Robust stabilization of time-delay systems with distributed delay control laws: necessary and sufficient conditions for a safe implementation

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Report TW 363, July 2003

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Keywords : delay equations, systems of neutral type, finite spectrum assignment, robustness.

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1 Introduction

Consider the linear finite-dimensional system with input delay

$$\dot{x}(t) = Ax(t) + Bu(t-\tau),\ x \in \mathbb{R}^d,\ u \in \mathbb{R}, \tag{1}$$

where we assume that the matrix A is not Hurwitz and the pair (A,B) controllable. An approach to stabilize the system (1), called the finite spectrum assignment (FSA) approach [14, 31], can be interpreted as follows: first a prediction of the state variable over one delay interval is generated and then a feedback of the predicted state is applied, thereby compensating the effect of the time-delay. This results in a closed-loop system with a finite number of eigenvalues, which can be assigned arbitrarily. Mathematically, with the feedback law

$$\begin{aligned} u(t) &= K^T x_p(t,t+\tau) \\ &= K^T \left(e^{A\tau}x(t) + \int_0^\tau e^{A\theta}Bu(t-\theta)d\theta\right), \end{aligned} \tag{2}$$

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where $x_p(t_1,t_2)$ is the prediction of $x(t)$ at $t=t_2$, based on values of x and u for $t\leq t_1$, the characteristic equation of the closed-loop system is given by

$$\det\left(\lambda I - A - BK^T\right)=0. \tag{3}$$

The elimination of the delay is employed in the so-called process model control techniques [32], as for example, the celebrated Smith Predictor [27]. In (1)-(2) it can also be interpreted as the effect of a model transformation, the Artstein's model reduction technique [1]. As shown in [14], the finite spectrum assignment approach can be generalized to more general systems, provided a spectral controllability condition is satisfied.

A difficulty in applying a control law of the form (2) consists of the practical implementation of the integral term, which needs to be calculated on-line. In this paper we approximate the distributed delay by a sum of point-wise delays by using a numerical quadrature rule, as suggested in [14], and investigate the effect of such semi-discretization on the stability of the closed-loop system, emphasizing the effect of an arbitrarily fine discretization. To state this more precise, first define $\mathcal{C}([0,\ \tau],\ \mathbb{C}^d)$ as the space of continuous functions from $[0,\ \tau]\subset\mathbb{R}$ to $\mathbb{C}^d$, equipped with the supremum norm. Then consider a sequence of operators $\{I_n\}_{n\geq 1}$ from $\mathcal{C}([0,\ \tau],\mathbb{C}^d)\rightarrow\mathbb{C}$, which are defined by

$$I_n(f)=\sum_{j=1}^{n} h_{j,n} f(\theta_{j,n}),\ \ h_{j,n}\in\mathbb{R},\ \theta_{j,n}\in[0,\ \tau], \tag{4}$$

for $f\in\mathcal{C}([0,\ \tau],\mathbb{C}^d)$ and satisfy the following convergence property:

$$\forall f\in\mathcal{C}([0,\ \tau],\mathbb{C}^d),\ \forall\varepsilon>0,\ \exists\bar{n}\in\mathbb{N}\ \text{s.t.}\ |I_n(f)-\textstyle\int_0^\tau f(\theta)d\theta|<\varepsilon,\ \forall n\geq\bar{n}. \tag{5}$$

When the quadrature formulae (4) are used to approximate the integral term in (2), we end up with a sequence of control-laws

$$u(t)\quad = K^T\left(e^{A\tau}x(t)+\textstyle\sum_{j=1}^{n} h_{j,n}e^{A\theta_{j,n}}Bu(t-\theta_{j,n})\right). \tag{6}$$

The effect of the discretization on stability has already been studied in the literature. In [29, 6] it was demonstrated with a scalar example that for some parameter values, the control law (6) may *not* stabilize the system (1), for *arbitrarily large values* of n. In [6, 25] the underlying instability mechanism was investigated and a necessary stability condition for large values of n was provided, expressed by the asymptotic stability of the solution semi-group of the functional difference equation

$$u(t)=K^T\int_0^\tau e^{A\theta}Bu(t-\theta)d\theta, \tag{7}$$

whose spectrum provides information on the position of the high-frequency modes of (1)-(6). However, we illustrate that this condition is not sufficient for stability. In fact, (in)stability may be sensitive firstly to the particular integration rule used in (6), more precisely the choice of the abscissa $\theta_{j,n}$, and secondly to *arbitrarily small* relative perturbations of these abscissa. Since small perturbations are inevitable in any practical application, the latter implies that a stability definition should take the robustness against such perturbations into account. Therefore, we say:

Definition 1 *The system (1)-(2) is robustly stable with respect to an implementation of the control law with the quadrature rule (4) when there exists a number $\bar{n}\in\mathbb{N}$ such that the closed-loop system (1)-(6) is asymptotically* *stable for all $n \geq \bar{n}$. Moreover, for each $n \geq \bar{n}$, there exist constants $\Delta\theta_{j,n} > 0$ such that the control-law*

$$u(t) = K^T \left(e^{A\tau} x(t) + \sum_{j=1}^{n} h_{j,n} K^T e^{A(\theta_{j,n}+\delta\theta_{j,n})} B u(t - (\theta_{j,n} + \delta\theta_{j,n})) \right)$$

achieves asymptotic stability for all $|\delta\theta_{j,n}| < \Delta\theta_{j,n}$.

We will provide a necessary *and* sufficient condition for the robust stability of the system (1)-(2). This *robust* stability condition turns out to be *independent* of the integration rule used. For simplicity of the notation we only consider perturbations on the $\theta_{j,n}$ in Definition 1, since we will show that (arbitrarily) small perturbations on the weights $h_{j,n}$ in (6) and small modelling errors on the system parameters A, B, τ cannot affect the asymptotic stability of the closed-loop system.

The fact that the control law (6) may not stabilize the system (1) for *arbitrarily large* values of *n*, despite the asymptotic stability of the ideal closed-loop system (1)-(2), and the sensitivity of stability w.r.t. *arbitrarily small* perturbations on the abscissa in (6) are phenomena, which are related to the lack of robustness of some boundary controlled hyperbolic PDEs, feedback controlled descriptor systems and neutral type systems against small delays in the control loop, as reported in e.g. [3, 4, 5, 10, 15, 16, 17, 18]. Also in the Smith predictor control scheme [27], sensitivity of stability w.r.t. infinitesimal modelling errors may occur, see [24] and the references therein.

The structure of the paper is as follows. After some preliminaries, we describe and illustrate the instability mechanism which may occur when approximating the integral term in (2). Next we derive necessary and sufficient conditions for the robust stability of (1)-(2), based on the stability theory of difference equations and neutral systems, developed in [2, 22]. Finally we take the obtained stability conditions under a closer look. We briefly discuss some examples and other stabilization approaches. In the conclusions we mention the main contributions of this paper.

2 Characteristic equation

The characteristic equation of the closed-loop systems (1)-(2) and (1)-(6) takes the form

$$\det\left(\begin{bmatrix} \lambda I - A & -Be^{-\lambda\tau} \\ -K^T e^{A\tau} & G(\lambda) \end{bmatrix} \right) = 0, \tag{8}$$

which can be transformed to

$$\det\left((\lambda I - A) G(\lambda) - B K^T e^{A\tau} e^{-\lambda\tau} \right) = 0. \tag{9}$$

When the control law (2) is implemented exactly, we have $G(\lambda) = G_\infty(\lambda)$, where

$$G_\infty(\lambda) \triangleq 1 - \int_0^\tau K^T e^{A\theta} B e^{-\lambda\theta} d\theta \tag{10}$$

$$= 1 + K^T \left(e^{(A-\lambda I)\tau} - I \right) (\lambda I - A)^{-1} B. \tag{11}$$

Substituting Equation (11) into (9) leads, after some calculations, to the characteristic Equation (3).

When the discretized control laws (6) are used, we have $G(\lambda) = G_n(\lambda)$, defined by

$$G_n(\lambda) \triangleq 1 - \sum_{j=1}^{n} h_{j,n} K^T e^{A\theta_{j,n}} B e^{-\lambda\theta_{j,n}}, \tag{12}$$

and Equation (9)-(12) can be rewritten as

$$\lambda^n \left(1-\sum_{j=1}^{n} h_{j,n} K^T e^{A\theta_{j,n}} B e^{-\lambda\theta_{j,n}}\right) + q(\lambda) = 0, \tag{13}$$

where for any $\alpha \in \mathbb{R}$, $\lim_{|\lambda|\to\infty,\ \Re(\lambda)\geq\alpha} q(\lambda)/\lambda^n = 0$. This is also the characteristic equation of the delay differential equation

$$\frac{d^n}{dt^n}\left(x(t)-\sum_{j=1}^{n} h_{j,n} K^T e^{A\theta_{j,n}} Bx(t-\theta_{j,n})\right) + q(\frac{d}{dt})x = 0. \tag{14}$$

Equation (9)-(12) is of *neutral* type. In fact, the approximation of the integral term forms a non-compact perturbation of the time-integration operator (solution semi-group) associated with the closed-loop system, which introduces an *essential spectrum*. As illustrated in [9, 2], precisely such an essential spectrum may cause a sensitivity of stability w.r.t. arbitrarily small modifications of the characteristic equation[2], such as perturbations on parameters. It corresponds to sequences of roots of (9)-(12) whose moduli tend to infinity, yet whose real parts have a *finite* limit.

Since the essential spectrum of the solution semi-group of (14) coincides with the essential spectrum of the solution semi-group of the functional difference equation

$$x(t) = \sum_{j=1}^{n} h_{j,n} K^T e^{A\theta_{j,n}} Bx(t-\theta_{j,n}), \tag{15}$$

see e.g. [8], we are led to the study of the robust stability of (15), or equivalently, of the behavior of the roots of its characteristic equation

$$G_n(\lambda) = 0. \tag{16}$$

Here, the robust stability of the difference equation is defined in an analogous way as in Definition 1, i.e. the stability of (15) should be robust against small perturbations on the parameters $\theta_{j,n}$.

In the rest of the paper we focus on the robust stability of the difference equation because in the proof of the main theorem, Theorem 3, we will show that for large n, the robust (in)stability of the closed-loop system (1)-(2) is completely determined by the robust stability of Equation (15). Notice the well known fact that the asymptotic stability of (15) is a necessary condition for the asymptotic stability of (14), see [8, 12], hence, we strengthen in some sense this result for the particular problem considered in this paper.

3 Instability mechanism

The difference equation (15) and, as a consequence, the system (1)-(6) may be unstable for *arbitrarily* large values of n. This is not in conflict with the stability of (1)-(2) and can intuitively be explained with the occurrence of unstable eigenvalues with a large modulus. When the approximation becomes better, some eigenvalues tend to the eigenvalues of the limit case, while the other have to move off to infinity. When some eigenvalues do so without leaving the right half plane, instability is persisted. This is now illustrated with an example. We also comment on the connection between the asymptotic stability of the difference equation (15) for large n, which is a necessary condition for the asymptotic stability of (14), and the asymptotic stability of Equation (7), expressed by the position of the roots of its characteristic equation

$$G_\infty(\lambda) = 0. \tag{17}$$

[2] in compact subsets of the complex plane

Thereby, we show that the stability of (7)-(17) is generally not sufficient for the stability of (15)-(16) and thus of (1)-(6) when n is sufficiently large.

Remark 2 *Equation (17) can be re-arranged as*

$$\det\left(\lambda I-(A+BK^T)+BK^Te^{A\tau}e^{-\lambda\tau}\right)/\det(\lambda I-A)=0,$$

and, hence, its solutions can be calculated by computing the eigenvalues of the DDE

$$\dot{x}=(A+BK^T)x-BK^Te^{A\tau}x(t-\tau), \tag{18}$$

which is possible using e.g. the software package DDE-Biftool [7], and removing the eigenvalues of A. Since this equation is of retarded type, its solution semi-group has no essential spectrum, which explains why the ideal closed-loop system (1)-(2) has only a finite number of eigenvalues.

Consider the system

$$\dot{x}(t)=x(t)+u(t-1), \tag{19}$$

and the control laws (2)-(6), where the controller gain satisfies $K=-3/2$ and the parameters of the integration rule are given by

$$\theta_{j,n}=\tfrac{j-1}{n},\quad h_{j,n}=\tfrac{1}{n},\quad j=1\dots n. \tag{20}$$

Note that this integration rule satisfies (5). In Figure 1 we plot the roots of (16) for $n=40$ and the roots of (17), illustrating the correspondence between the rightmost roots. However, in Figure 2 (above) we plot the roots of (16) for $n=40$ on a different scale. This reveals the fact that, due to the approximation of the integral, the spectrum has become *periodic* w.r.t. shifts in the imaginary parts of the eigenvalues and, hence, the corresponding solution semi-group has got an essential spectrum. Precisely the (introduced) eigenvalues with a large imaginary part are also approximate eigenvalues of the closed-loop system (1)-(6). In Figure 2 (below) the roots of (16) are shown for $n=60$ also. By increasing n, the rightmost roots of the limit case (17) are better approximated and meanwhile the periodicity in the shifts of the imaginary parts increases (more precisely, it equals $2\pi ni$). This implies that the eigenvalues, introduced by the approximation, move off to infinity.

The fact that large qualitative changes of stability properties for arbitrarily small perturbations are caused by the introduction or the behavior of an essential spectrum [22] and the analysis with the above example could indicate that the stability of the difference equation (15) might be determined by the real part of the rightmost solution of (17), i.e. the eigenvalues of Equation (7), for any integration rule, as suggested in [6, 25]. However, such a conclusion is false and the stability of (7)-(17) is thus not sufficient for the stability of (1)-(6). Indeed, when in the example the integration rule is modified to

$$\theta_{j,n}=\begin{cases}\frac{j-1}{n}, & j \text{ even}\\ \frac{j-4/5}{n}, & j \text{ odd}\end{cases},\quad h_{j,n}=\tfrac{1}{n},\quad j=1\dots n \tag{21}$$

which also satisfies the convergence property (5), the roots of (16) for $n=40$ and $n=60$ are shown in Figure 3 and we have instability. Although the rightmost roots of (17) are well approximated also, their real parts do no longer determine the stability of the corresponding difference equation (15). In the next section we will derive a necessary and sufficient condition for the *robust* stability of (1)-(6) (following Definition 1), which turns out to be *independent* of the type of integration rule used. For our scalar example (19) with $K=-3/2$, this condition is not satisfied. Although the first integration rule (20) leads to a stable essential spectrum, stability can always be destroyed by *infinitesimal* perturbations of the abscissa, as mathematically proven in the next Section.

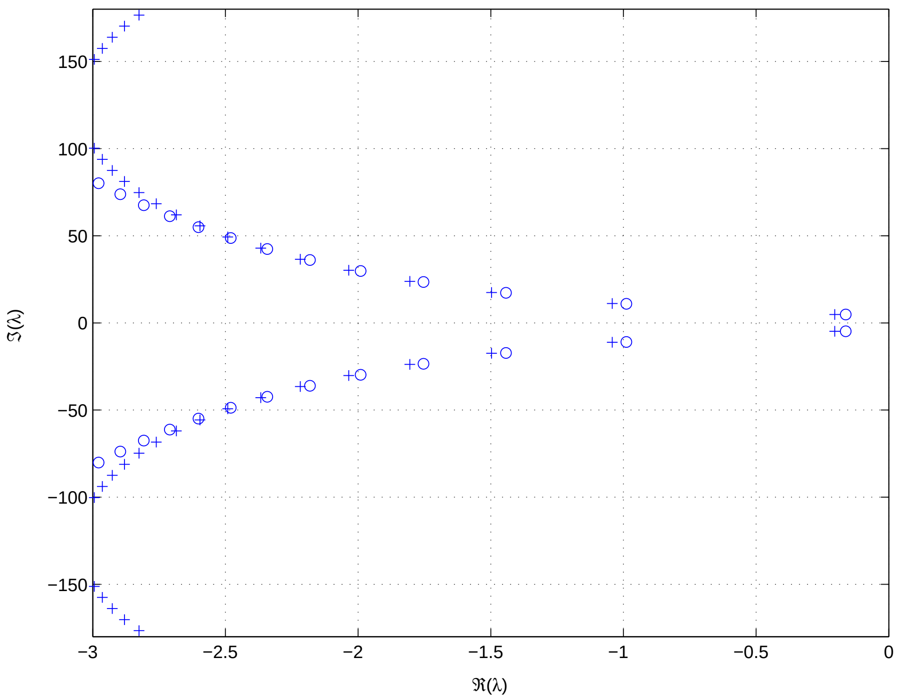


Figure 1: Roots of (17), indicated by circles, and roots of (16) for $n = 40$, corresponding to the system (19)-(20). The roots of (17) are calculated with the software package DDE-Biftool [7], after transformation to (18). The roots of (16) are calculated based on the observation that this equation is a polynomial in $\exp(-1/n)$.

4 Conditions for robust stability

In order to check the (robust) stability of the difference equation (15), we have to analyze the set

$$Z_n = \{\Re(\lambda) : \ G_n(\lambda) = 0\},$$

whose properties are described in [2, 22] and now briefly rehearsed. The difference equation (15) can be interpreted as an equation with n different delays $\theta_{j,n}, \ j = 1 \ldots n$. The stability is influenced by the rational dependency structure of these delays. To express this structure, assume that the n delays $\theta_{j,n}$ depend on $m \leq n$ rationally independent delays[3] $(r_1, \ldots, r_m)$, i.e. there exist *integer* numbers $\gamma_{j,k}, \ j = 1 \ldots n, \ k = 1 \ldots m$, such that

$$\theta_{j,n} = \sum_{k=1}^{m} \gamma_{j,k} r_k, \ j = 1 \ldots n.$$

Then the closure of the set $\bar{Z}_n$ satisfies $\bar{Z}_n = \mathcal{Z}_n^m(\Theta_n)$, where $\Theta_n = (\theta_{1,n}, \ldots, \theta_{n,n})$ and

$$\begin{aligned} \mathcal{Z}_n^m(\Theta_n) = \ & \{\alpha \in \mathbb{R} : \ \exists \varphi_1 \ldots \varphi_m \in [0, \ 2\pi) \text{ such that} \\ & 1 - \textstyle\sum_{j=1}^n h_{j,n} K^T e^{A\theta_{j,n}} B e^{-\alpha\theta_{j,n}} e^{-i\sum_{k=1}^m \gamma_{j,k}\varphi_k} = 0\}. \end{aligned} \qquad (22)$$

[3]The delays $(r_1, \ldots, r_m)$ are rationally independent when $\sum_{k=1}^m \alpha_k r_k = 0, \ \alpha_k \in \mathbb{Z}$ implies $r_k = 0, \ \forall k = 1 \ldots m$. For instance two delays are rationally independent when their ratio is an irrational number.

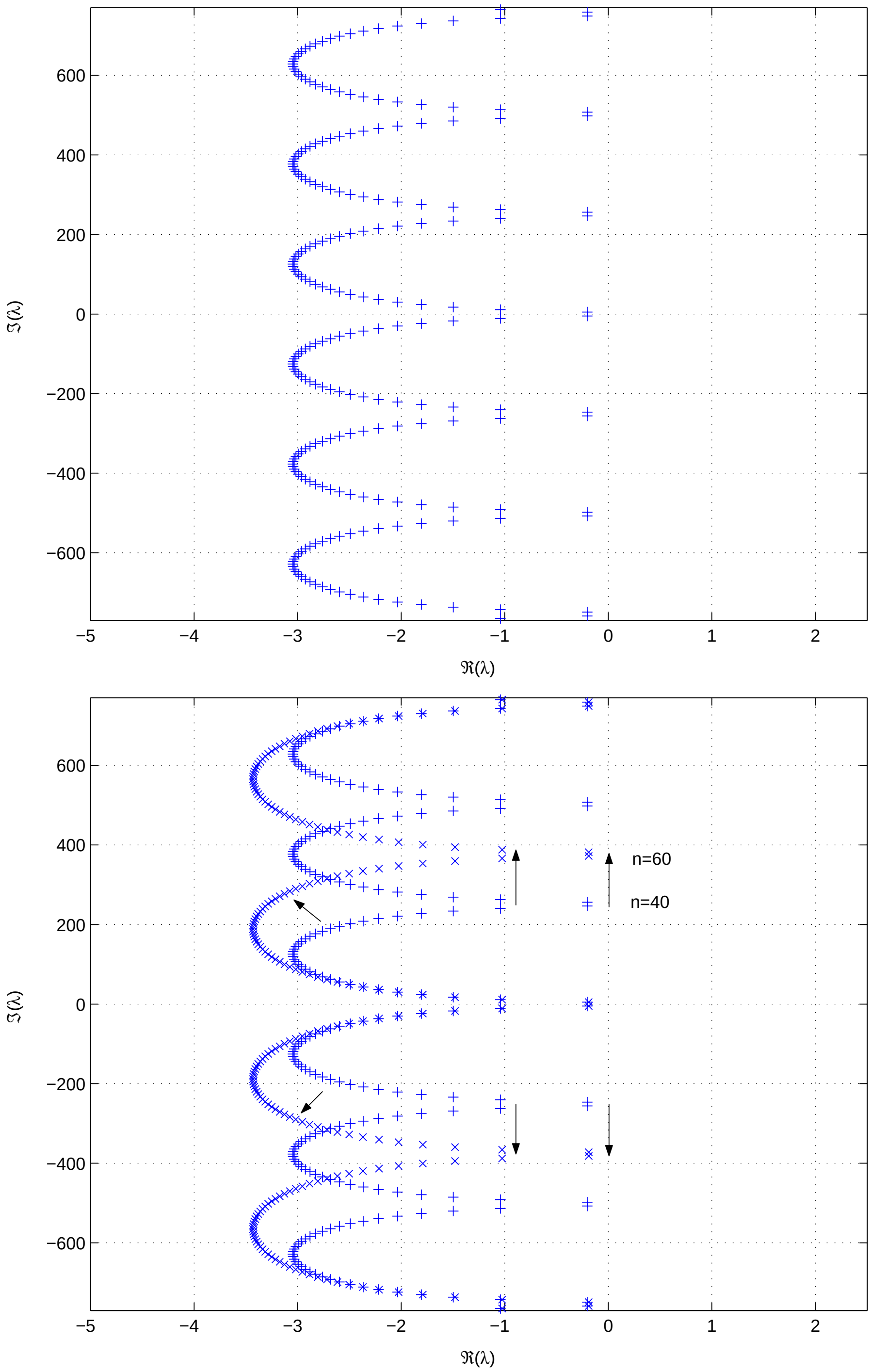
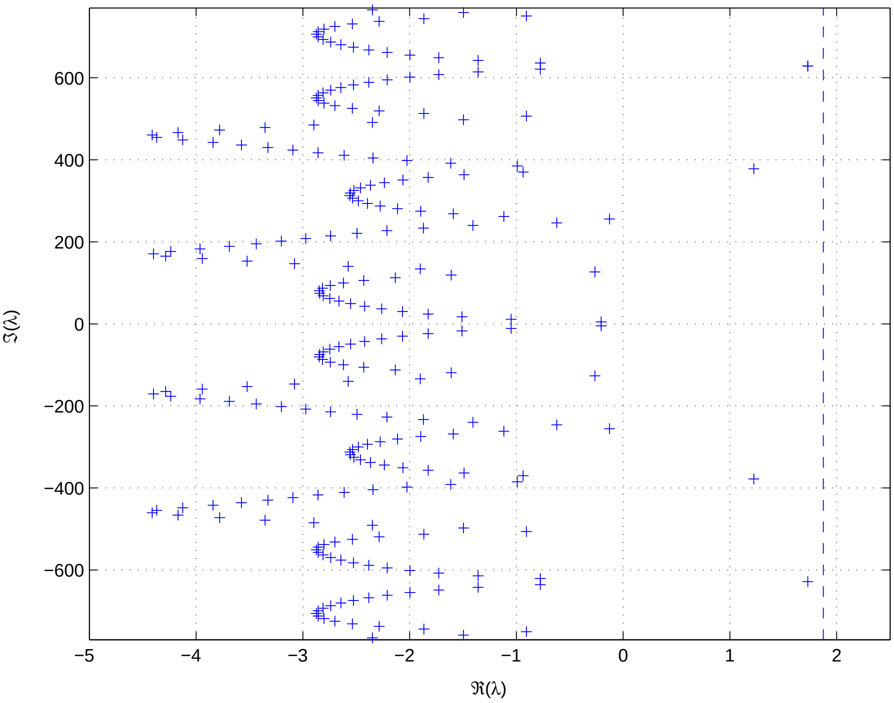


Figure 2: (Above) Roots of (16) for $n = 40$. A detail of this plot is shown in Figure 1. (Below) Roots of (16) for both $n = 40$, indicated with '+' and $n = 60$, indicated with 'x'. The system under investigation is (19)-(20).



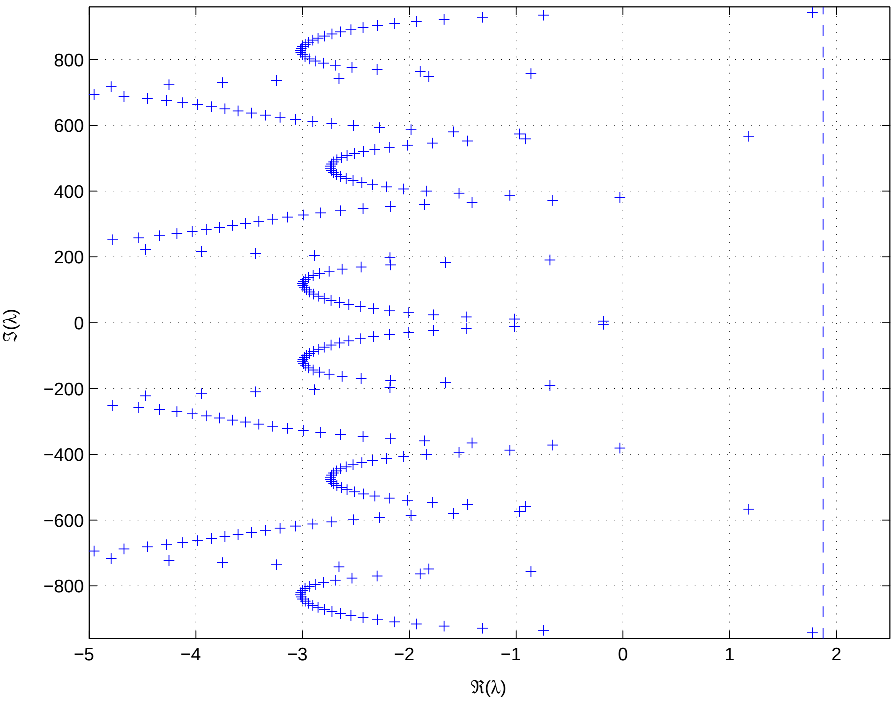


Figure 3: Roots of (16) for the system (19) when using the modified integration rule (21) with $n = 40$ (above) and $n = 60$ (below). Unlike the situation displayed in Figure 2, stability for large n is no longer determined by the roots of Equation (17).

This result follows from substituting $\lambda = \alpha + i\beta$ in the characteristic equation of (15) and the observation that with a suitable choice of β, $(\beta r_1, \dots, \beta r_m)$ mod 2π is arbitrarily close to any given $(\varphi_1, \dots, \varphi_m)$ by Kronecker's Theorem [11, Theorem 444]. Note that for given m, the dependency structure of the delays and, as a consequence, the set $\mathcal{Z}_n^m$ also depend on the particular values of the integers $\gamma_{j,k}$, but for reasons of simplicity we suppress this explicit dependence in the notation.

The set (22) always consists of a finite number of intervals. When $m = 1$, the n delays are *fully dependent* or *commensurate*. Then the spectrum of (15) is periodic w.r.t. shifts in the imaginary parts of the eigenvalues and the set $\mathcal{Z}_n^1(\Theta_n)$ consists of a finite number of points, as illustrated with the examples in the previous section. For $m > 1$, the spectrum is quasi-periodic w.r.t. changes in the imaginary parts. When $m = n$, the delays are *fully independent*. Furthermore, we have for all m,

$$\mathcal{Z}_n^m(\Theta_n) \subseteq \mathcal{Z}_n^n(\Theta_n). \tag{23}$$

Since arbitrarily small perturbations of the n delays $\theta_{j,n}$ may change their rational dependency structure, they may affect stability, as follows from (22). This paradox is explained by the lower semi-continuity of the map $\Theta_n \to Z_n(\Theta_n)$ w.r.t. the Hausdorff metric and concerns the behavior of the eigenvalues with a *large* imaginary part. For more details we refer to [2, 22].

We seek conditions for which the difference equation (15) is *robustly* stable. Because the n delays $\theta_{j,n}$ can always be perturbed to fully independent delays by applying *arbitrarily* small changes, and the relation (23), robust stability is determined by the maximal value $\alpha_{\max}(n)$ of $\mathcal{Z}_n^n(\Theta_n)$, defined by the equation

$$1 - \sum_{j=1}^{n} h_{j,n} |K^T e^{A\theta_{j,n}} B| e^{-\alpha_{\max}(n)\theta_{j,n}} = 0, \tag{24}$$

which converges[4] to α_M, the solution of

$$1 - \int_0^{\tau} |K^T e^{A\theta} B| e^{-\alpha_M} d\theta = 0, \tag{25}$$

as $n \to \infty$. Hence, the robust (in)stability of the difference equation (15) for large n is determined by the condition $\alpha_M < 0$ ($\alpha_M > 0$). Rewriting this condition and taking into account the relation between the spectrum of (15) and the spectrum of (14), or equivalently, the spectrum of the closed loop system (1)-(6) we obtain the main result:

Theorem 3 *Consider the system (1)-(2) where $A + BK^T$ is Hurwitz. Let*

$$S = \int_0^{\tau} |K^T e^{A\theta} B| d\theta.$$

If $S < 1$, then the system (1)-(2) is robustly stable according to Definition 1.
If $S > 1$, then the system (1)-(2) is not robustly stable.

Proof. When $S > 1$, the difference Equation (15), is not robustly stable for large values of n. This implies the lack of robust stability of (14) and of (1)-(2), as follows from the arguments spelled out before. The case where $S < 1$ deserves further attention. Here, note that the difference equation (15) only provides information on the essential spectrum of the solution semi-group of (14), while the latter also has a point spectrum.

[4] To see this, note that the left hand side of (24), resp. of (25) is a strictly increasing function of $\alpha_{\max}$, resp. α_M.

When $S < 1$, we have $\alpha_M < 0$. Take a number ε satisfying $\alpha_M < \varepsilon < 0$ and such that $A + BK^T$ has no eigenvalues in the half plane $\mathbb{C}_\varepsilon \triangleq \{\lambda \in \mathbb{C} : \Re(\lambda) \geq \varepsilon\}$. Equation (13) can be rewritten as

$$G_n(\lambda) = -q(\lambda)/\lambda^n. \tag{26}$$

Because $S < 1$ it is easy to prove that, for large n, $|G_n(\lambda)|$ can be uniformly bounded from below over $\mathbb{C}_\varepsilon$ by a strictly positive constant, independent of n. Furthermore, such a bound can be chosen which is *robust* against small perturbations[5] on the parameters $\theta_{j,n}$. On the other hand, the right-hand side of (26) tends to zero for large $|\lambda|$. This implies the existence of numbers $M, \bar{n} > 0$ such that all the solutions of (26), i.e. the eigenvalues of the closed-loop system (1)-(6), in $\mathbb{C}_\varepsilon$ satisfy $|\lambda| \leq M$ when $n \geq \bar{n}$, also when small perturbations are taken into account. Define the *compact* set $\mathcal{S} \triangleq \{\lambda \in \mathbb{C}_\varepsilon : |\lambda| \leq M\}$. Since on $\mathcal{S}$, the function $\det\left((\lambda I - A)G_n(\lambda) - BK^T e^{A\tau} e^{-\lambda\tau}\right)$ uniformly converges to the function $\det\left((\lambda I - A)G(\lambda) - BK^T e^{A\tau} e^{-\lambda\tau}\right) = \det(\lambda I - A - BK^T)$ as $n \to \infty$, we can apply [22, Theorem A.1], which states that the two functions have the same number of zeros in $\mathcal{S}$ when n is sufficiently large. Because S is compact, small perturbations cannot change this result. Therefore, the characteristic equation of the system (1)-(6) has no solutions in $\mathcal{S}$ for large n and, as a consequence, in $\mathbb{C}_\varepsilon$, and is (robustly) asymptotically stable. ∎

Remark 4 *The robust stability condition of Theorem 3 is independent of the type of quadrature rule used, only the convergence property (5) is assumed.*

Remark 5 *For $S = 1$ we can conclude on the existence of sequences of characteristic roots of (1)-(6) which approach the imaginary axis. However, the way of approaching the imaginary axis (from the left/from the right/oscillatory) may also depend on other system parameters than those of the difference equation (15), which only describes the limit case.*

Remark 6 *So far we have only considered small perturbations on the abscissa $\theta_{j,n}$ in (6). From expression (22) it follows that arbitrarily small perturbations of the weights $h_{j,n}$ cannot affect asymptotic (in)stability. The structure of Equation (8) reveals that stability is also robust against small modelling errors on the system parameters A, B and τ.*

Example 7 *The system (19) with $K = -3/2$ is not robustly stable, since $S = 3/2(e-1) > 1$ and $\alpha_M \approx 1.8749$. In Figure 3 this value of α_M is indicated with the dashed line. For both the integration rules (20) and (21) the 'delays' in (15) are commensurate and the set Z_n consists of a finite number of points, as illustrated with Figures 2 and 3. For the rule (20) the delays have as greatest common divisor $1/n$, while this is $1/(5n)$ for (21). Therefore, the delays are 'more independent' in the second case and this intuitively explains why the set $\bar{Z}_n$ contains more points and better approximates the interval (with right end-point α_M), which would have been obtained when the delays were perturbed to fully independent delays.*

5 Robust stabilization

In order to design a robustly stable FSA controller for the system (1), the following synthesis problem needs to be solved:

Problem 8 *Find a feedback gain K such that*

$$A + BK^T \text{ is Hurwitz} \tag{27}$$

[5]Precisely this is the point where the stability condition of Equation (7) fails.

and

$$S = \int_0^{\tau} |K^T e^{A\theta} B| d\theta < 1. \quad (28)$$

We now comment on some theoretical properties, provide examples and briefly discuss some alternative approaches.

5.1 Properties

Instrumental to the feasibility study of Problem 8 is the following technical result:

Lemma 9 *Assume that A has eigenvalues in the closed right-half plane. Then for any (fixed) value of K, the system (1)-(2) is not robustly stable for large values of the time-delay.*

Proof. Define $\lambda_m = \arg\max\{\Re(\lambda) : \lambda \in \sigma(A)\}$. For any $\lambda \in \mathbb{C}^+$ with $\Re(\lambda) > \Re(\lambda_m)$, we have

$$\begin{array}{ll} \int_0^\infty |K^T e^{A\theta} B| d\theta & \geq \int_0^\infty |K^T e^{A\theta} B|.|e^{-\lambda\theta}| d\theta \\ & \geq \left| \int_0^\infty K^T e^{A\theta} B e^{-\lambda\theta} d\theta \right| \\ & \geq \left| K^T (\lambda I - A)^{-1} B \right| \end{array} . \quad (29)$$

The last expression can be interpreted as the transfer function of the system (K^T, A, B). When $A + BK^T$ is Hurwitz, the pair (K^T, A) is detectable and the unstable poles of A appear in this transfer function. Consequently, by letting $\lambda \to \lambda_m$, the right-hand side of Equation (29) becomes arbitrarily large. Hence $\int_0^\infty |K^T e^{A\theta} B| d\theta = \infty$ and the robust stability condition of Theorem 3 is violated for large τ. ■

The next theorem concerns the solvability of Problem 8.

Theorem 10 *When A has all its eigenvalues in the* closed *left half plane, robust stability can be achieved for any value of the time-delay with a control law of the form (2). When A has eigenvalues in the* open *right half plane, then robust stabilization is not possible for large values of the delay.*

Proof. When A has its eigenvalues in the closed-left half plane, there always exists a sequence $\{K_n\}_{n\geq 1}$ with $K_n \to 0$ as $n \to \infty$ such that $A + BK_n^T$ is Hurwitz, which is a standard result in the low gain theory of ordinary differential equations [28, 13]. Because $K_n \to 0$, also the condition (28) can be satisfied for any delay value.

The proof of the second part is by contradiction. Therefore, assume that the statement is false, i.e.

$$\begin{array}{l} \exists \{\tau_n\}_{n\geq 1}, \{K_n\}_{n\geq 1} \text{ with } \lim_{n\to\infty} \tau_n = \infty \text{ such that} \\ A + BK_n^T \text{is Hurwitz and } \int_0^{\tau_n} |K_n^T e^{A\theta} B| d\theta < 1. \end{array} \quad (30)$$

This always leads to a contradiction.

<u>Case 1:</u> the sequence $\{K_n\}_{n\geq 1}$ is bounded.

Assume that this sequence is converging with limit K (in the other case one can always construct a converging subsequence). For $K = 0$ we have a contradiction between the Hurwitz stability of $A + BK_n$ for large n and the instability of A. When $K \neq 0$, there exist numbers $\bar{\tau}, \bar{n}$ such that

$$\int_0^{\bar{\tau}} |K^T e^{A\theta} B| d\theta > 2$$

(by the arguments used in the proof of Lemma 9) and

$$\left| \int_0^{\bar{\tau}} |K^T e^{A\theta} B| - |K_n^T e^{A\theta} B| d\theta \right| < 1, \ \forall n \geq \bar{n}.$$

Hence, we have for large n:

$$\int_0^{\tau_n} |K_n^T e^{A\theta} B| d\theta \geq \int_0^{\bar{\tau}} |K_n^T e^{A\theta} B| d\theta = \int_0^{\bar{\tau}} |K^T e^{A\theta} B| d\theta + \int_0^{\bar{\tau}} |K_n^T e^{A\theta} B| - |K^T e^{A\theta} B| d\theta > 1,$$

which contradicts the assumption.
<u>Case 2:</u> the sequence $\{K_n\}_{n\geq 1}$ is unbounded.
Assume without loosing generality that the sequence $\{F_n\}_{n\geq 1} = \left\{ \frac{K_n}{\|K_n\|} \right\}_{n\geq 1}$ is converging with limit F. Then the second statement of (30) can be written as

$$\int_0^{\tau_n} |F_n^T e^{A\theta} B| d\theta < \frac{1}{\|K_n\|}.$$

Since $F_n \to F$ and $\|K_n\| \to \infty$, we have

$$\int_0^{\infty} |F^T e^{A\theta} B| d\theta = 0, \tag{31}$$

and, as a consequence,

$$g(t) \triangleq F^T e^{At} B = 0, \ \ \forall t \geq 0.$$

This implies $g(0+) = 0, \ g'(0+) = 0, \cdots, g^{n-1}(0+) = 0$, which can be written as

$$F^T \left[B \, AB \, A^2 B \ldots A^{n-1} B \right] = 0.$$

From the controllability of (A, B) it follows that $F = 0$ and we have again a contradiction since $\|F\| = 1$. ■

The following theorem provides information on the structure of the solutions of (27)-(28).

Theorem 11 *Assume that Problem 8 is feasible. Then the set of robustly stabilizing feedback gains is bounded.*

Proof. The proof is by contradiction. Therefore, assume that there is a sequence $\|K_n\|_{n\geq 1}$ with $\|K_n\| \to \infty$ as $n \to \infty$, which satisfies the conditions of Problem 8, including

$$\int_0^{\tau} |K_n^T e^{A\theta} B| d\theta < 1. \tag{32}$$

Assume that the sequence $\{F_n\}_{n\geq 1} = \left\{ \frac{K_n}{\|K_n\|} \right\}_{n\geq 1}$ has a limit F. The inequality (32) can be written as $\int_0^{\tau} |F_n^T e^{A\theta} B| d\theta < 1/\|K_n\|$. Since $\|K_n\| \to \infty$ and $F_n \to F$, it follows that $\int_0^{\tau} |F^T e^{A\theta} B| d\theta = 0$. Hence, we have $F^T e^{At} B = 0, \ \forall t \in [0, \ \tau]$, which implies $F^T e^{At} B = 0, \ \forall t \in [0, \ \infty)$. As shown in the proof of the previous theorem, this leads to a contradiction. ■

Remark 12 *The stabilizability properties of Theorems 10 and 11 are comparable to those in the case of static state feedback discussed in [23, 21, 20, 19], where a controller of the form*

$$u(t) = K^T x(t) \tag{33}$$

is used. This results in the closed-loop system

$$\dot{x} = Ax + BK^T x(t-\tau). \tag{34}$$

Now there are infinitely many closed-loop eigenvalues, which have to be controlled with a finite number of controller parameters. This introduces the limitations on the stabilizability. For more details we refer to [19].

Note that, at first sight, we compare the theoretical stabilizability properties of (33) with the robust stabilizability properties of (2) (for an exact implementation there are no limitations). However, this comparison is justified because Equation (34) is of retarded type and, hence, its stability is robust against small perturbations of the parameters [8], i.e. the theoretical stability conditions are equal to the robust stability conditions.

5.2 Examples

As a first example consider the (robust) stabilizability of the scalar system

$$\dot{x} = ax + bu(t-\tau),$$

with a control law of the form (2), i.e. $u = kx_p(t, t+\tau)$. Then the stabilizability conditions (27)-(28) are expressed by

$$\begin{cases} a + bk < 0, \\ e^{a\tau} < 1 + \frac{a}{|bk|}. \end{cases}$$

In order to maximize the values of a and τ for which these conditions are satisfied, we can let $|k| \to |a/b|+$. This leads to the stabilizability condition

$$e^{a\tau} < 2 \;\to\; a\tau < 0.69315,$$

while for a simple state feedback controller, $u(t) = kx(t)$, the stabilizability condition is given by $a\tau < 1$, see [23].

As a second example we investigate the robust stabilizability of the system (1)-(2), where

$$A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $K = [k_1 \; k_2]^T$ as a function of the delay τ. The stabilizability conditions (27) and (28) are given respectively by

$$\begin{cases} k_2 < -2, \\ k_2 > 1/2\, k_1 - 5/4, \end{cases} \tag{35}$$

and

$$\int_0^\tau \left|(e^\theta \sin(\theta/2))k_1 + (e^\theta \cos(\theta/2))k_2\right| d\theta < 1. \tag{36}$$

Observe that conditions (35) imply that k_1 and k_2 are both negative. Then for $\tau \leq \pi$, condition (36) reduces to

$$-k_1 \int_0^\tau (e^\theta \sin(\theta/2))d\theta - k_2 \int_0^\tau (e^\theta \cos(\theta/2))d\theta < 1.$$

Performing the integrals gives the delay dependent linear restriction

$$-k_1\left(e^\tau(4\sin\frac{1}{2}\tau - 2\cos\frac{1}{2}\tau) + 2\right) - k_2\left(e^\tau(4\cos\frac{1}{2}\tau + 2\sin\frac{1}{2}\tau) - 4\right) < 5. \tag{37}$$

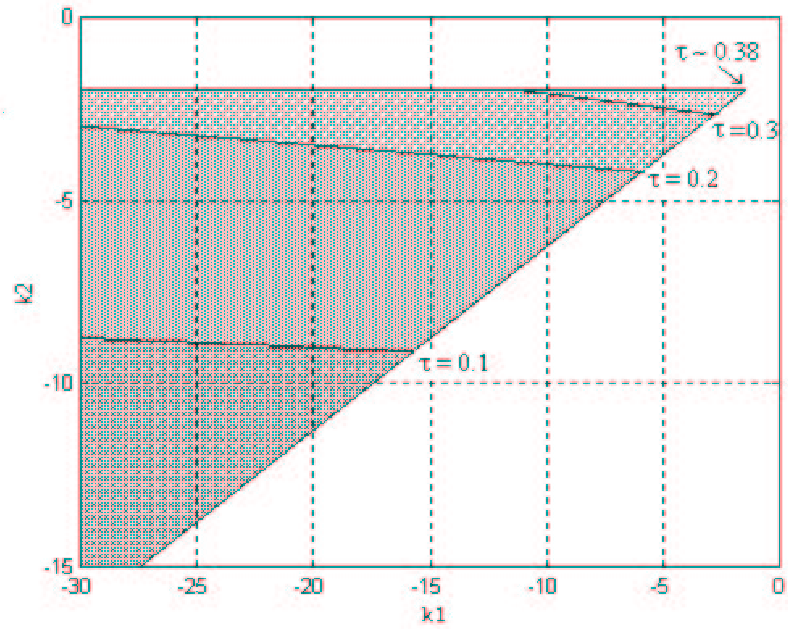


Figure 4: Stability regions using the distributed control law (2), for different values of the time delay.

Hence, the region where robust stability of the closed-loop system is insured can be described by the linear restrictions (35), which are the stabilizability conditions for the system without delay (and those when the integral in control law (2) is performed exactly), and the linear restriction (36), which is associated to robustness requirements and depends on the value of the delay. The above is illustrated in Figure 4. One observes that the robustness issue drastically reduces the scope of finite spectrum assignment control laws.

For comparison purposes, the stabilizability of the above system with the simple linear controller (33) is studied. The closed-loop quasipolynomial is

$$\lambda^2 - 2\lambda - \lambda k_2 e^{-\lambda\tau} + \frac{5}{4} + k_2 e^{-\lambda\tau} - \frac{1}{2} k_1 e^{-\lambda\tau}.$$

A D-subdivision analysis leads to the stability-instability boundaries for $\lambda = 0$ and $\lambda = j\omega$, respectively described by

$$\frac{5}{4} + k_2 - \frac{1}{2} k_1 = 0$$

and

$$k_1 = \frac{(5+4\omega^2)\sin(\omega\tau) + (-3\omega - 4\omega^3)\cos(\omega\tau)}{2\omega}$$
$$k_2 = \frac{(5-4\omega^2)\sin(\omega\tau) - 8\omega\cos(\omega\tau)}{4\omega}$$

As depicted in Figure 5 these boundaries define regions of stability that also depend on the delay.

In the case of the distributed control law, the linear restrictions (35),(37) intersect when $\tau = 0.383$, hence, for larger delays the robust stability region is empty. For static state feedback, according to the results developed in [21], the maximal delay value for stabilizability is $\tau = 0.62$. In the two above examples the stabil-

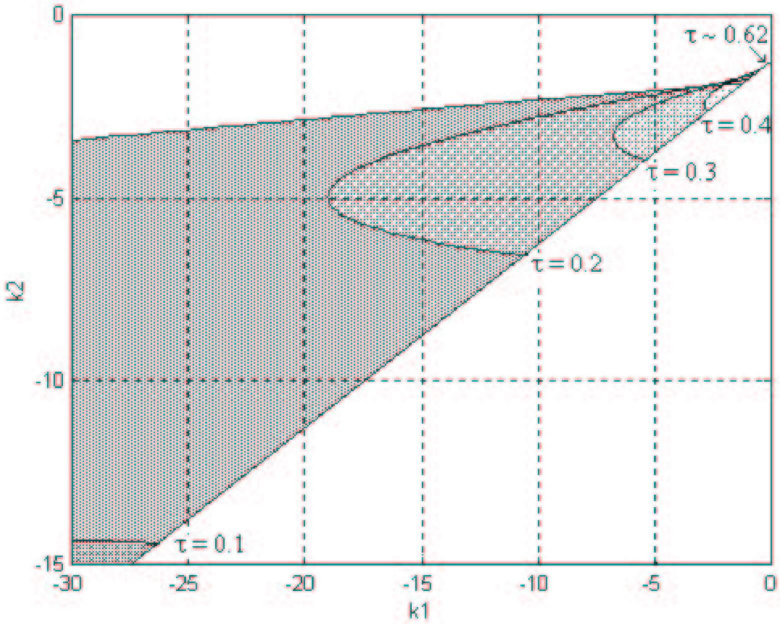


Figure 5: Stability regions using the static state feedback law (33), for different values of the time delay.

izability result for a simple (delayed) state feedback is less conservative than the one for a (robust) distributed delay control law.

5.3 Alternative approaches

The main advantage of the control-law (2) lies in the fact that all the closed-loop eigenvalues can be assigned arbitrarily. Disadvantages are the difficulty in implementing the control law (the integral should be calculated on-line) and the limitations concerning the robust stability of the closed-loop system. We mention possible alternatives or modifications.

1. Using the static state feedback controller (33), the robust stabilizability properties are comparable. They may even be better, as illustrated in Subsection 5.2. Furthermore, such a controller is very simple, easy to implement, and a numerical tool is available to calculate the feedback gain [23].

2. As explained before, the instability mechanism is a high frequency mechanism. Actually instability is caused by the throughput at infinity of past inputs in Equation (6). One could remove this problem with a low-pass filter and use e.g. a control law of the form

$$\begin{array}{l} \dot{v}(t) = -fv + fK^T\left(e^{A\tau}x(t) + \sum_{j=1}^{n} h_{j,n} e^{A\theta_{j,n}} Bu(t-\theta_{j,n})\right), \\ u(t) = v(t). \end{array} \quad (38)$$

   It is easy to show that in any complex half plane, $\Re(\lambda) \geq r,\ r \in \mathbb{R}$, there are only a finite number of closed-loop eigenvalues. As $n, f \to \infty$, only d eigenvalues stay in this half plane and converge to the eigenvalues of $A+BK^T$. These eigenvalues are not sensitive w.r.t. arbitrarily small parameter perturbation and (38) is robustly stabilizing. Here we should remark that we have only investigated the robustness in an asymptotic sense in this paper, i.e. for *arbitrarily* small perturbations and a large number of discretization points.

3. When the input u is kept *piecewise constant* in time-intervals of length Δ, as proposed in [30], the system (1) is completely equivalent with a discrete system. When $\Delta = \tau/p,\ p \in \mathbb{N}$, this discrete system has the form
$$x(k+1) = A_d x(k) + B_d u(k-p), \quad A_d = e^{A\Delta},\ B_d = \int_0^{\Delta} e^{A(\Delta-s)} B ds. \tag{39}$$
For small Δ this system has a large 'tail', which can be compensated with a prediction, analogously to the continuous time case: with the control law
$$\begin{array}{rl} u(k) & = K_d^T x_p(k, k+p) \\ & = K_d^T \left(A_d^p x(k) + \sum_{n=1}^{p} A_d^{n-1} B_d u(k-n) \right), \end{array} \tag{40}$$
the characteristic equation of the closed-loop system is given by $\det\left(zI - (A_d + B_d K_d^T)\right) = 0$. Because the system (39)-(40) is fully discrete, the maximal possible frequency is given by $1/(2\Delta)$ (the Nyquist-Shannon criterion) and, therefore, sensitivity of stability w.r.t. arbitrarily small perturbations of the parameters of (40) is not possible.

   Note that the control-law (40) can be considered as a discretization of the continuous control law (2), using a particular quadrature rule. However, in [30] it is illustrated that one cannot use any quadrature rule satisfying (5) in the discretization and obtain stability for sufficiently small values of Δ. For instance, applying Simpson's rule may lead to instability for small values of Δ. The instability mechanism is related to the one in the continuous time case, where approximations lead to eigenvalues with arbitrarily large frequencies. In the discrete time-case, unstable modes may occur with the maximal frequency[6] $1/(2\Delta)$, which tends to infinity as $\Delta \to 0$.

6 The multiple input case

So far we have only considered singe input systems. By using the same arguments as spelled out in Section 4 it is easy to show that in the multiple input case robust (in)stability is determined by the condition $\alpha_M < 0$ ($\alpha_M > 0$), where
$$\alpha_M = \underset{\left\{\begin{array}{l}\alpha \in \mathbb{R} \\ g:\ [0,\ \tau] \to [0,\ 2\pi)\end{array}\right.}{\arg\max} \ \alpha, \quad \text{subject to}: \ \det\left(I - \int_0^{\tau} K^T e^{A\theta} B e^{-\alpha\theta} e^{-ig(\theta)} d\theta\right) = 0. \tag{41}$$
Notice that the function g may be discontinuous, also in the optimum. For instance, in the scalar case the optimal value is given by
$$g(\theta) = \left\{\begin{array}{ll} 0, & K^T e^{A\theta} B \geq 0, \\ \pi, & K^T e^{A\theta} B < 0. \end{array}\right.$$
The condition (41) is hard to check. However, a *sufficient* condition for robust stability is given by
$$\int_0^{\tau} \|K^T e^{A\theta} B\| d\theta < 1.$$

[6] Such a mode corresponds to a negative real eigenvalue of the fully discretized system.

7 Conclusions

We have analysed the stability of finite spectrum assignment based controllers, when the distributed delay in the control law is approximated with a sum of point-wise delays. In our opinion the main contributions are:

1. Unlike the existing literature on this topic, we have provided *necessary and sufficient* conditions for the stability of the approximation. These conditions are stronger than the necessary conditions, proposed in a recent paper [25]. Since the instability mechanism is related to the behavior of the essential spectrum of the solution semi-group of neutral functional differential equations, stability may be sensitive to the type of integration rule used and to *infinitesimal* perturbations on parameters and therefore, stability needs to be defined in a robust sense (cf. Definition 1). We have derived properties of the obtained stability conditions, emphasizing the relation with these of static state feedback controllers, and illustrated them with some examples.

2. Using eigenvalue plots we have graphically illustrated the instability mechanism and the necessity of taking small perturbations into account in the stability analysis.

3. The robustness analysis of this paper can be considered as a practical application of the stability theory of difference equations and neutral type systems, developed in [2, 22] and the reference therein.

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