t_{n-q,j}, Y_j^{(n-q)}), \quad j = 1, 2, \tag{7.3}$$

Table 7.3

Convergence order of extended GL methods for (7.2)*: 3rd order MRK method combined with 1st order method* (7.3)*, combined with 2nd order method* (7.4)*, combined with 3rd order method* (5.15)*, and combined with 4th order method* (5.14)*.*

m	(7.3)	(7.4)	(5.15)	(5.14)
8	1.6318	3.5703	4.1053	4.2630
16	1.4976	3.2562	3.8449	3.9556
32	1.4102	3.0393	3.6858	3.7715
64	1.3489	2.8821	3.5787	3.6492

Table 7.4

Convergence order of extended GL methods for (7.2)*: 4th order MRK method combined with 1st order method* (7.3)*, combined with 2nd order method* (7.4)*, combined with 3rd order method* (5.15)*, and combined with 4th order method* (5.14)*.*

m	(7.3)	(7.4)	(5.15)	(5.14)
8	1.6320	3.5895	4.3429	4.0022
16	1.4976	3.2555	4.1125	3.9982
32	1.4102	3.0354	3.9351	3.9964
64	1.3489	2.8804	3.8030	3.9960

combined with the second order quadrature formula:

$$Z_j^{(n)} = h\sum_{q=0}^{m}{}'' \, g\left(t_{n,j}, t_{n-q,j}, Y_j^{(n-q)}\right), \quad j = 1, 2, \tag{7.4}$$

and combined with the fourth order quadrature formula (5.14). The estimated convergence orders are given in Table 7.3. For completeness, we also added the second column from Table 7.2, where the method was combined with a third order quadrature method. Finally, in Table 7.4, we present similar results for the fourth order MRK method from Example 5.2, combined the four quadrature rules of different orders.

All of these results seem to indicate that the order of the extended GL method is equal to the minimum of the orders of its two components: the underlying GL method and the quadrature rule. A proof for this observation, and a discussion or derivation of the conditions under which it would hold, remain to be found.

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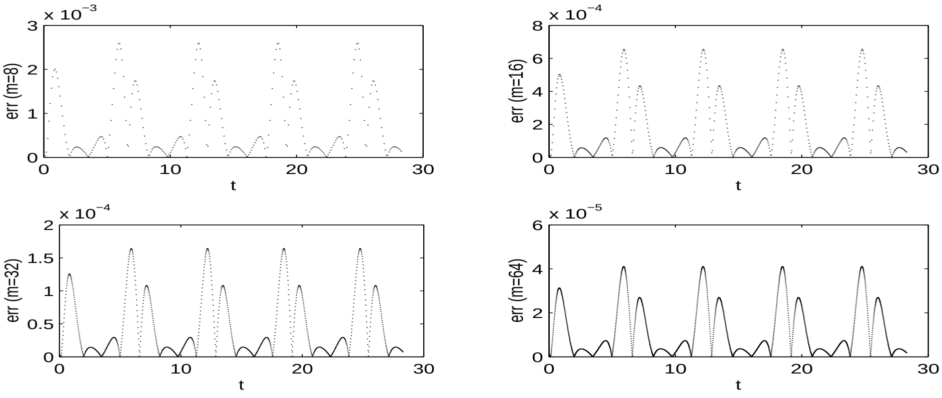

Fig. 7.1. *Error of the method from Example 5.1 with* $m = 8, 16, 32, 64$ *for* (7.1) *on* $[0, 9\pi]$.

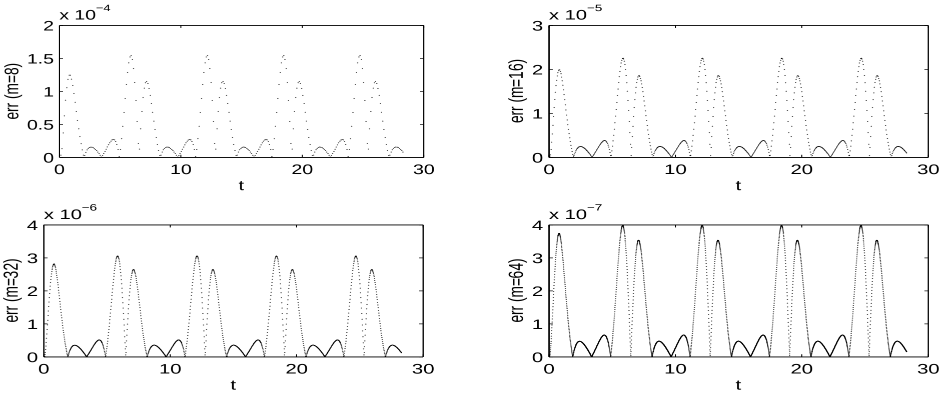

Fig. 7.2. *Error of the method from Example 5.3 with* $m = 8, 16, 32, 64$ *for* (7.1) *on* $[0, 9\pi]$.

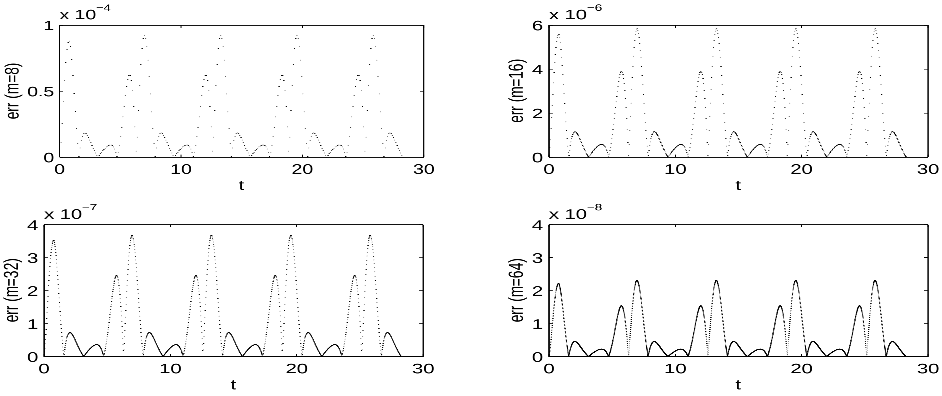

Fig. 7.3. *Error of the method from Example 5.2 with* $m = 8, 16, 32, 64$ *for* (7.1) *on* $[0, 9\pi]$.

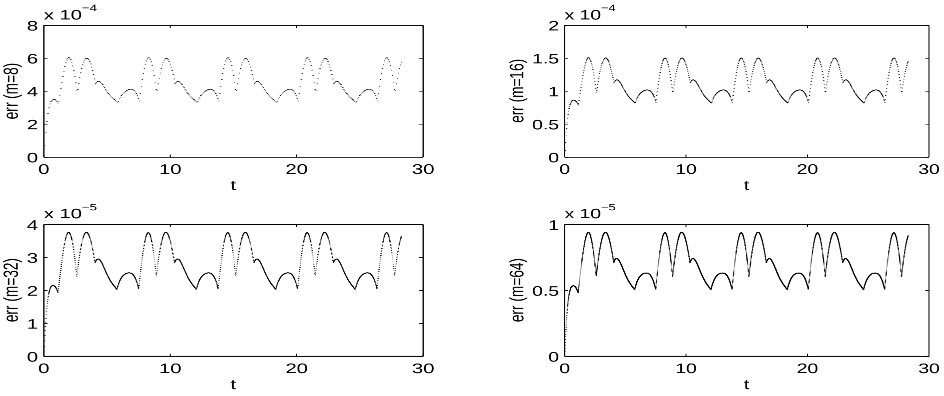

FIG. 7.4. *Error of the method from Example 5.1 with* $m = 8, 16, 32, 64$ *for* (7.2) *on* $[0, 9\pi]$.

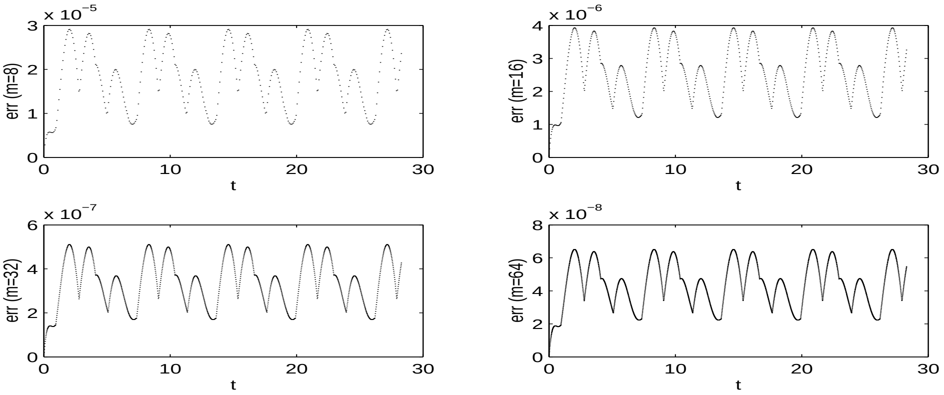

FIG. 7.5. *Error of the method from Example 5.3 with* $m = 8, 16, 32, 64$ *for* (7.2) *on* $[0, 9\pi]$.

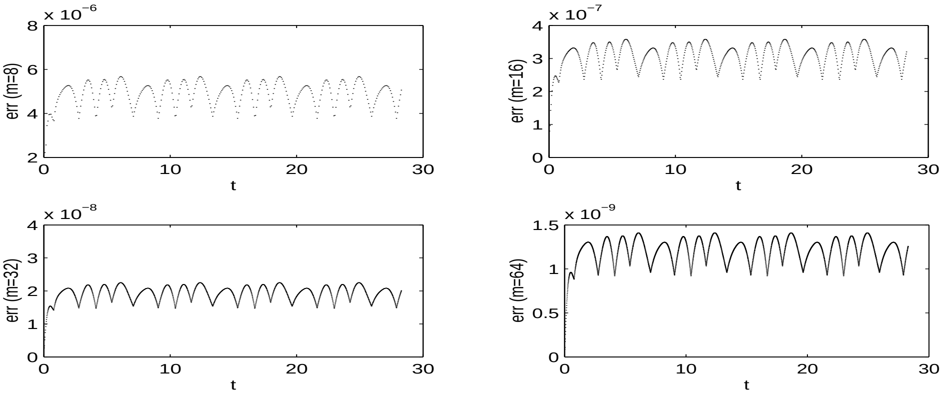

FIG. 7.6. *Error of the method from Example 5.2 with* $m = 8, 16, 32, 64$ *for* (7.2) *on* $[0, 9\pi]$.