Stability Analysis of Runge-Kutta Methods for Nonlinear Volterra Delay-integro-differential Equations

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1 Introduction

The last several decades have witnessed a large development in the computational implementation and the theoretical analysis of numerical methods for integro-differential equations of Volterra type. An extensive collection of results has been presented in the monographs by Linz [14], Brunner & van der Houwen [7] and Baker [1]. The aforementioned references focus on integro-differential equations without delay. However, it is well-known that certain real-life problems require models of Volterra *delay*-integro-differential equation type (VDIDEs) for an adequate description (cf. [5]). Hence, recently, researchers have also turned their attention to the study of VDIDEs. Baker & Ford [2], Koto [13], Huang & Vandewalle [12], and Luzyanina, Engelborghs & Roose [15] have dealt with the linear stability of numerical methods for VDIDEs. Baker [3], Brunner [6], and Enright & Hu [9] have studied the convergence of linear multistep methods and continuous Runge-Kutta methods, respectively.

Up to now, only few results have been presented in the literature on the *nonlinear stability* of numerical methods for VDIDEs. Baker & Tang [4] investigated nonlinear stability of continuous Runge-Kutta methods for equations with unbounded delays. Their results are primarily applicable and relevant for nonstiff problems since their approach is based on a classical Lipschitz condition. Zhang and Vandewalle [16] considered BDF methods applied to a class of stiff VDIDEs, which they called *class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ (see also §2), and obtained

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some analytical and numerical stability results. In the present paper, we continue this study and treat the nonlinear stability of two classes of adapted Runge-Kutta (RK) methods for the above problem class of equations. Both global and asymptotic stability criteria of the presented methods will be derived.

The paper is structured as follows. In §2 we recall the problem class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ and formulate two classes of discretization schemes. One is a RK method with linear-compound-quadrature formula; the other is a RK method with Pouzet quadrature formula (cf. [7]). A slightly different scheme from the second one was first introduced by Koto [13] and employed to study *linear* numerical stability. Some concepts and Lemmas which play a key role in the derivation of the stability results are given in §3. The main results are presented in §4 and §5, where we describe the global and asymptotic stability criteria for the above two classes of methods. Finally, we end with some concluding remarks in §6. There we point out that the discretization schemes based on the Gauss, Radau IA, Radau IIA and Lobatto IIIC formulae are all globally and asymptotically stable under certain conditions.

2 A class of VDIDEs and their Runge-Kutta discretization

Consider the following complex N-dimensional system of VDIDEs with constant delay $\tau > 0$,

$$\begin{cases} y'(t) = f(t, y(t), G(t, y(t-\tau), \int_{t-\tau}^{t} g(t, x, y(x))dx)), & t \in [t_0, +\infty), \\ y(t) = \varphi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{2.1}$$

where the mappings f, G, k and φ are smooth enough, such that system (2.1) has a unique smooth solution $y(t)$, and satisfies the conditions

$$\Re\langle f(t, y_1, z) - f(t, y_2, z), y_1 - y_2\rangle \leq \alpha\|y_1 - y_2\|^2, \tag{2.2}$$

$$\|f(t, y, z_1) - f(t, y, z_2)\| \leq \beta\|z_1 - z_2\|, \tag{2.3}$$

$$\|G(t, y_1, z_1) - G(t, y_2, z_2)\| \leq \sigma_1\|y_1 - y_2\| + \sigma_2\|z_1 - z_2\|, \tag{2.4}$$

$$\|g(t, x, z_1) - g(t, x, z_2)\| \leq \gamma\|z_1 - z_2\|, \quad (t, x) \in \mathbb{D}, \tag{2.5}$$

in which $t \in [t_0, +\infty)$, $\mathbb{D} = \{(t, x) : t \in [t_0, +\infty), x \in [t - \tau, t]\}$, $y, y_1, y_2, z, z_1, z_2 \in \mathbb{C}^N$, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote a given inner product and the corresponding induced norm in the complex N-dimensional space $\mathbb{C}^N$. The constants $-\alpha, \beta, \sigma_1, \sigma_2$ and γ are given and nonnegative. In the present paper, all the problems of type (2.1) with (2.2) -(2.5) will be called *problems of class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$. Some examples of the problems of class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ have been given in paper [16]. For the subsequent stability analysis, we will also need to consider systems with different initial condition

$$\begin{cases} \tilde{y}'(t) = f(t, \tilde{y}(t), G(t, \tilde{y}(t-\tau), \int_{t-\tau}^{t} g(t, x, \tilde{y}(x))dx)), & t \in [t_0, +\infty), \\ \tilde{y}(t) = \psi(t), & t \in [t_0 - \tau, t_0]. \end{cases} \tag{2.6}$$

In paper [16], Zhang & Vandewalle pointed out that if systems (2.1) and (2.6) belong to class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ with

$$\alpha + \beta(\sigma_1 + \sigma_2\gamma\tau) < 0, \tag{2.7}$$

the following two analytical stability results hold:

$$\|y(t) - \tilde{y}(t)\| \leq \max_{\theta\in[t_0-\tau,\ t_0]} \|\varphi(\theta) - \psi(\theta)\|, \quad \forall t \geq t_0, \quad \text{and} \quad \lim_{t\to+\infty} \|y(t) - \tilde{y}(t)\| = 0. \tag{2.8}$$

To arrive at the discretization schemes for (2.1), we first recall the s-stage underlying Runge-Kutta (RK) method

$$\begin{cases} y_i^{(n)} = y_n + h \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, y_j^{(n)}), \quad i = 1, 2, ..., s, \\ y_{n+1} = y_n + h \sum\limits_{j=1}^{s} b_j f(t_n + c_j h, y_j^{(n)}), \quad n \geq 0 \end{cases} \tag{2.9}$$

for ODEs systems of the form

$$\begin{cases} y'(t) &= f(t, y(t)), \quad t > 0, \\ y(0) &= y_0. \end{cases}$$

Method (2.9) is characterized by the abscissae c_j, the weights b_j and the coefficients a_{ij}, where we always assume that the methods (2.9) satisfy the classical consistency conditions together with a common restriction on the magnitude of c_i,

$$\sum_{i=1}^{s} b_i = 1 \quad \text{and} \quad 0 \leq c_i \leq 1, \quad i = 1, 2, \ldots, s.$$

We will further denote the underlying RK method (2.9) by the classical Butcher tableau, with $A = (a_{ij}) \in \mathbb{R}^{s\times s}$, $b = (b_1, b_2, \ldots, b_s)^T$ and $c = (c_1, c_2, \ldots, c_s)^T \in \mathbb{R}^s$.

Adapting method (2.9) to VDIDE (2.1) yields the following discretization scheme

$$\begin{cases} y_i^{(n)} = y_n + h \sum\limits_{j=1}^{s} a_{ij} f(t_j^{(n)}, y_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)})), \quad i = 1, 2, ..., s, \\ y_{n+1} = y_n + h \sum\limits_{j=1}^{s} b_j f(t_j^{(n)}, y_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)})), \quad n \geq 0, \end{cases} \tag{2.10}$$

where the time stepsize is given by $h = \tau/m$, with m a given positive integer; the time points are given by $t_n = t_0 + nh$ and $t_j^{(n)} = t_n + c_j h$, and $y_i^{(n)}$, $z_j^{(n)}$ and y_n are approximations to

$$y(t_i^{(n)}), \quad z(t_j^{(n)}) := \int_{t_j^{(n-m)}}^{t_j^{(n)}} g(t_j^{(n)}, x, y(x))dx, \quad \text{and} \quad y(t_n),$$

respectively. As to the computation of the integral $z_j^{(n)}$, we distinguish two alternatives. The first is based on using the linear-compound-quadrature (LCQ) formula

$$z_j^{(n)} = h \sum_{q=n-m+1}^{n} [\nu g(t_j^{(n)}, t_j^{(q-1)}, y_j^{(q-1)}) + (1-\nu) g(t_j^{(n)}, t_j^{(q)}, y_j^{(q)})], \quad j = 1, 2, \ldots, s, \tag{2.11}$$

where ν is a parameter on the interval $[0,1]$. The second approach adopts the so-called Pouzet quadrature (PQ) formula (cf. Brunner & van der Houwen [7], and Koto [13])

$$\begin{aligned} z_j^{(n)} &= h\sum_{r=1}^{s} a_{jr} g(t_j^{(n)}, t_r^{(n)}, y_r^{(n)}) + h\sum_{q=1}^{m}\sum_{r=1}^{s} b_r g(t_j^{(n)}, t_r^{(n-q)}, y_r^{(n-q)}) \\ &\quad - h\sum_{r=1}^{s} a_{jr} g(t_j^{(n)}, t_r^{(n-m)}, y_r^{(n-m)}) \end{aligned} \tag{2.12}$$

which is produced by discretizing the integral of the function $g(t_n + c_i h, x, y(x))$ over the interval $x \in [t_n + c_i h - \tau, t_n + c_i h]$, split into the following three parts:

$$\int_{t_n}^{t_n+c_ih} g(t_n + c_ih, x, y(x))dx + \int_{t_{n-m}}^{t_n} g(t_n + c_ih, x, y(x))dx - \int_{t_{n-m}}^{t_{n-m}+c_ih} g(t_n + c_ih, x, y(x))dx.$$

Moreover, we always set $y_0 = y(t_0)$ and, for $-m \le n < 0$, we take

$$y_j^{(n)} = y(t_j^{(n)}), \quad z_j^{(n)} = \int_{t_j^{(n)}-\tau}^{t_j^{(n)}} g(t_j^{(n)}, x, \varphi(x))dx, \quad y_n = y(t_n).$$

As such, we have defined two classes of discretization schemes for (2.1): method (2.10) with (2.11) and method (2.10) with (2.12). The former will further be called *RK method with LCQ formula* and the latter *RK method with PQ formula*. When any of the above methods is applied to system (2.6), the obtained solution sequences approximating $y(t_j^{(n)}), z(t_j^{(n)})$ and $y(t_n)$ will be denoted by $\tilde{y}_j^{(n)}, \tilde{z}_j^{(n)}$ and $\tilde{y}_n$. Also, we set $\tilde{y}_0 = \tilde{y}(t_0)$ and, for $-m \le n < 0$,

$$\tilde{y}_j^{(n)} = \tilde{y}(t_j^{(n)}), \quad \tilde{z}_j^{(n)} = \int_{t_j^{(n)}-\tau}^{t_j^{(n)}} g(t_j^{(n)}, x, \psi(x))dx, \quad \tilde{y}_n = \tilde{y}(t_n).$$

3 Introductory concepts and basic lemmas

This section will recall and present some concepts and lemmas that will be important for the presentation of our main results in section §4 and §5.

Definition 3.1 *(cf. [8]) The underlying RK method* (2.9) *is called* (k,l)*-algebraically stable if there exist real constants* $k > 0$, l *and a nonnegative diagonal matrix* $D = diag(d_1, d_2, \dots, d_s) \in \mathbb{R}^{s\times s}$ *such that the matrix* M *is nonnegative definite, where*

$$M = \begin{pmatrix} k - 1 - 2le^TDe & e^TD - b^T - 2le^TDA \\ De - b - 2lA^TDe & DA + A^TD - bb^T - 2lA^TDA \end{pmatrix} \in \mathbb{R}^{(s+1)\times(s+1)},$$

$A = (a_{ij}) \in \mathbb{R}^{s\times s}$, $b = (b_1, b_2 \dots, b_s)^T \in \mathbb{R}^s$ *and* $e = (1, 1, \dots, 1)^T \in \mathbb{R}^s$. *In particular, an (1,0)-algebraically stable method is called algebraically stable.*

Some (k,l)-algebraically stable underlying RK methods are given in the references by Burrage & Butcher [8] and Hairer & Wanner [10]. Using this concept, Huang et al. [11]

investigated the nonlinear stability of RK methods for *non-distributed* delay differential equations. Here, we will adopt this concept to analyze the nonlinear stability of RK methods for *distributed* delay differential equations (i.e., VDIDEs).

First, we introduce the following notational conventions:

$$\mathcal{Y}_n = y_n - \tilde{y}_n, \quad \mathcal{Y}_j^{(n)} = y_j^{(n)} - \tilde{y}_j^{(n)}, \quad \mathcal{Z}_j^{(n)} = z_j^{(n)} - \tilde{z}_j^{(n)},$$

$$\mathcal{F}_j^{(n)} = f(t_j^{(n)}, y_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)})) - f(t_j^{(n)}, \tilde{y}_j^{(n)}, G(t_j^{(n)}, \tilde{y}_j^{(n-m)}, \tilde{z}_j^{(n)})).$$

The following stability behavior of RK methods (2.10) with LCQ formula (2.12) and RK methods with PQ formula (2.12) will be investigated in the subsequent sections.

Definition 3.2 *Method* (2.10) *with LCQ formula (2.12) or PQ formula* (2.12) *is called globally stable for problems of class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *if, there exists a constant* $\mathcal{H} > 0$*, which depends only on* $\alpha, \beta, \sigma_1, \sigma_2, \gamma, \tau$ *and the method, such that*

$$\|\mathcal{Y}_n\| \leq \mathcal{H} \max_{t_0 - \tau \leq t \leq t_0} \|\varphi(t) - \psi(t)\|, \quad \forall n \geq 1. \tag{3.1}$$

Since the stability constant $\mathcal{H}$ doesn't depend on the time variable t, the above concept of global stability is a type of long-term stability. In other studies, when a similar numerical stability for other differential equations was investigated, one often asked $\mathcal{H} = 1$. However, it is our opinion that this excludes many excellent numerical methods.

Definition 3.3 *Method* (2.10) *with LCQ formula (2.12) or PQ formula* (2.12) *is called asymptotically stable for problems of class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *if*

$$\lim_{n \to \infty} \|\mathcal{Y}_n\| = 0. \tag{3.2}$$

When presenting our stability analysis, the two (very technical) Lemmas proven below will play a key role.

Lemma 3.4 *Suppose that the underlying RK method* (2.9) *is* (k, l)*-algebraically stable with* $0 < k \leq \check{k} \leq 1$*, where* $\check{k}$ *is independent of the index* $\mathbf{n}$ *of the method, and the conditions* (2.2)-(2.4) *hold. Then the induced method* (2.10) *satisfies for all* $n \geq 0$

$$\begin{aligned} \|\mathcal{Y}_{n+1}\|^2 \quad \leq \quad & \check{k}^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\check{k}^m + h\beta\sigma_1\} \sum_{i=0}^{n} \check{k}^{n-m-i} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\ & + \beta\sigma_1\tau\check{k}^{n-m+1} \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} + h\beta\sigma_2 \sum_{i=0}^{n} \check{k}^{n-i} \sum_{j=1}^{s} d_j \|\mathcal{Z}_j^{(i)}\|^2. \end{aligned} \tag{3.3}$$

Remark 3.5 *In particular, when* $\check{k} = 1$, (3.3) *can be read*

$$\begin{aligned} \|\mathcal{Y}_{n+1}\|^2 \quad \leq \quad & \|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(\mathbf{2}\sigma_1 + \sigma_2)) - 2l] \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\ & + \beta\sigma_1\tau \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} + h\beta\sigma_2 \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Z}_j^{(i)}\|^2 \end{aligned} \tag{3.4}$$

Proof. By a direct computation and $(k,l)-$algebraic stability, one has (see also [8])

$$\|\mathcal{Y}_{n+1}\|^2 - k\|\mathcal{Y}_n\|^2 - 2\sum_{j=1}^{s} d_j \Re\langle \mathcal{Y}_j^{(n)}, h\mathcal{F}_j^{(n)} - l\mathcal{Y}_j^{(n)}\rangle = -\sum_{i=1}^{s+1}\sum_{j=1}^{s+1} m_{ij}\langle \omega_i, \omega_j\rangle \le 0, \tag{3.5}$$

where $M = (m_{ij})$, $\omega_1 = \mathcal{Y}_n, \omega_{i+1} = h\mathcal{F}_i^{(n)}$ $(i = 1, 2, \dots, s)$. Hence, one has

$$\|\mathcal{Y}_{n+1}\|^2 \le k\|\mathcal{Y}_n\|^2 + 2\sum_{j=1}^{s} d_j \Re\langle \mathcal{Y}_j^{(n)}, h\mathcal{F}_j^{(n)} - l\mathcal{Y}_j^{(n)}\rangle. \tag{3.6}$$

It follows from (2.2)-(2.4) that

$$\begin{aligned}
2\Re\langle \mathcal{Y}_j^{(n)}, h\mathcal{F}_j^{(n)}\rangle &= 2h\Re\langle \mathcal{Y}_j^{(n)}, f(t_j^{(n)}, y_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)})) - f(t_j^{(n)}, \tilde{y}_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)}))\rangle \\
&+ 2h\Re\langle \mathcal{Y}_j^{(n)}, f(t_j^{(n)}, \tilde{y}_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)})) - f(t_j^{(n)}, \tilde{y}_j^{(n)}, G(t_j^{(n)}, \tilde{y}_j^{(n-m)}, \tilde{z}_j^{(n)}))\rangle \\
&\le 2h\alpha\|\mathcal{Y}_j^{(n)}\|^2 + 2h\|\mathcal{Y}_j^{(n)}\|\|f(t_j^{(n)}, \tilde{y}_j^{(n)}, G(t_j^{(n)}, y_j^{(n-m)}, z_j^{(n)})) \\
&\qquad - f(t_j^{(n)}, \tilde{y}_j^{(n)}, G(t_j^{(n)}, \tilde{y}_j^{(n-m)}, \tilde{z}_j^{(n)}))\| \\
&\le 2h\alpha\|\mathcal{Y}_j^{(n)}\|^2 + 2h\beta\|\mathcal{Y}_j^{(n)}\|(\sigma_1\|\mathcal{Y}_j^{(n-m)}\| + \sigma_2\|\mathcal{Z}_j^{(n)}\|) \\
&\le h[2\alpha + \beta(\sigma_1 + \sigma_2)]\|\mathcal{Y}_j^{(n)}\|^2 + h\beta\sigma_1\|\mathcal{Y}_j^{(n-m)}\|^2 + h\beta\sigma_2\|\mathcal{Z}_j^{(n)}\|^2
\end{aligned} \tag{3.7}$$

where the latter is obtained by using the inequality $2uv \le u^2 + v^2$ $(\forall u, v \in \mathbb{R})$. Substituting both (3.7) and the condition $0 < k \le \check{k}$ into (3.6) gives

$$\|\mathcal{Y}_{n+1}\|^2 \le \check{k}\|\mathcal{Y}_n\|^2 + [h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(n)}\|^2 + h\beta\sum_{j=1}^{s} d_j[\sigma_1\|\mathcal{Y}_j^{(n-m)}\|^2 + \sigma_2\|\mathcal{Z}_j^{(n)}\|^2].$$

An induction to the above inequality generates the following result

$$\begin{aligned}
\|\mathcal{Y}_{n+1}\|^2 &\le \check{k}^{n+1}\|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\sum_{i=0}^{n}\check{k}^{n-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 \\
&+ h\beta\sum_{i=0}^{n}\check{k}^{n-i}\sum_{j=1}^{s} d_j[\sigma_1\|\mathcal{Y}_j^{(i-m)}\|^2 + \sigma_2\|\mathcal{Z}_j^{(i)}\|^2].
\end{aligned} \tag{3.8}$$

Also, it holds that

$$\begin{aligned}
h\sum_{i=0}^{n}\check{k}^{n-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i-m)}\|^2 &= h\sum_{i=0}^{n-m}\check{k}^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 + h\sum_{i=-m}^{-1}\check{k}^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 \\
&\le h\sum_{i=0}^{n}\check{k}^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 + \tau\check{k}^{n-m+1}\sum_{j=1}^{s} d_j\max_{-m\le i\le -1}\|\mathcal{Y}_j^{(i)}\|^2,
\end{aligned} \tag{3.9}$$

where we have used conditions $mh = \tau$ and $0 < \check{k} \le 1$. A combination of (3.8) and (3.9) infers (3.3). This completes the proof. □

Lemma 3.6 *Suppose that* $\{A_i\}_{i=0}^n$ *and* $\{B_i\}_{i=-m}^n$ *are two arbitrary nonnegative real sequences. Then the following inequalities hold:*

$$\sum_{i=0}^{n}(A_i\sum_{j=0}^{m}B_{i-j}) \leq \sum_{j=0}^{m}\sum_{i=0}^{n}A_{i+j}B_i + (\sum_{j=1}^{m}\sum_{i=1}^{j}A_{j-i})\max_{-m\leq q\leq -1}\{B_q\}, \qquad \forall n,\ m\geq 0 \tag{3.10}$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i-j} \leq (m+1)\sum_{i=0}^{n}B_i + \frac{m(m+1)}{2}\max_{-m\leq q\leq -1}\{B_q\}, \qquad \forall n,\ m\geq 0. \tag{3.11}$$

Proof. Inequality (3.10) is proven first, by rewriting its left hand side as follows,

$$\begin{aligned}
\sum_{i=0}^{n}(A_i\sum_{j=0}^{m}B_{i-j}) &= \sum_{i=0}^{n}A_iB_i + \sum_{j=1}^{m}\sum_{i=0}^{n}A_iB_{i-j} \\
&= \sum_{i=0}^{n}A_iB_i + \sum_{j=1}^{m}(\sum_{i=j}^{n}A_iB_{i-j} + \sum_{i=0}^{j-1}A_iB_{i-j}) \\
&= \sum_{i=0}^{n}A_iB_i + \sum_{j=1}^{m}(\sum_{i=0}^{n-j}A_{i+j}B_i + \sum_{i=1}^{j}A_{j-i}B_{-i}) \\
&= \sum_{j=0}^{m}\sum_{i=0}^{n-j}A_{i+j}B_i + \sum_{j=1}^{m}\sum_{i=1}^{j}A_{j-i}B_{-i}
\end{aligned}$$

The nonnegativity of A_i and B_i allows to bound the latter expression by the right hand side of (3.10). Inequality (3.11) is obtained by setting $A_i = 1$ for all i in inequality (3.10). □

A trivial modification of inequalities (3.10) and (3.11) will also be used further on in this paper, namely the inequalities

$$\sum_{i=0}^{n}(A_i\sum_{j=1}^{m}B_{i-j}) \leq \sum_{j=1}^{m}\sum_{i=0}^{n}A_{i+j}B_i + (\sum_{j=1}^{m}\sum_{i=1}^{j}A_{j-i})\max_{-m\leq q\leq -1}\{B_q\}, \qquad \forall n,\ m\geq 0 \tag{3.12}$$

and

$$\sum_{i=0}^{n}\sum_{j=1}^{m}B_{i-j} \leq m\sum_{i=0}^{n}B_i + \frac{m(m+1)}{2}\max_{-m\leq q\leq -1}\{B_q\}, \qquad \forall n,\ m\geq 0. \tag{3.13}$$

4 Stability of RK methods with LCQ formula

This section will deal with both the global and the asymptotic stability of Runge Kutta methods with linear compound quadrature formula.

Theorem 4.1 *Suppose the underlying RK method (2.9) is (k,l)-algebraically stable for a non-negative diagonal matrix $D = diag(d_1, d_2, \ldots, d_s) \in \mathbb{R}^{s\times s}$, where $0 < k \le 1$. Then the induced RK method (2.10) with LCQ formula (2.11) is globally stable for class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ with stability constant*

$$\mathcal{H} = \sqrt{1 + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2)\sum_{j=1}^{s} d_j} \tag{4.1}$$

whenever

$$h[2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2] \le 2l. \tag{4.2}$$

Proof. By LCQ formula (2.11) and condition (2.5), we have

$$\begin{aligned} \|\mathcal{Z}_j^{(i)}\|^2 &\le h^2[\nu\gamma \sum_{q=i-m+1}^{i} \|\mathcal{Y}_j^{(q-1)}\| + (1-\nu)\gamma \sum_{q=i-m+1}^{i} \|\mathcal{Y}_j^{(q)}\|]^2 \\ &\le h^2\gamma^2(\sum_{q=i-m}^{i} \|\mathcal{Y}_j^{(q)}\|)^2 \\ &\le h^2\gamma^2(m+1)\sum_{q=i-m}^{i} \|\mathcal{Y}_j^{(q)}\|^2 \\ &= h^2\gamma^2(m+1)\sum_{q=0}^{m} \|\mathcal{Y}_j^{(i-q)}\|^2, \end{aligned} \tag{4.3}$$

in which the Cauchy inequality has been used. Substituting (4.3) into (3.4) yields

$$\|\mathcal{Y}_{n+1}\|^2 \le \|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2)) - 2l]\sum_{i=0}^{n}\sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \tag{4.4}$$

$$+\beta\sigma_1\tau\sum_{j=1}^{s} d_j \max_{-m\le i\le -1}\{\|\mathcal{Y}_j^{(i)}\|^2\} + h^3\beta\sigma_2\gamma^2(m+1)\sum_{i=0}^{n}\sum_{j=1}^{s} d_j \sum_{q=0}^{m}\|\mathcal{Y}_j^{(i-q)}\|^2. \tag{4.5}$$

With inequality (3.11) in Lemma 3.6, it holds

$$\sum_{i=0}^{n}\sum_{q=0}^{m}\|\mathcal{Y}_j^{(i-q)}\|^2 \le (m+1)\sum_{i=0}^{n}\|\mathcal{Y}_j^{(i)}\|^2 + \frac{m(m+1)}{2}\max_{-m\le i\le -1}\{\|\mathcal{Y}_j^{(i)}\|^2\}. \tag{4.6}$$

Embedding (4.6), (4.2) and condition $mh = \tau$ into (4.5) yields

$$\begin{aligned}
\|\mathcal{Y}_{n+1}\|^2 \leq\ & \|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2)) - 2l] \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\
& + \beta\sigma_1\tau \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} + h^3\beta\sigma_2\gamma^2(m+1)^2 \sum_{j=1}^{s} d_j \sum_{i=0}^{n} \|\mathcal{Y}_j^{(i)}\|^2 \\
& + h^3\beta\sigma_2\gamma^2 \frac{m(m+1)^2}{2} \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} \\
\leq\ & \|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2)) - 2l] \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\
& + \beta\sigma_1\tau \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} + 4m^2h^3\beta\sigma_2\gamma^2 \sum_{j=1}^{s} d_j \sum_{i=0}^{n} \|\mathcal{Y}_j^{(i)}\|^2 \\
& + 2(mh)^3\beta\sigma_2\gamma^2 \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\}
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{Y}_{n+1}\|^2 \leq\ & \|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2) - 2l] \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\
& + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2) \sum_{j=1}^{s} d_j \max_{-m \leq i \leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} \\
\leq\ & [1 + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2) \sum_{j=1}^{s} d_j] \max_{t_0 - \tau \leq t \leq t_0} \|\varphi(t) - \psi(t)\|^2, \quad \forall n \geq 0. \qquad (4.7)
\end{aligned}$$

This leads to the value of $\mathcal{H}$ given in (4.1); hence the method is globally stable. □

Since an underlying RK method is algebraically stable iff $b_j \geq 0$ $(j = 1, 2, \ldots, s)$ and matrix $DA + A^T D - bb^T$ is nonnegative definite, where $D = \mathrm{diag}(b_1, b_2, \ldots, b_s) \in \mathbb{R}^{s \times s}$ (cf. [10]), we can derive the following corollary based on Theorem 4.1.

Corollary 4.2 *Suppose that the underlying RK method* (2.9) *is algebraically stable. Then the induced RK method* (2.10) *with LCQ formula* (2.11) *is globally stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *with stability constant* (4.1) *with* $d_j = b_j$ *whenever*

$$\beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2 \leq -2\alpha. \qquad (4.8)$$

Next, we study the *asymptotic stability.* For this we have the following theorem.

Theorem 4.3 *Suppose the underlying RK method* (2.9) *is* (k, l)*-algebraically stable for a nonnegative diagonal matrix* $D = diag(d_1, d_2, \ldots, d_s) \in \mathbb{R}^{s \times s}$*, where* $0 < k < 1$*. Then the* *induced RK method* (2.10) *with LCQ formula* (2.11) *is asymptotically stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *whenever*

$$h[2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2] < 2l. \tag{4.9}$$

Proof. We define the quantity θ as

$$\theta = \max\left\{k, \ \left[\frac{h\beta(\sigma_1 + 4\sigma_2\gamma^2\tau^2)}{2l - h[2\alpha + \beta(\sigma_1 + \sigma_2)]}\right]^{\frac{1}{m}}\right\}. \tag{4.10}$$

With $0 < k < 1$ and (4.9), it can be deduced that $0 < \theta < 1$. Because $0 < k \leq \theta$, it follows from (3.3) in Lemma 3.4 that

$$\begin{aligned} \|\mathcal{Y}_{n+1}\|^2 &\leq \theta^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\theta^m + h\beta\sigma_1\} \sum_{i=0}^{n} \theta^{n-m-i} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\ &\quad + \beta\sigma_1\tau\theta^{n-m+1} \sum_{j=1}^{s} d_j \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_j^{(i)}\|^2\} + h\beta\sigma_2 \sum_{i=0}^{n}\theta^{n-i}\sum_{j=1}^{s} d_j\|\mathcal{Z}_j^{(i)}\|^2 \\ &\leq \theta^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\theta^m + h\beta\sigma_1\} \sum_{i=0}^{n} \theta^{n-m-i} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\ &\quad + \beta\sigma_1\tau\theta^{n-m+1} \sum_{j=1}^{s} d_j \max_{t_0-\tau\leq t\leq t_0} \|\varphi(t) - \psi(t)\|^2 + h\beta\sigma_2 \sum_{i=0}^{n}\theta^{n-i}\sum_{j=1}^{s} d_j\|\mathcal{Z}_j^{(i)}\|^2. \end{aligned}$$

Also, by the latter inequality, and by (4.3) and (3.10), we have

$$\begin{aligned} h\sum_{i=0}^{n}\theta^{n-i}\sum_{j=1}^{s} d_j\|\mathcal{Z}_j^{(i)}\|^2 &\leq h^3\gamma^2(m+1)\theta^n \sum_{j=1}^{s} d_j \sum_{i=0}^{n}\theta^{-i}\sum_{q=0}^{m}\|\mathcal{Y}_j^{(i-q)}\|^2 \\ \leq h^3\gamma^2(m+1)\theta^n \sum_{j=1}^{s} d_j [\sum_{i=0}^{n}\sum_{q=0}^{m}\theta^{-(i+q)}\|\mathcal{Y}_j^{(i)}\|^2 &+ (\sum_{q=1}^{m}\sum_{i=1}^{q}\theta^{-(q-i)}) \max_{-m\leq\hat{q}\leq-1}\{\|\mathcal{Y}_j^{(\hat{q})}\|^2\}] \\ &\leq h^3\gamma^2(m+1)^2 \sum_{i=0}^{n}\theta^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 + h^3\gamma^2(m+1)m^2\theta^{n-m+1}\sum_{j=1}^{s} d_j \max_{-m\leq\hat{q}\leq-1}\{\|\mathcal{Y}_j^{(\hat{q})}\|^2\} \\ &\leq 4m^2h^3\gamma^2 \sum_{i=0}^{n}\theta^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 + 2m^3h^3\gamma^2\theta^{n-m+1}\sum_{j=1}^{s} d_j \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t) - \psi(t)\|^2 \\ &= 4h\tau^2\gamma^2 \sum_{i=0}^{n}\theta^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 + 2\tau^3\gamma^2\theta^{n-m+1}\sum_{j=1}^{s} d_j \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t) - \psi(t)\|^2. \end{aligned} \tag{4.11}$$

Substituting (4.11) into the bound for $\|\mathcal{Y}_{n+1}\|^2$ leads to a further upper bound for $\|\mathcal{Y}_{n+1}\|^2$:

$$\begin{aligned} &\theta^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\theta^m + h\beta(\sigma_1 + 4\sigma_2\gamma^2\tau^2)\} \sum_{i=0}^{n}\theta^{n-m-i}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 \\ &+ \beta\tau\theta^{n-m+1}(\sigma_1 + 2\sigma_2\gamma^2\tau^2)\sum_{j=1}^{s} d_j \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t) - \psi(t)\|^2. \end{aligned} \tag{4.12}$$

Since by

$$h[2\alpha + \beta(\sigma_1 + \sigma_2)] \leq h[2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2] < 2l$$

and (4.10) we have

$$[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\theta^m + h\beta(\sigma_1 + 4\sigma_2\gamma^2\tau^2) < 0.$$

Bound (4.12) therefore implies

$$\|\mathcal{Y}_{n+1}\|^2 \leq \theta^{n+1}\|\mathcal{Y}_0\|^2 + \beta\tau\theta^{n-m+1}(\sigma_1 + 2\sigma_2\gamma^2\tau^2)\sum_{j=1}^{s} d_j \max_{t_0-\tau\leq t\leq t_0} \|\varphi(t) - \psi(t)\|^2. \quad (4.13)$$

This, together with $0 < \theta < 1$, leads to $\lim_{n\to\infty} \|\mathcal{Y}_n\| = 0$. □

5 Stability of RK methods with PQ formula

This section will focus on the global and the asymptotic stability of RK methods (2.10) with PQ formula (2.12). First, we define a quantity that is frequently used later on:

$$\mathcal{L} := \min_{1\leq r\leq s}\{d_r\},$$

in which d_r $(1 \leq r \leq s)$ is indicated in Definition 3.1.

Theorem 5.1 *Suppose the underlying RK method* (2.9) *is* (k, l)*-algebraically stable for a positive diagonal matrix* $D = diag(d_1, d_2, \ldots, d_s) \in \mathbb{R}^{s\times s}$*, where* $0 < k \leq 1$*. Then the induced RK method* (2.10) *with PQ formula* (2.12) *is globally stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *with stability constant*

$$\mathcal{H} = \sqrt{1 + \beta\tau[\sigma_1 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}(\sum_{j=1}^{s} d_j \sum_{r=1}^{s}(|a_{jr}|^2 + |b_r|^2))]\sum_{r=1}^{s} d_r} \quad (5.1)$$

whenever

$$h\{2\alpha + \beta[2\sigma_1 + \sigma_2 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}\sum_{j=1}^{s} d_j \sum_{r=1}^{s}(2|a_{jr}|^2 + |b_r|^2)]\} \leq 2l. \quad (5.2)$$

Proof. With condition (2.5) and PQ formula (2.12), we have

$$\begin{aligned}
\|\mathcal{Z}_j^{(i)}\|^2 &\leq [h\gamma(\sum_{r=1}^{s}|a_{jr}|\|\mathcal{Y}_r^{(i)}\| + \sum_{q=1}^{m}\sum_{r=1}^{s}|b_r|\|\mathcal{Y}_r^{(i-q)}\| + \sum_{r=1}^{s}|a_{jr}|\|\mathcal{Y}_r^{(i-m)}\|)]^2 \\
&\leq 3h^2\gamma^2[(\sum_{r=1}^{s}|a_{jr}|\|\mathcal{Y}_r^{(i)}\|)^2 + (\sum_{q=1}^{m}\sum_{r=1}^{s}|b_r|\|\mathcal{Y}_r^{(i-q)}\|)^2 + (\sum_{r=1}^{s}|a_{jr}|\|\mathcal{Y}_r^{(i-m)}\|)^2] \\
&\leq 3h^2\gamma^2[(\sum_{r=1}^{s}|a_{jr}|^2)(\sum_{r=1}^{s}\|\mathcal{Y}_r^{(i)}\|^2) + m\sum_{q=1}^{m}(\sum_{r=1}^{s}|b_r|^2)(\sum_{r=1}^{s}\|\mathcal{Y}_r^{(i-q)}\|^2) \\
&\quad +(\sum_{r=1}^{s}|a_{jr}|^2)(\sum_{r=1}^{s}\|\mathcal{Y}_r^{(i-m)}\|^2)], \quad (5.3)
\end{aligned}$$

where we have repeatedly used the Cauchy inequality and the induced inequality

$$(\sum_{i=1}^{n} \mathcal{A}_i)^2 \leq n \sum_{i=1}^{n} \mathcal{A}_i^2, \quad \forall n \in \mathbb{N}, \ \forall \mathcal{A}_i \in \mathbb{R}. \tag{5.4}$$

Inserting (5.3) into (3.4) generates

$$\begin{aligned}\|\mathcal{Y}_{n+1}\|^2 \leq{}& \|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2)) - 2l] \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\ &+ \beta\sigma_1\tau \sum_{j=1}^{s} d_j \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} + 3h^3\beta\sigma_2\gamma^2[(\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=0}^{n} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2) \\ &+ m(\sum_{j=1}^{s} d_j)(\sum_{r=1}^{s} |b_r|^2)(\sum_{r=1}^{s} \sum_{i=0}^{n} \sum_{q=1}^{m} \|\mathcal{Y}_r^{(i-q)}\|^2) + (\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=0}^{n} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i-m)}\|^2)]. \end{aligned} \tag{5.5}$$

It follows from inequality (3.13) that

$$\sum_{i=0}^{n} \sum_{q=1}^{m} \|\mathcal{Y}_r^{(i-q)}\|^2 \leq m \sum_{i=0}^{n} \|\mathcal{Y}_r^{(i)}\|^2 + \frac{m(m+1)}{2} \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_r^{(i)}\|^2\}. \tag{5.6}$$

Moreover, it holds that

$$\sum_{i=0}^{n} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i-m)}\|^2 = \sum_{i=-m}^{n-m} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2 \leq \sum_{i=0}^{n} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2 + m \sum_{r=1}^{s} \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_r^{(i)}\|^2\}. \tag{5.7}$$

Substituting (5.6), (5.7) and $h = \frac{\tau}{m} \leq \tau$ into (5.5) yields a new upper bound for $\|\mathcal{Y}_{n+1}\|^2$:

$$\begin{aligned}&\|\mathcal{Y}_0\|^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2)) - 2l] \sum_{i=0}^{n} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 + \beta\sigma_1\tau \sum_{j=1}^{s} d_j \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} \\ &+ 3h^3\beta\sigma_2\gamma^2[2(\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=0}^{n} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2) + m^2(\sum_{j=1}^{s} d_j)(\sum_{r=1}^{s} |b_r|^2)(\sum_{i=0}^{n} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2) \\ &+ \frac{m^2(m+1)}{2}(\sum_{j=1}^{s} d_j)(\sum_{r=1}^{s} |b_r|^2)(\sum_{r=1}^{s} \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_r^{(i)}\|^2\}) \\ &+ m(\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{r=1}^{s} \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_r^{(i)}\|^2\})]\end{aligned}$$

$$
\begin{aligned}
\leq\ & \|\mathcal{Y}_0\|^2 + [h(2\alpha+\beta(2\sigma_1+\sigma_2)) - 2l]\sum_{i=0}^{n}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 \\
& +\beta\sigma_1\tau\sum_{j=1}^{s} d_j \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_j^{(i)}\|^2\} + \frac{3h^3\beta\sigma_2\gamma^2}{\mathcal{L}}[2(\sum_{j=1}^{s} d_j\sum_{r=1}^{s}|a_{jr}|^2)(\sum_{i=0}^{n}\sum_{r=1}^{s} d_r\|\mathcal{Y}_r^{(i)}\|^2) \\
& +m^2(\sum_{j=1}^{s} d_j)(\sum_{r=1}^{s}|b_r|^2)(\sum_{i=0}^{n}\sum_{r=1}^{s} d_r\|\mathcal{Y}_r^{(i)}\|^2) + m^3(\sum_{j=1}^{s} d_j)(\sum_{r=1}^{s}|b_r|^2)(\sum_{r=1}^{s} d_r \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_r^{(i)}\|^2\}) \\
& +m(\sum_{j=1}^{s} d_j\sum_{r=1}^{s}|a_{jr}|^2)(\sum_{r=1}^{s} d_r \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_r^{(i)}\|^2\})] \\
\leq\ & \|\mathcal{Y}_0\|^2 + [h(2\alpha+\beta(2\sigma_1+\sigma_2)) - 2l]\sum_{i=0}^{n}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2 + \beta\sigma_1\tau\sum_{j=1}^{s} d_j \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_j^{(i)}\|^2\} \\
& +\frac{3h\beta\sigma_2\gamma^2\tau^2}{\mathcal{L}}[(\sum_{j=1}^{s} d_j\sum_{r=1}^{s}(2|a_{jr}|^2+|b_r|^2)](\sum_{i=0}^{n}\sum_{r=1}^{s} d_r\|\mathcal{Y}_r^{(i)}\|^2) \\
& +\frac{3\beta\sigma_2\gamma^2\tau^3}{\mathcal{L}}[\sum_{j=1}^{s} d_j\sum_{r=1}^{s}(|a_{jr}|^2+|b_r|^2)](\sum_{r=1}^{s} d_r \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_r^{(i)}\|^2\}) \\
=\ & \|\mathcal{Y}_0\|^2 + \{h[2\alpha+\beta(2\sigma_1+\sigma_2+\frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}\sum_{j=1}^{s} d_j\sum_{r=1}^{s}(2|a_{jr}|^2+|b_r|^2))] - 2l\}(\sum_{i=0}^{n}\sum_{j=1}^{s} d_j\|\mathcal{Y}_j^{(i)}\|^2) \\
& +\beta\tau\{\sigma_1+\frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}[\sum_{j=1}^{s} d_j\sum_{r=1}^{s}(|a_{jr}|^2+|b_r|^2)]\}(\sum_{r=1}^{s} d_r \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_r^{(i)}\|^2\}). \qquad (5.8)
\end{aligned}
$$

Applying (5.2) to (5.8) shows

$$
\begin{aligned}
\|\mathcal{Y}_{n+1}\|^2 \ \leq\ & \|\mathcal{Y}_0\|^2 + \beta\tau\{\sigma_1+\frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}[\sum_{j=1}^{s} d_j\sum_{r=1}^{s}(|a_{jr}|^2+|b_r|^2)]\}(\sum_{r=1}^{s} d_r \max_{-m\leq i\leq -1}\{\|\mathcal{Y}_r^{(i)}\|^2\}) \\
\leq\ & \{1+\beta\tau[\sigma_1+\frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}(\sum_{j=1}^{s} d_j\sum_{r=1}^{s}(|a_{jr}|^2+|b_r|^2))]\sum_{r=1}^{s} d_r\} \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|^2.
\end{aligned}
$$

This implies inequality (5.1) and hence the method is globally stable. □

Corollary 5.2 *Suppose the underlying RK method* (2.9) *with* $b_j > 0$ *is algebraically stable. Then the induced RK method* (2.10) *with PQ formula* (2.12) *is globally stable for the class* $\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ *with stability constant* (5.1) *with* $d_j = b_j$, *whenever*

$$
\beta[2\sigma_1+\sigma_2+\frac{3\sigma_2\gamma^2\tau^2}{\tilde{\mathcal{L}}}\sum_{j=1}^{s} b_j\sum_{r=1}^{s}(2|a_{jr}|^2+|b_r|^2)] \leq -2\alpha, \ \ \textit{with}\ \ \tilde{\mathcal{L}} = \min_{1\leq r\leq s}\{b_r\}. \qquad (5.9)
$$

Next, we study the *asymptotic stability* of the RK methods with PQ formula. For this, we have the following theorem.

Theorem 5.3 *Suppose the underlying RK method* (2.9) *is* (k,l)*-algebraically stable for a positive diagonal matrix* $D = diag(d_1, d_2, \dots, d_s) \in \mathbb{R}^{s\times s}$*, where* $0 < k < 1$*. Then the induced RK method* (2.10) *with PQ formula* (2.12) *is asymptotically stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *whenever*

$$h\{2\alpha + \beta[2\sigma_1 + \sigma_2 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}} \sum_{j=1}^{s} d_j \sum_{r=1}^{s} (2|a_{jr}|^2 + |b_r|^2)]\} < 2l. \tag{5.10}$$

Proof. We define the quantity η as

$$\eta = \max\left\{ k, \quad \left[\frac{h[\sigma_1 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}} \sum\limits_{j=1}^{s} d_j \sum\limits_{r=1}^{s} (2|a_{jr}|^2 + |b_r|^2)]}{2l - h[2\alpha + \beta(\sigma_1 + \sigma_2)]} \right]^{\frac{1}{m}} \right\}. \tag{5.11}$$

It follows from both $0 < k < 1$ and (5.10) that $0 < \eta < 1$. Since $0 \le k \le \eta$, we can bound $\|\mathcal{Y}_{n+1}\|^2$ by using (3.3) in Lemma 3.4,

$$\begin{aligned}
\|\mathcal{Y}_{n+1}\|^2 \le\ & \eta^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\eta^m + h\beta\sigma_1\} \sum_{i=0}^{n} \eta^{n-m-i} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\
& + \beta\sigma_1\tau\eta^{n-m+1} \sum_{j=1}^{s} d_j \max_{-m\le i\le -1} \{\|\mathcal{Y}_j^{(i)}\|^2\} + h\beta\sigma_2 \sum_{i=0}^{n} \eta^{n-i} \sum_{j=1}^{s} d_j \|\mathcal{Z}_j^{(i)}\|^2 \\
\le\ & \eta^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\eta^m + h\beta\sigma_1\} \sum_{i=0}^{n} \eta^{n-m-i} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)}\|^2 \\
& + \beta\sigma_1\tau\eta^{n-m+1} \sum_{j=1}^{s} d_j \max_{t_0-\tau\le t\le t_0} \|\varphi(t) - \psi(t)\|^2 + h\beta\sigma_2 \sum_{i=0}^{n} \eta^{n-i} \sum_{j=1}^{s} d_j \|\mathcal{Z}_j^{(i)}\|^2.
\end{aligned} \tag{5.12}$$

Also, by (5.3), (3.12) and by using the condition $h = \frac{\tau}{m} \le \tau$ we can bound one of the terms in the above inequality,

$$\begin{aligned}
h \sum_{i=0}^{s} \eta^{n-i} \sum_{j=1}^{s} d_j \|\mathcal{Z}_j^{(i)}\|^2 \le 3h^3\gamma^2 \sum_{i=0}^{s} \eta^{n-i} \sum_{j=1}^{s} d_j [(\sum_{r=1}^{s} |a_{jr}|^2)(\sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2) \\
+ m \sum_{q=1}^{m} (\sum_{r=1}^{s} |b_r|^2)(\sum_{r=1}^{s} \|\mathcal{Y}_r^{(i-q)}\|^2) + (\sum_{r=1}^{s} |a_{jr}|^2)(\sum_{r=1}^{s} \|\mathcal{Y}_r^{(i-m)}\|^2)]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 3h^3\gamma^2\{(\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=0}^{n} \eta^{n-i} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2) \\
& + m(\sum_{r=1}^{s} |b_r|^2)(\sum_{j=1}^{s} d_j) \sum_{r=1}^{s} [\sum_{q=1}^{m} \sum_{i=0}^{n} \eta^{n-(i+q)} \|\mathcal{Y}_r^{(i)}\|^2 \\
& + (\sum_{q=1}^{m} \sum_{i=1}^{q} \eta^{n-(q-i)})(\max_{-m\leq i\leq -1} \{\|\mathcal{Y}_r^{(i)}\|^2\})] + (\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=-m}^{n-m} \eta^{n-m-i} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2)\} \\
\leq\ & 3h^3\gamma^2\{(\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=0}^{n} \eta^{n-i} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2) \\
& + m(\sum_{r=1}^{s} |b_r|^2)(\sum_{j=1}^{s} d_j)(m \sum_{i=0}^{n} \eta^{n-m-i} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2 + m^2 \eta^{n-m+1} \sum_{r=1}^{s} \max_{-m\leq i\leq -1} \{\|\mathcal{Y}_r^{(i)}\|^2\}) \\
& + (\sum_{j=1}^{s} d_j \sum_{r=1}^{s} |a_{jr}|^2)(\sum_{i=0}^{n} \eta^{n-m-i} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2 + \sum_{i=-m}^{-1} \eta^{n-m-i} \sum_{r=1}^{s} \|\mathcal{Y}_r^{(i)}\|^2)\} \\
\leq\ & \frac{3h\gamma^2\tau^2}{\mathcal{L}}[\sum_{j=1}^{s} d_j \sum_{r=1}^{s} (2|a_{jr}|^2 + |b_r|^2)](\sum_{i=0}^{n} \eta^{n-m-i} \sum_{r=1}^{s} d_r \|\mathcal{Y}_r^{(i)}\|^2) \\
& + \frac{3\gamma^2\tau^3\eta^{n-m+1}}{\mathcal{L}}[\sum_{j=1}^{s} d_j \sum_{r=1}^{s} (|a_{jr}|^2 + |b_r|^2)](\sum_{r=1}^{s} d_r \max_{-1\leq i\leq -m} \{\|\mathcal{Y}_r^{(i)}\|^2\}) \\
\leq\ & \frac{3h\gamma^2\tau^2}{\mathcal{L}}[\sum_{j=1}^{s} d_j \sum_{r=1}^{s} (2|a_{jr}|^2 + |b_r|^2)](\sum_{i=0}^{n} \eta^{n-m-i} \sum_{r=1}^{s} d_r \|\mathcal{Y}_r^{(i)}\|^2) \\
& + \frac{3\gamma^2\tau^3\eta^{n-m+1}}{\mathcal{L}}[\sum_{j=1}^{s} d_j \sum_{r=1}^{s} (|a_{jr}|^2 + |b_r|^2)](\sum_{r=1}^{s} d_r) \max_{t_0-\tau\leq t\leq t_0} \|\varphi(t) - \psi(t)\|^2. \qquad (5.13)
\end{aligned}
$$

Inserting (5.13) into (5.12) generates

$$
\begin{aligned}
& \|\mathcal{Y}_{n+1}\|^2 \leq \eta^{n+1}\|\mathcal{Y}_0\|^2 + \{[h(2\alpha + \beta(\sigma_1 + \sigma_2)) - 2l]\eta^m \qquad (5.14) \\
& + h\beta[\sigma_1 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}(\sum_{j=1}^{s} d_j \sum_{r=1}^{s} (2|a_{jr}|^2 + |b_r|^2))]\}(\sum_{i=0}^{n} \eta^{n-m-i} \sum_{j=1}^{s} d_j \|\mathcal{Y}_j^{(i)})\|^2 \\
& + \beta\tau\eta^{n-m+1}\{\sigma_1 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}[\sum_{j=1}^{s} d_j \sum_{r=1}^{s} (|a_{jr}|^2 + |b_r|^2)]\}(\sum_{r=1}^{s} d_r) \max_{t_0-\tau\leq t\leq t_0} \|\varphi(t) - \psi(t)\|^2.
\end{aligned}
$$

Moreover, by (5.10) it holds that

$$
h[2\alpha + \beta(\sigma_1 + \sigma_2)] \leq h\{2\alpha + \beta[2\sigma_1 + \sigma_2 + \frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}} \sum_{j=1}^{s} d_j \sum_{r=1}^{s} (2|a_{jr}|^2 + |b_r|^2)]\} < 2l. \qquad (5.15)
$$

Combining this with (5.11) leads to

$$[h(2\alpha+\beta(\sigma_1+\sigma_2))-2l]\eta^m+h\beta[\sigma_1+\frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}(\sum_{j=1}^{s}d_j\sum_{r=1}^{s}(2|a_{jr}|^2+|b_r|^2))]<0. \quad (5.16)$$

Hence, it follows from (5.14) that $\|\mathcal{Y}_{n+1}\|^2$ is bounded by

$$\eta^{n+1}\|\mathcal{Y}_0\|^2+\beta\tau\eta^{n-m+1}\{\sigma_1+\frac{3\sigma_2\gamma^2\tau^2}{\mathcal{L}}\sum_{j=1}^{s}d_j\sum_{r=1}^{s}(|a_{jr}|^2+|b_r|^2)\}\sum_{r=1}^{s}d_r\max_{t_0-\tau\le t\le t_0}\|\varphi(t)-\psi(t)\|^2.$$

This, together with $0\le\eta\le 1$, shows $\lim_{n\to\infty}\|\mathcal{Y}_n\|=0$: the method is asymptotically stable. □

6 Application to some classical underlying RK schemes

The global and asymptotic stability results derived in the previous sections are applicable to the VDIDE methods induced by various common RK methods. Based on Corollary 4.2 and Corollary 5.2, we can summarize our findings in the two theorems given below. For their proof we need only note the fact that the underlying RK methods of type Gauss, Radau IA, Radau IIA and Lobatto IIIC are all algebraically stable and satisfy $b_j>0$ $(j=1,2,\dots,s)$. (cf. [10]).

Theorem 6.1 *Suppose the underlying RK method* (2.9) *is of type Gauss, Radau IA, Radau IIA or Lobatto IIIC. Then the induced RK method* (2.10) *with LCQ formula* (2.11) *is globally stable for the class* $\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ *with stability constant* (4.1) *with* $d_j=b_j$ *whenever condition* (4.8) *holds.*

Theorem 6.2 *Suppose the underlying RK method* (2.9) *is of type Gauss, Radau IA, Radau IIA and Lobatto IIIC. Then the induced RK method* (2.10) *with PQ formula* (2.12) *is globally stable for the class* $\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ *with stability constant* (5.1) *with* $d_j=b_j$ *whenever condition* (5.9) *holds.*

As for the asymptotic stability of the methods, we can rely on Theorems 4.3 and 5.3 to provide us with effective criteria for judging the asymptotic behavior of the methods. To illustrate this, we present some examples.

Example 6.1 First, we consider the two-stage Lobatto IIIC formula:

$$\begin{array}{c|cc} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}\ . \quad (6.1)$$

Burrage and Butcher [8] showed that this formula is $(\frac{1}{(1-l)^2},l)$-algebraically stable for the nonnegative diagonal matrix $D=\mathrm{diag}(\frac{1}{2(1-l)},\frac{1}{2(1-l)})$, where $l<1$. Whenever $l<0$, we have

$$0<k:=\frac{1}{(1-l)^2}<1.$$

Hence, we derive from Theorem 4.3 that method (2.10) generated by the LCQ formula (2.11) and the Lobatto IIIC formula (6.1) is asymptotically stable for the class $\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ when

$$h[2\alpha+\beta(2\sigma_1+\sigma_2)+4\beta\sigma_2\gamma^2\tau^2] < 2l < 0.$$

By Theorem 5.3 we have that the method generated by the PQ formula (2.12) and the Lobatto IIIC formula (6.1) is asymptotically stable for the class $\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ when

$$h[2\alpha+\beta(2\sigma_1+\sigma_2+9\sigma_2\gamma^2\tau^2)] < 2l < 0.$$

Example 6.2 Next, we consider the two-stage Radau IIA formula

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array} \quad . \tag{6.2}$$

Burrage and Butcher [8] have deduced that this formula is $(\frac{16}{(5-2l)^2},l)$-algebraically stable for the nonnegative diagonal matrix $D=\mathrm{diag}(\frac{9}{(3-l)(5-2l)},\frac{2}{5-2l})$ when $l\le\frac{9-3\sqrt{17}}{8}$. The method is $(\frac{(3+4l)^2}{(3-2l)(3+4l-2l^2)},l)$-algebraically stable for the nonnegative diagonal matrix given by $D=\mathrm{diag}(\frac{(3+4l)^2}{4(3+4l-2l^2)},\frac{3+4l}{4(3+4l-2l^2)})$ when $\frac{9-3\sqrt{17}}{8}<l<\frac{2}{3}$. Also, we have that

$$0<k:=\frac{16}{(5-2l)^2}<1$$

whenever $l<\frac{1}{2}$ or $l>\frac{9}{2}$, and

$$0<k:=\frac{(3+4l)^2}{(3-2l)(3+4l-2l^2)}<1$$

whenever $\frac{15-3\sqrt{33}}{4}<l<0$ or $l>\frac{15+3\sqrt{33}}{4}$. Combining all of the above, we know that the Radau IIA formula is $(\frac{16}{(5-2l)^2},l)$-algebraically stable and satisfies $0<k:=\frac{16}{(5-2l)^2}<1$ whenever $l\le\frac{9-3\sqrt{17}}{8}$, and $(\frac{(3+4l)^2}{(3-2l)(3+4l-2l^2)},l)$-algebraically stable and satisfies $0<k:=\frac{(3+4l)^2}{(3-2l)(3+4l-2l^2)}<1$ whenever $\frac{9-3\sqrt{17}}{8}<l<0$. Therefore, our theorems imply that the method with LCQ formula (2.11) is asymptotically stable for the class $\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ whenever one of the following two conditions holds:

$$(I) \qquad h[2\alpha+\beta(2\sigma_1+\sigma_2)+4\beta\sigma_2\gamma^2\tau^2]<2l\le\frac{9-3\sqrt{17}}{4},$$

$$(II) \qquad \max\{\frac{9-3\sqrt{17}}{4},\ h[2\alpha+\beta(2\sigma_1+\sigma_2)+4\beta\sigma_2\gamma^2\tau^2]\}<2l<0.$$

The generated method with PQ formula (2.12) is asymptotically stable for the problem class

$\mathbb{DI}(\alpha,\beta,(\sigma_1,\sigma_2),\gamma)$ whenever one of the following two conditions holds:

$$(III) \qquad h[2\alpha+\beta(2\sigma_1+\sigma_2+\tfrac{3\sigma_2\gamma^2\tau^2(161-30l)}{8\mathcal{L}(5-2l)(3-l)})]<2l\le\tfrac{9-3\sqrt{17}}{4} \qquad with$$

$$\mathcal{L}=\begin{cases}\frac{2}{5-2l}, & -\frac{3}{2}\le l\le\frac{9-3\sqrt{17}}{8},\\ \frac{9}{(3-l)(5-2l)}, & l\le-\frac{3}{2};\end{cases}$$

$$(IV) \qquad \max\{\tfrac{9-3\sqrt{17}}{4},\ h[2\alpha+\beta(2\sigma_1+\sigma_2+(\tfrac{29+22l}{2})\sigma_2\gamma^2\tau^2)]\}<2l<0.$$

In a similar way, we can show that methods induced by Gauss and Radau IA type formulae, with LCQ or PQ methods, possess asymptotic stability under certain suitable conditions.

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