A note on wave number dependence of wavelet matrix compression for integral equations with oscillatory kernel

Daan Huybrechs
Jo Simoens
Stefan Vandewalle

Report TW 356, April 2003 (revised January 2004)

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

A note on wave number dependence of wavelet matrix compression for integral equations with oscillatory kernel

Daan Huybrechs
Jo Simoens
Stefan Vandewalle

Report TW 356, April 2003 (revised January 2004)

Department of Computer Science, K.U.Leuven

Abstract

This paper analyzes the effect of large wave numbers on the wavelet method for integral equations arising in electromagnetic applications. It is shown that the compression of the stiffness matrix deteriorates with increasing wave number, a characteristic that has been reported before in the literature. Here, however, the exact dependence on the wave number is calculated analytically for the two dimensional Helmholtz problem.

Keywords : integral equations, boundary element method, wavelets, Helmholtz equation.
AMS(MOS) Classification : Primary : 35J05, Secondary : 42C40, 65N38.

A note on wave number dependence of wavelet matrix compression for integral equations with oscillatory kernel

Daan Huybrechs, Jo Simoens and Stefan Vandewalle

Katholieke Universiteit Leuven
Department of Computerscience
Celestijnenlaan 200A, B-3001 Leuven, Belgium.
{Daan.Huybrechs,Jo.Simoens,Stefan.Vandewalle}@cs.kuleuven.ac.be

Abstract

This paper analyzes the effect of large wave numbers on the wavelet method for integral equations arising in electromagnetic applications. It is shown that the compression of the stiffness matrix deteriorates with increasing wave number, a characteristic that has been reported before in the literature. Here, however, the exact dependence on the wave number is calculated analytically for the two dimensional Helmholtz problem.

1 Introduction

For the calculation of electromagnetic fields in scattering problems, integral equations are often preferred over partial differential equations. A partial differential equation describes the field on a domain surrounding the multi-dimensional obstacle, whereas the integral equation is typically defined only on the lower dimensional boundary. The latter approach leads to meshes that are easier to construct with fewer unknowns after discretization. The system of equations obtained after application of a finite element method, here called the boundary element method, is dense, however. As a result, classical direct and iterative solvers have high computational complexities for such problems.

One method to overcome the complexity of the dense solve was developed following the ideas in the paper of Beylkin, Coifman and Rokhlin [2]. There, it was noted that the integral operator can be represented almost sparsely by using a suitable wavelet basis, rather than the classical nodal boundary element basis functions. The stiffness matrix in that basis can be compressed to one with only $\mathcal{O}(N(\log N)^\alpha)$ elements, where N is the number of discretisation points. This gives rise to a fast matrix-vector product; the solution can then often be calculated efficiently by using iterative solvers. The method was extended by Dahmen, Prößdorf and Schneider to cover operators of nonzero order, by introducing a suitable preconditioner that bounds the condition number uniformly in N [7]. Further improvements to the method were then presented by Petersdorff and Schwab in [15]. After a rigorous analysis to match the error in every step to the discretisation error of the overall scheme, the final reduction that achieves linear complexity was established by Schneider [14].

In this paper, we investigate the behaviour of the wavelet method for oscillatory problems. It is known [3, 17] that the wavelet-based matrix compression becomes less effective for higher frequencies. More precisely, by means of intuitive arguments, given in the previous references, it has been demonstrated that there is a linear relation between the matrix-fill and the frequency. Here, we analyse this effect rigorously for the Helmholtz equation in two dimensions. We quantify the achievable compression by mathematical deduction and are able to establish an upper bound that is close to, but not entirely, linear. This is then illustrated by means of numerical results.

2 Preliminaries

2.1 The problem and its discretization

The problem considered in this paper is the Fredholm integral equation of the first kind

$$(Kv)(x) := \int_\Gamma K(x,y)v(y)ds(y) = f(x) \quad \forall x \in \Gamma, \tag{1}$$

with Γ a manifold in $\mathbb{R}^d$, $K(x,y)$ the kernel function, $f(x)$ a given function on Γ and $v(y)$ the unknown density function. We will deal in particular with the integral formulation of the two-dimensional Helmholtz equation

$$\Delta v + k^2 v = 0,$$

with wave number k. This equation can be written in the form (1) with kernel

$$K(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \tag{2}$$

see [5], over the boundary Γ of a two-dimensional domain. It has order $r = -1$.

Following the boundary element method (BEM), a solution is assumed of the form

$$v_J(y) = \sum_{k=1}^{N_J} u_{Jk}\phi_{Jk}(y)$$

with $N_J = 2^J$, where the set of functions ϕ_{Jk}, $k = 1, \ldots, N_J$, is a basis for a linear space Φ_J of functions defined on Γ. The discretisation of (1) obtained with a Galerkin-type approach leads to a linear system of the form

$$S_J u_J = f_J \tag{3}$$

with

$$S_{J,(k,l)} = \langle K\phi_{Jl}, \phi_{Jk}\rangle \quad \text{and} \quad f_{Jk} = \langle f, \phi_{Jk}\rangle\ , \tag{4}$$

where $\langle f, g\rangle$ denotes the classical $L_2(\Gamma)$-inner product. When using the nodal basis for the Helmholtz equation, the matrix S_J in (3) is a dense matrix with condition number of the order $\mathcal{O}(N_J)$ [13]. Direct solution methods for the system require $\mathcal{O}(N_J^3)$ operations. An iterative method based on matrix-vector products will require $\mathcal{O}(N_J^2)$ operations per step, with the number of steps dependent on N_J. This high computational complexity can be accounted for by using a suitable wavelet basis.

2.2 The wavelet approach

Our analysis will be based on the use of a periodisation of the CDF-family of biorthogonal wavelets $\psi_{jk}(t)$ on scale j with index k as basis functions [4]. We will denote the dual wavelets by $\tilde{\psi}_{jk}$. The scaling function corresponds to the hat function that is used in the nodal basis. The stiffness matrix M_J in the wavelet basis is related to S_J through the wavelet transformation matrix T_J, i.e. $M_J = T_J S_J T_J^T$.

Since the wavelets are biorthogonal, the number of vanishing moments $\tilde{d}$ of the primal wavelets can be varied independently of the approximation order d. In the boundary element method, d determines the convergence rate, while $\tilde{d}$ determines the compression of the stiffness matrix. For optimal compression, the inequality $\tilde{d} > d - r = d + 1$ should hold [6].

The wavelets are defined in the parameter domain $t \in [0, 1]$, and lifted to Γ using a periodic parameterization $\kappa(t) : [0, 1] \rightarrow \Gamma$ with $|\partial\kappa(t)| \neq 0$. We write $\hat{\psi}_{jk}(x) = \psi_{jk}(\kappa^{-1}(x))$, $x \in \Gamma$. Using these wavelet functions as boundary elements, most of the entries in the stiffness matrix are small. The matrix can be approximated by a sparse matrix with only $O(N_J)$ nonzero elements [8]. A priori estimates for the size of the elements have been developed, so the discarded elements need not be computed. With appriopriate integration routines, the compressed matrix can be calculated directly in $\mathcal{O}(N_J)$ operations [14]. The error introduced by the compression is of the same order as the discretisation error of the entire scheme. The accuracy and convergence of the solution is therefore uncompromised.

Additionally, the condition number of the stiffness matrix can be bounded uniformly in N with a simple diagonal preconditioner. The combination of the preconditioning with a fast matrix-vector product makes the calculation of the solution with iterative methods possible in linear time [8].

3 Wavelet compression for high wave numbers

3.1 Outline

A wavelet basis is called optimal for the numerical solution of (1) when the amount of work is proportional to the number of unknowns. This optimality is an asymptotic characteristic, relevant for large values of N. The meaning of 'large' depends in general on the shape of the boundary, the kernel function and its parameters. This dependence is reflected in the value of the proportionality constant $C = C(\Gamma, K)$, as a function of Γ (the boundary) and K (the kernel function). For a Helmholtz problem in two dimensions with a fixed boundary, we will study the size of the proportionality constant as a function of the wave number, i.e., we will derive the function $C(k)$. It is known [3, 17] that $C(k)$ increases with k, and the qualitative analysis in [17] makes the assumption of a linear dependence plausible for a straight line domain. In this section, we shall prove through a rigorous analysis that the dependence is indeed close to, but not entirely linear, and this for more general boundaries with a smooth parameterization.

A detailed overview of the wavelet method can be found in [8]. We will proceed here by taking the effect of the wave number into account in every step of the derivation of the method. We find an upper bound, that is shown to be sharp with numerical results. First, we derive in §3.2 an estimate for the derivatives of the kernel (2). This estimate will be used to bound the size of the matrix elements. Next, in §3.3, a similar estimate is constructed in the parameter domain $t \in [0,1]$. Then, in §3.4 and §3.5, we estimate the size of the elements in the stiffness matrix. Finally, in §3.6, the density of the stiffness matrix after compression is analysed. The result of this derivation is formulated in Theorem 6 of §3.7. An upper bound is found that is shown to be sharp by means of some numerical results presented in §3.8.

3.2 Estimates for the derivatives of the kernel

Our estimates for the size of the elements in the stiffness matrix originate in the Caldéron-Zygmund estimate for the derivatives of Schwartz kernels, see [6],

$$\Big|\frac{\partial^{|\alpha|+|\beta|}K}{\partial x^\alpha \partial y^\beta}(x,y)\Big| \leq C_1(k)|x-y|^{-(n+r+|\alpha|+|\beta|)}, \quad x \neq y, \quad x,y \in \Gamma. \tag{5}$$

The orders α and β of the derivative are written in multi-index form, i.e., $\partial x^\alpha = \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$ with $|\alpha| = \alpha_1 + \alpha_2$. The integer n is the dimension of the boundary manifold, and r is the order of the operator.

It is well-known that the 2D Helmholtz kernel (2) satisfies (5) with $n = 1$ and $r = -1$, for some constant C_1 [14]. In order to obtain the dependence of C_1 on the wave number, we must explicitly calculate these derivatives. The result is summarized in the following lemma.

Lemma 1. *The function* $K(x,y) := \frac{i}{4}H_0^{(1)}(k|x-y|)$ *satisfies* (5) *with* $n = 1$*,* $r = -1$*, and*

$$C_1(k) = \mathcal{O}(k^{|\alpha|+|\beta|-\frac{1}{2}}), \quad k \to \infty. \tag{6}$$

Proof. In order to estimate the left hand side of (5), we define $z(x,y) := |x-y|$ and $f(z) := H_0^{(1)}(kz)$. We then apply the chain rule and product rule for derivatives. The resulting sum contains contributions that are derivatives of f w.r.t. z, and partial derivatives of z. The latter are independent of k; the former are k-dependent. A recursive argument shows that the p-th order derivative of f with respect to z contains a term

$$(-1)^{\frac{p}{2}}k^p H_0^{(1)}(kz) \quad \text{or} \quad (-1)^{\frac{p-1}{2}}k^p H_1^{(1)}(kz)$$

for p even, resp. for p odd. The term with the highest order derivative of f is

$$\frac{\partial^{|\alpha|+|\beta|}f}{\partial z^{|\alpha|+|\beta|}}\big(\frac{\partial z}{\partial x_1}\big)^{\alpha_1}\big(\frac{\partial z}{\partial x_2}\big)^{\alpha_2}\big(\frac{\partial z}{\partial y_1}\big)^{\beta_1}\big(\frac{\partial z}{\partial y_2}\big)^{\beta_2}.$$

First, assume $p := |\alpha| + |\beta|$ to be even. The sum then contains the term

$$T := (-1)^{\frac{p}{2}}k^p H_0^{(1)}(kz)\frac{(x_1-y_1)^{\alpha_1}}{z^{\alpha_1}}\frac{(x_2-y_2)^{\alpha_2}}{z^{\alpha_2}}\frac{(x_1-y_1)^{\beta_1}}{z^{\beta_1}}\frac{(x_2-y_2)^{\beta_2}}{z^{\beta_2}}. \tag{7}$$

We know from (5) that T is bounded on Γ by Dz^{-p} with $D > 0$ a constant that depends on k. We have

$$|T|z^p \leq k^p|H_0^{(1)}(kz)|z^p,$$

so that

$$|T|z^p \leq k^p|H_0^{(1)}(kL)|L^p \quad \text{with} \quad L = \max|x-y|, \ \forall x,y \in \Gamma.$$

The dependence (6) now follows from the asymptotic expression [1]

$$H_{\nu}^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{4}-\frac{\nu\pi}{2})}, \quad x \to \infty, \tag{8}$$

and the fact that the term T has the highest exponent of k.

The argument for p odd is completely analogous. □

3.3 Estimates for the derivatives in the parameter domain

For the application of the wavelet based solution method, we will need an expression of type (5) along the boundary curve, represented as function on the one-dimensional parameter space $t \in [0,1]$. This can be readily obtained by applying the chain rule for derivatives, and by using (5) for every term in the sum. Define

$$G(t,\tau) := K(\kappa(t), \kappa(\tau)), \quad t, \tau \in [0,1] \tag{9}$$

with $\kappa : [0,1] \to \Gamma$ the parameterization of Γ. We then look for an expression of the form

$$\left|\frac{\partial^{\alpha+\beta} G}{\partial t^{\alpha} \partial \tau^{\beta}}(t,\tau)\right| \leq C_2(k)|\kappa(t)-\kappa(\tau)|^{-(n+r+\alpha+\beta)}, \quad t \neq \tau, \quad t, \tau \in [0,1]. \tag{10}$$

The orders of the derivative α and β are now scalars.

Defining $\kappa(t) = (x_1(t), x_2(t))$ and $\kappa(\tau) = (y_1(\tau), y_2(\tau))$, we apply the product and chain rule to (9). An upper bound for each partial derivative of K is known through Lemma 1. We assume the parameterization sufficiently smooth, so that the derivatives of κ are bounded. They are, of course, independent of k. It is clear that the highest order derivative of K determines the asymptotic behaviour around the diagonal $\kappa(t) = \kappa(\tau)$, where the term $|\kappa(t)-\kappa(\tau)|^{-(n+r+\alpha+\beta)}$ grows to infinity. We arrive at

$$\frac{\partial^{\alpha+\beta} G}{\partial t^{\alpha} \partial \tau^{\beta}} = \mathcal{O}(|\kappa(t)-\kappa(\tau)|^{-(n+r+\alpha+\beta)}), \quad t-\tau \to 0.$$

The asymptotic behaviour of $C_2(k)$ is determined by the largest exponent of k in the constants of the upper bounds for every term. This means that the exponent is again $\alpha+\beta-1/2$. Thus, we have proved the following lemma.

Lemma 2. *The function* $G(t,\tau) := \frac{i}{4} H_0^{(1)}(k|\kappa(t)-\kappa(\tau)|)$ *with* $\kappa : [0,1] \to \Gamma$ *satisfies* (10) *with*

$$C_2(k) = \mathcal{O}(k^{\alpha+\beta-\frac{1}{2}}), \quad k \to \infty.$$

3.4 Estimates for the size of the elements of the stiffness matrix

If the wavelets in t and τ have disjunct support, one can show [6] that the size of the corresponding element in the stiffness matrix is bounded by

$$|\langle K\hat{\psi}_{jk}, \hat{\psi}_{j'k'}\rangle| \leq C_3(k) \frac{2^{-(j+j')(\frac{n}{2}+\tilde{d})}}{\operatorname{dist}(\operatorname{supp}\, \hat{\psi}_{jk}, \operatorname{supp}\, \hat{\psi}_{j'k'})^{n+2\tilde{d}+r}}. \tag{11}$$

The support of the wavelet $\hat{\psi}_{jk}$ is denoted by $\operatorname{supp}\hat{\psi}_{jk} \subset \Gamma$. The bound (11) decreases very rapidly with increasing distance between the supports of the wavelet functions.

The bound for elements with overlapping supports can also be small, when the difference in scale between the wavelets is large enough, and the smaller wavelet is contained entirely in an interval defined by two successive singular points of the larger wavelet. The singular points are those points in the support where the basis function or its derivatives are discontinuous. For $j' > j$, one has [14]

$$|\langle K\hat{\psi}_{jk}, \hat{\psi}_{j'k'}\rangle| \leq C_4(k) \frac{2^{-j'(\frac{n}{2}+\tilde{d})} 2^{j\frac{n}{2}}}{\operatorname{dist}(\operatorname{sing\,supp}\, \hat{\psi}_{jk}, \operatorname{supp}\, \hat{\psi}_{j'k'})^{\tilde{d}+r}}. \tag{12}$$

The denominator is the distance between the support of the wavelet on the finer scale j', and the nearest singular point of the wavelet on scale j. A similar expression exists for $j > j'$.

The analysis of the matrix compression is based on the estimates (11) and (12). The constants in these expressions depend on the wave number. Using the results of the last paragraph, we can now quantify that dependence. To that end, we will first repeat the derivation of (11).

The kernel function in the double integral (4), taken to the parameter domain, is developed into a Taylor expansion around a point in the support of ψ_{jk}. For a wavelet ψ_{jk} with $\tilde{d}$ vanishing moments, the first $\tilde{d}$ terms of the expansion will vanish,

$$\langle K\hat{\psi}_{jk}, \hat{\psi}_{j'k'}\rangle = \langle G\psi_{jk}, \psi_{j'k'}\rangle = \int_0^1 \int_0^1 \frac{\partial^{\tilde{d}} G}{\partial t^{\tilde{d}}}(t', \tau) \frac{(t'-t)^{\tilde{d}}}{\tilde{d}!}\, \psi_{jk}(t)\psi_{j'k'}(\tau)|\kappa'(t)||\kappa'(\tau)|\, dt\, d\tau$$

with $t' \in \operatorname{supp}\psi_{jk}$. Doing the same for τ gives

$$\begin{aligned}\langle G\psi_{jk}, \psi_{j'k'}\rangle &= \int_0^1 \int_0^1 \frac{\partial^{2\tilde{d}} G}{\partial t^{\tilde{d}} \partial \tau^{\tilde{d}}}(t', \tau') \frac{(t'-t)^{\tilde{d}}}{\tilde{d}!} \frac{(\tau'-\tau)^{\tilde{d}}}{\tilde{d}!}\, \psi_{jk}(t)\psi_{j'k'}(\tau)|\kappa'(t)||\kappa'(\tau)|\, dt\, d\tau \\ &\leq \frac{C_2(k)}{\operatorname{dist}(\operatorname{supp}\hat{\psi}_{jk}, \operatorname{supp}\hat{\psi}_{j'k'})^{2\tilde{d}}} \int_0^1 \int_0^1 \frac{(t'-t)^{\tilde{d}}}{\tilde{d}!} \frac{(\tau'-\tau)^{\tilde{d}}}{\tilde{d}!}\, \psi_{jk}(t)\psi_{j'k'}(\tau)|\kappa'(t)||\kappa'(\tau)|\, dt\, d\tau,\end{aligned}$$

with $\tau' \in \operatorname{supp}\psi_{j'k'}$. Knowing that $\operatorname{supp}\psi_{jk} \sim 2^{-j}$ and $\int |\psi_{jk}(t)|dt \sim 2^{-j/2}$, and assuming a sufficiently smooth parameterization, we arrive at

$$\langle K\hat{\psi}_{jk}, \hat{\psi}_{j'k'}\rangle \leq \frac{C_2(k)}{\operatorname{dist}(\operatorname{supp}\hat{\psi}_{jk}, \operatorname{supp}\hat{\psi}_{j'k'})^{2\tilde{d}}}\, A\, 2^{-j\tilde{d}}\, 2^{-j'\tilde{d}}\, 2^{-j/2}\, 2^{-j'/2}.$$

with A a constant, independent of k.

The result has the same form as (11) with $n = 1$ and $r = -1$. The dependence on the wave number k is similar to that of $C_2(k)$, with $\alpha = \beta = \tilde{d}$. We have

$$C_3(k) = \mathcal{O}(k^{2\tilde{d}-1/2}). \tag{13}$$

3.5 Second estimate

The derivation of estimate (12) for wavelets with overlapping support is somewhat more involved, and was first established in [14]. The property of the vanishing moments can be used only once, for the smaller wavelet that is fully contained within the singular points of the other wavelet. The use of this property as was done above in the double integral arising from $\langle G\psi_{jk}, \psi_{j'k'}\rangle$, is not immediately possible due to the singularity in the kernel function. We note, however, that the result of the application of the integral operator G to a smooth function $f \in C_0^\infty$ is also smooth. The restriction of ψ_{jk} to the interval in the parameter domain that contains $\psi_{j'k'}$ can be extended to a smooth function $f \in C_0^\infty$, with $\operatorname{supp}(f) \sim 2^{-j}$ and $\|f\|_{H^s(\mathbb{R})} \le c\, 2^{js}$ [8]. After applying operator G to f, we can again use the property of vanishing moments to establish the estimate (12). A concise mathematical proof is given in [8].

Define $\psi_{jk}(t) = f(t) + \tilde{f}(t)$, such that the support of $\tilde{f}(t)$ does not overlap with the support of $\psi_{j'k'}$. We analyze the wave number dependence of the estimate (12) for the functions f and $\tilde{f}$.

To this end, we need to derive the dependence on k of the derivatives of Gf, since

$$\begin{aligned}\langle Gf, \psi_{j'k'}\rangle &= \int_0^1 \frac{\partial^{\tilde{d}} Gf}{\partial \tau^{\tilde{d}}}(\tau')\frac{(\tau'-\tau)^{\tilde{d}}}{\tilde{d}!}\,\psi_{j'k'}(\tau)|\kappa'(\tau)|\,d\tau \\ &\le A\,2^{-j'(\tilde{d}+1/2)} \sup_{\tau\in\operatorname{supp}(\psi_{j'k'})}\left|\frac{\partial^{\tilde{d}} Gf}{\partial \tau^{\tilde{d}}}(\tau)\right| \end{aligned} \qquad (14)$$

with $\tau' \in \operatorname{supp}(\psi_{j'k'})$. An explicit formula for the derivative of the function Gf, is given in [12, Chapter 3 (3.4.5)] for the special case where the kernel $G(t,\tau)$ is only a function of $(t-\tau)$. An expression can also be found for the more general case where the kernel depends on $(\kappa(t) - \kappa(\tau))$. We prove it here specifically for the kernel (9).

Theorem 3. *Define* $g(\tau) := \int_a^b G(t,\tau)v(t)dt$ *with* $v \in C^1$ *and* $G(t,\tau)$ *as* (9). *Then* $\forall \tau \in (a,b)$,

$$g'(\tau) = \int_a^b G(t,\tau)v'(t)dt + \int_a^b (\frac{\partial G}{\partial t} + \frac{\partial G}{\partial \tau})v(t)dt + G(a,\tau)v(a) - G(b,\tau)v(b). \qquad (15)$$

Proof. We first show that both integrals in the right hand side of (15) exist. The first integrand is improperly integrable. To show the existence of the second integral, define $r(t,\tau) = |\kappa(t) - \kappa(\tau)|$. Note that $r(t,t) = 0$ and $\frac{\partial r}{\partial t}(t,t) + \frac{\partial r}{\partial \tau}(t,t) = 0$, and also $\frac{\partial r}{\partial t}(t,t+\delta) + \frac{\partial r}{\partial \tau}(t,t+\delta) = \mathcal{O}(\delta)$. Hence, the function

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial \tau} = \frac{i}{4}\frac{\partial H_0^{(1)}}{\partial r}(\frac{\partial r}{\partial t} + \frac{\partial r}{\partial \tau})$$

is continuous in $t = \tau$, and thus also the second integral exists.

To prove the expression for the derivative, we note that

$$\begin{aligned} g(\tau+\delta) &= \int_a^b G(t,\tau+\delta)v(t)dt = \int_{a-\delta}^{b-\delta} G(t+\delta,\tau+\delta)v(t+\delta)dt \\ &= \int_{a-\delta}^{a} G(t+\delta,\tau+\delta)v(t+\delta)dt + \int_{a}^{b-\delta} G(t+\delta,\tau+\delta)v(t+\delta)dt, \end{aligned}$$

and

$$G(t+\delta,\tau+\delta) = G(t,\tau) + \delta(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial \tau}) + \mathcal{O}(\delta^2).$$

We can also write $g(\tau) = \int_a^{b-\delta} G(t,\tau)v(t)dt + \int_{b-\delta}^{b} G(t,\tau)v(t)dt$. Now, by the definition of the derivative, $g'(\tau) = \lim_{\delta\to 0}\frac{g(\tau+\delta)-g(\tau)}{\delta}$ yields the result (15).

□

Higher order derivatives of Gf can be found by applying Theorem 3 recursively, with $v(t) := f(t)|\kappa'(t)|$. The factor $|\kappa'(t)|$ is independent of j and k, and therefore will not influence the estimate. Assuming $f \in C_0^\infty$ with support contained in $[a,b]$, we have $v^{(i)}(a) = v^{(i)}(b) = 0$. The higher order derivative of $g = Gf$ is then given by

$$g^{(n)}(\tau) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) \int_a^b ((\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})^i G(t,\tau)) \frac{d^{n-i}}{dt^{n-i}}(f(t)|\kappa'(t)|)dt. \tag{16}$$

Again, each integral on the right hand side exists, since $(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})^i r(t, t+\delta) = \mathcal{O}(\delta)$, $i >= 0$. We combine (16) with (10) to find a wave number dependence for $\left|\frac{\partial^{\tilde{d}} Gf}{\partial \tau^{\tilde{d}}}(z)\right|$ of $\mathcal{O}(k^{\tilde{d}-1/2})$. To establish a bound for the right hand side of (14), we first note that by the Sobolev embedding theorem $\sup|f^{(i)}(t)| = \|f\|_{W^{\infty,i}} \leq c_1 \|f\|_{H^{i+1/2}} = c_2\, 2^{j(i+1/2)}$. Now, using the fact that $\mathrm{dist}(\mathrm{sing\,supp}\ \hat{\psi}_{jk}, \mathrm{supp}\ \hat{\psi}_{j'k'}) \leq c2^{-j}$, $g^{(\tilde{d})}(\tau)$ can be bounded by

$$A \int_a^b 2^{j(\tilde{d}+1/2)} dt \leq B 2^{j(\tilde{d}+1/2-1)} \leq C 2^{j/2}\, \mathrm{dist}(\mathrm{sing\,supp}\ \hat{\psi}_{jk}, \mathrm{supp}\ \hat{\psi}_{j'k'})^{-(\tilde{d}-1)}.$$

Combined with (14), this concludes an estimate of the form (12),

$$|\langle Gf, \psi_{j'k'}\rangle| \leq C_4(k) \frac{2^{-j'(\tilde{d}+1/2)} 2^{j/2}}{\mathrm{dist}(\mathrm{sing\,supp}\ \hat{\psi}_{jk}, \mathrm{supp}\ \hat{\psi}_{j'k'})^{\tilde{d}-1}}.$$

It remains to bound the part $|\langle G\tilde{f}, \psi_{j'k'}\rangle|$. This is more straightforward, as the integrand is not singular. Using the fact that $|\tilde{f}(t)| \leq c2^{j/2}$ and applying (10) once, we can proceed like in (14) (with f replaced by $\tilde{f}$),

$$\begin{aligned} |\langle G\tilde{f}, \psi_{j'k'}\rangle| &\leq A\, 2^{-j'(\tilde{d}+1/2)} \left| \frac{\partial^{\tilde{d}} G\tilde{f}}{\partial \tau^{\tilde{d}}}(\tau') \right| \\ &\leq B\, 2^{-j'(\tilde{d}+1/2)} \int_{\mathrm{supp}(\tilde{f})} |\tilde{f}(t)|\, \mathrm{dist}(\kappa(t), \mathrm{supp}(\hat{\psi}_{j'k'}))^{-\tilde{d}} dt \\ &\leq C_4(k)\, 2^{-j'(\tilde{d}+1/2)}\, 2^{j/2}\, \mathrm{dist}(\mathrm{sing\,supp}\ \hat{\psi}_{jk}, \mathrm{supp}\ \hat{\psi}_{j'k'})^{-(\tilde{d}-1)}. \end{aligned}$$

With these results, we have shown that estimate (12) holds, with a constant that depends on the wave number with the order

$$C_4(k) = \mathcal{O}(k^{\tilde{d}-1/2}). \tag{17}$$

3.6 Density of the compressed stiffness matrix

Based on estimates (11) and (12), a compression scheme can be devised to approximate the stiffness matrix in the wavelet basis by a sparse matrix. In an optimal scheme, the error introduced by the compression is matched to the discretisation error of the entire method. The compression is a two steps procedure. To that end, one defines two thresholds [14]

$$\delta_{j,j'} = \max\{a 2^{-\min\{j,j'\}}, a 2^{(J(2d'-r)-(j+j')(\tilde{d}+d'))/(2\tilde{d}+r)}\} \tag{18}$$

$$\delta^S_{j,j'} = \max\{a' 2^{-\max\{j,j'\}}, a' 2^{(J(2d'-r)-\max\{j,j'\}\tilde{d}-(j+j')d')/(\tilde{d}+r)}\}. \tag{19}$$

The two constants a and a' determine the amount of compression, and have to be selected carefully, see [14]. Too large a value for a and a' will lead to a denser matrix; not all of its elements may be needed for the required accuracy of the solution. Too small a value results in loss of convergence.

The first treshold (18), based on estimate (11), gives rise to a sparse matrix M^ϵ_J with elements

$$m^\epsilon_{(j,k),(j',k')} := \begin{cases} m_{(j,k),(j',k')} & \text{if } \operatorname{dist}(\operatorname{supp}\psi_{jk}, \operatorname{supp}\psi_{j'k'}) \leq \delta_{j,j'}, \\ 0 & \text{otherwise.} \end{cases}$$

The number of remaining elements is almost linear in N_J, up to a logarithmic factor. The second compression removes that logarithmic factor, by making a sparse matrix $\hat{M}_J$ with only $\mathcal{O}(N_J)$ elements defined by

$$\hat{m}_{(j,k),(j',k')} := \begin{cases} m^\epsilon_{(j,k),(j',k')} & \text{if } j' \leq j \text{ and } \operatorname{dist}(\operatorname{supp}\psi_{jk}, \operatorname{sing\,supp}\psi_{j'k'}) \leq \delta^S_{j,j'}, \\ m^\epsilon_{(j,k),(j',k')} & \text{if } j \leq j' \text{ and } \operatorname{dist}(\operatorname{sing\,supp}\psi_{jk}, \operatorname{supp}\psi_{j'k'}) \leq \delta^S_{j,j'}, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

The system of equations (3) now becomes

$$\hat{M}_J \hat{u}_J = f_J.$$

Define $E_{j,j'} := M_{j,j'} - M^\epsilon_{j,j'}$ as the error that is introduced by the first compression in the block matrix corresponding to the scales j and j', and $F_{j,j'} := M^\epsilon_{j,j'} - \hat{M}_{j,j'}$ as the error by the second compression. Then it is shown [8] that

$$\|E_{j,j'}\| \leq C\, a^{-2\tilde{d}-r} 2^{2Jr/2} 2^{-2d'(J-\frac{j+j'}{2})} \tag{21}$$

$$\|F_{j,j'}\| \leq C\, (a')^{-\tilde{d}-r} 2^{2Jr/2} 2^{-2d'(J-\frac{j+j'}{2})}. \tag{22}$$

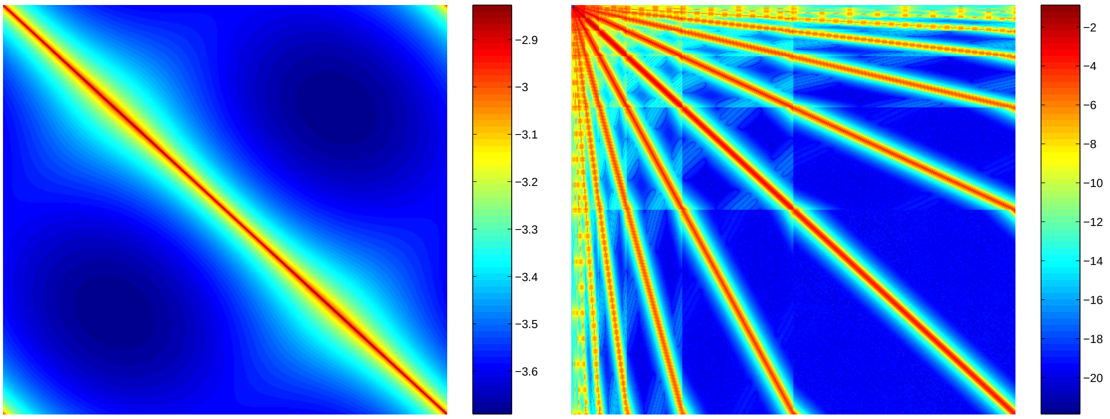
Based on these expressions, one can show that the compressed scheme is consistent with the original operator equation, and retains the order of convergence. If the errors were bounded uniformly in k, it would ensure that the compression error is independent of the wavenumber.

This means that we must choose the parameters a and a' in a suitable way. We note that expression (21) is established by summing the corresponding estimates of the form (11) for the discarded elements. This introduces a dependence on k of the order $\mathcal{O}(k^{2\tilde{d}-1/2})$, that is transferred unchanged to (21). It can be compensated by an asymptotical behaviour of $a = \mathcal{O}(k^p)$ if

$$k^{-2(\tilde{d}-1/2)p}\, k^{2\tilde{d}-1/2} = \mathcal{O}(1) \Leftrightarrow -2(\tilde{d}-1/2)p + 2\tilde{d} - 1/2 \leq 0$$

or

$$p \geq 1 + \frac{1}{4\tilde{d}-2}. \tag{23}$$



(a) Nodal basis

(b) Wavelet basis

Figure 1: The stiffness matrix in the nodal basis and the wavelet basis with six vanishing moments. The colour is an indication of the size of the corresponding element.

The compression error is thus bounded uniformly in k if $a = \mathcal{O}(k^{1+\frac{1}{4\tilde{d}-2}})$. The asymptotical behaviour of the parameter a needs to be slightly larger than linear in k, but improves somewhat as the number of vanishing moments increases.

The second compression is handled similarly, leading to

$$p \geq 1 + \frac{1}{2\tilde{d} - 2}. \tag{24}$$

It is important to note here that we only consider the compression of the stiffness matrix. The wave number also influences the condition number, even after preconditioning, and will therefore have an impact, e.g. on the convergence of iterative solution methods. This means that, for large values of k, the system may become increasingly ill conditioned. The uniform bound on the compression error that we have derived ensures however that, for a specific value of k, the compressed scheme retains the convergence properties of the corresponding uncompressed Galerkin scheme.

3.7 Wave number dependence of the wavelet compression

The number of nonzero elements in the compressed matrices M_J^ϵ and $\hat{M}_J$, depends on the parameters a and a' in (18) and (19). Their values determine the sparsity structure of the submatrices $\hat{M}_{jj'}$ in the stiffness matrix.

The tresholds indicate a minimal distance between the supports of wavelets corresponding to a matrix element. They are chosen such that the error introduced by discarding elements matches the discretisation error. The allowable error varies for each combination of scales j and j', and in general the rougher scales require higher accuracy, while the elements corresponding to finer scales can be less accurate. Combining the estimates for the matrix elements, and the tresholds used for discarding some of them, reveals that the compressed

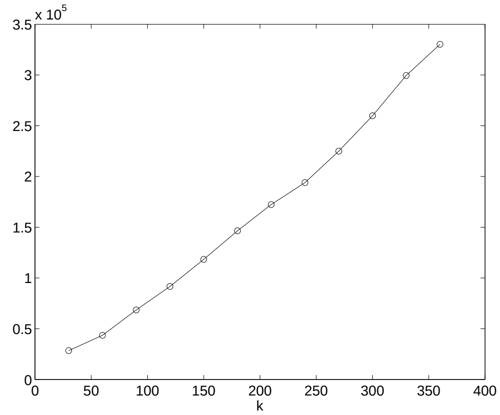


Figure 2: The number of significant elements for fixed $N = 1024$ and increasing k. The problem is a circle with radius 0.5.

stiffness matrix keeps only $\mathcal{O}(N)$ elements.

As the tresholds increase, the condition on the distance between wavelets becomes stronger, and as a result the matrix will be less sparse. We see from (18) that the required minimal distance between the support of the wavelets corresponding to a matrix element, is directly proportional to a. While the shape of the boundary Γ and the parameterization $\kappa(t)$ will have an influence here, we can say in first order approximation that the number of elements kept is also linear in a.

Lemma 4. *The number of nonzero elements in* $\hat{M}_J$*, as defined by* (20)*, is* $\mathcal{O}(a) + \mathcal{O}(a')$ *as a function of the wave number* k*.*

We have investigated the necessary asymptotical behaviour of the parameters a and a' for large values of k. The results (23) and (24) lead to another lemma.

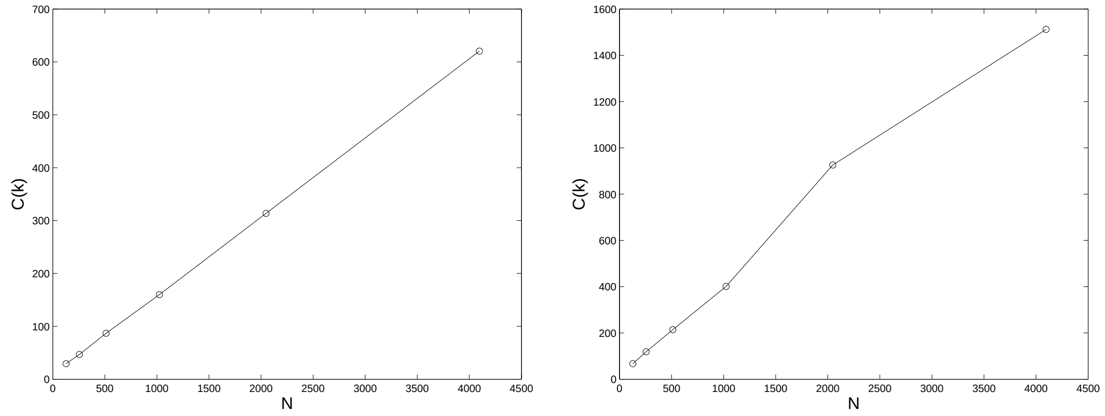
Lemma 5. *In order to achieve compression that maintains the convergency properties of the uncompressed Galerkin scheme, the parameters* a *and* a' *in* (18) *and* (19) *have to be chosen such that*

$$a = \mathcal{O}(k^{1+\frac{1}{4\tilde{d}-2}}) \quad \text{and} \quad a' = \mathcal{O}(k^{1+\frac{1}{2\tilde{d}-2}}).$$

The combination of the previous lemmas leads to the following statement.

Theorem 6. *The number of nonzero elements in the wavelet compressed stiffness matrix with optimal choice of* a *and* a' *in the threshold constants* (18) *and* (19) *increases asymptotically linear in* N*, with a proportionality constant of the order* $\mathcal{O}(k^{1+1/(2\tilde{d}-2)})$*.*

Note that the dependence on k of the proportionality constant means that, with increasing wave numbers, the stiffness matrix fills up to become a dense matrix. It is common practice to increase the number of boundary elements N linearly with the wave number k, i.e., oscillations are represented with a fixed number of unknowns per period. In that case, the actual number of nonzero elements in the compressed stiffness matrix grows asymptotically as $\mathcal{O}(N^2)$, i.e. the matrix looses any significant sparsity.



(a) Circle

(b) Ellipse

Figure 3: Numerical values for $C(k)$ for a Helmholtz problem on a circle and an ellipse, with 10 points per wavelength.

3.8 Numerical results

Figure 1 shows the stiffness matrix for the Helmholtz problem on an ellipse in the nodal basis, and in the wavelet basis. Most entries in the wavelet transformed matrix are discardable. The larger elements are located around the diagonal, for every combination of scales.

For a fixed wave number and increasing N, the number of significant elements in the stiffness matrix is linear in N. For a fixed N and increasing k, the number of significant elements will increase almost linear in k. This is shown in Figure 2.

However, some care has to be taken when comparing compression ratios for increasing values of the wave number k. For larger values of k, the solution oscillates more rapidly and the discretisation error increases. In order to obtain comparable accuracy of the computed solution, N has to increase with k. It is common practice [17] to choose a fixed number of discretisation points per wavelength $\lambda = 2\pi/k$, making N proportional to k. In the following calculations we have chosen 10 points per wavelength.

We have chosen a simple a posteriori compression criterion, similar to the one in the paper by Beylkin, Coifman and Rokhlin [2], where the wavelet compression was originally suggested. We discard elements smaller than a threshold fixed for the entire matrix, rather than the level dependent thresholds that are needed for linear complexity without a logarithmic factor. This gives a reasonable approximation of the more complicated compression scheme, and allows for a more easy comparison when the wave number varies. The threshold is chosen such that the error introduced by the compression is of the same order as the discretisation error. Following [17], we chose

$$\tau = \frac{\delta}{N_J} \|S_J\|_\infty .$$

The parameter δ can be chosen so as to obtain a certain accuracy; here it is set to 0.1.

The numerical results show the dependence on the wave number. Figure 3 gives the numerically determined values of $C(k)$, defined here as the number of nonzero elements in the compressed matrix divided by N_J, the number of unknowns. The boundary Γ is a circle around the origin with radius 0.5 for the picture on the left, and an ellipse around the origin with axis lenghs of 0.3 along the X-axis and 0.5 along the Y-axis, for the picture on the right. The behaviour is in each case seen to be almost linear, as predicted by the theory.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications, New York, 1965.

[2] G. Beylkin, R. R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms I. *Comm. Pure Appl. Math.*, 44:141–183, 1991.

[3] W. C. Chew, J. M. Chin, C. C. Lu, E. Michielssen, and J. M. Song. Fast solution methods in electromagnetics. *IEEE Trans. Antennas Propag.*, 45(3):533–543, 1997.

[4] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 55:485–560, 1992.

[5] D. Colton and R. Kress. *Integral Equation Methods in Scattering Theory.* Wiley, New York, 1983.

[6] W. Dahmen. Wavelet and multiscale methods for operator equations. In *Acta Numerica*, volume 6, pages 55–228. Cambridge Univ. Press, Cambridge, 1997.

[7] W. Dahmen, S. Prößdorf, and R. Schneider. Multiscale methods for pseudo-differential equations on smooth closed manifolds. In C. K. Chui, L. Montefusco, and L. Puccio, editors, *Wavelets: Theory, Alorithms and Applications*, volume 5 of *Wavelet Analysis and Its Applications*, pages 385–424. Academic Press, 1994.

[8] W. Dahmen, H. Harbrecht, and R. Schneider. Compression techniques for boundary integral equations – optimal complexity estimates. IGPM report 02-06, RWTH Aachen, 2002.

[9] W. Dahmen, B. Kleemann, S. Prößdorf, and R. Schneider. Multiscale methods for the solution of the Helmholtz and Laplace equations. In W. Wendland, editor, *Boundary Element Methods, Reports from the Final Conference of the Priority Research Programme 1989-1995 of the German Research Foundation*, Berlin, 1996. Springer-Verlag.

[10] W. Dahmen and A. Kunoth. Multilevel preconditioning. *Numerische Mathematik*, 63:315–344, 1992.

[11] W. Dahmen, A. Kunoth, and R. Schneider. Operator equations, multiscale concepts and complexity. In J. Renegar, M. Shub, and S. Smale, editors, *Lectures in Applied Mathematics*, volume 32, pages 225–261. American Mathematical Society, 1996.

[12] W. Hackbusch. *Integral Equations: Theory and Numerical Treatment*, volume 120 of *Internat. Series of Numer. Math.* Birkhäuser, 1995.

[13] B. Kleemann, A. Rathsfeld, and R. Schneider. Multiscale methods for boundary integral equations and their application to boundary value problems in scattering theory and geodesy. In W. Hackbusch and G. Wittum, editors, *Proceedings of the 12th GAMM-Seminar Kiel on Boundary Elements: Implementation and Analysis of Advanced Algorithms*, volume 54 of *Notes on Numerical Fluid Mechanics*, Braunschweig, Wiesbaden, 1996. Vieweg-Verlag.

[14] R. Schneider. Multiskalen- und wavelet-matrixkompression: Analysisbasierte Methoden zur effizienten Lösung großer vollbesetzter Gleichungssysteme. Habilitationsschrift, Darmstadt, 1998.

[15] T. von Petersdorff and C. Schwab. Wavelet approximations for first kind boundary integral equations on polygons. *Numer. Math.*, 74(4), 1996.

[16] T. von Petersdorff and C. Schwab. Fully discrete multiscale Galerkin BEM. In W. Dahmen, A. Kurdila, and P. Oswald, editors, *Multiscale Wavelet Methods for PDEs*. Academic Press, 1999.

[17] R. L. Wagner and W. Cho Chew. A study of wavelets for the solution of electromagnetic integral equations. *IEEE Trans. Antennas Propag.*, 43(8):802–810, 1995.