Computing orthogonal rational functions on the halfline

J. Van Deun *A. Bultheel*

Report TW 346, October 2002

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Computing orthogonal rational functions on the halfline

J. Van Deun *A. Bultheel*

Report TW 346, October 2002

Department of Computer Science, K.U.Leuven

Abstract

We derive formulas relating the recurrence coefficients for orthogonal rational functions on the halfline $[0, \infty]$ and the interval $[-1, 1]$. With the aid of these formulas we can limit our attention to the case of the interval to compute functions on the halfline as well.

Keywords : orthogonal rational functions
AMS(MOS) Classification : 42C05

Computing orthogonal rational functions on the halfline

J. Van Deun and A. Bultheel

Department of Computer Science, K.U.Leuven, Belgium

E-mail: {joris.vandeun–adhemar.bultheel}@cs.kuleuven.ac.be

Abstract

We derive formulas relating the recurrence coefficients for orthogonal rational functions on the halfline $[0, \infty]$ and the interval $[-1, 1]$. With the aid of these formulas we can limit our attention to the case of the interval to compute functions on the halfline as well.

1 Introduction

The main purpose of this note is to provide formulas relating orthogonal rational functions on the halfline $[0, \infty]$ to orthogonal rational functions on the interval $[-1, 1]$. Then if we can compute the latter functions, we can also easily obtain the former ones (or, with some obvious modifications, orthogonal rational functions on *any* finite interval or halfline).

More specifically, we will deduce formulas that give the recurrence coefficients for orthogonal rational functions on the halfline in terms of the coefficients for functions on the interval. Once the recurrence coefficients are known, the computation of the functions is straightforward.

2 Preliminaries

The real line is denoted by $\mathbb{R}$ and the extended real line by $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

By a measure μ we will mean a positive bounded Borel measure whose support $\text{supp}(\mu) \subset \hat{\mathbb{R}}$ is an infinite set and normalized such that $\mu(\hat{\mathbb{R}}) = 1$. The inner product in the metric space $L_2(\mu)$ is then defined as

$$\langle f\,, g\rangle = \int f\overline{g}d\mu. \tag{1}$$

Next we will introduce the spaces of rational functions with real poles. Let a sequence $A = \{\alpha_1, \alpha_2, \dots\} \subset \hat{\mathbb{R}} \setminus \{0\}$ be given such that $A \cap \text{supp}(\mu) = \emptyset$. As a consequence we cannot have $\text{supp}(\mu) = \hat{\mathbb{R}}$. Define factors

$$Z_n(z) = \frac{z}{1 - z/\alpha_n}$$

and basis functions

$$b_0 = 1, \quad b_n(z) = b_{n-1}(z)Z_n(z), \quad n = 1, 2, \dots$$

Then the space of rational functions with poles in A is defined as

$$\mathcal{L}_n = \text{span}\{b_0, \dots, b_n\}.$$

Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n and define

$$\pi_n(z) = \prod_{k=1}^{n}(1 - z/\alpha_k),$$

then we may write equivalently

$$\mathcal{L}_n = \{p_n/\pi_n, p_n \in \mathcal{P}_n\}.$$

Orthonormalizing the basis $\{b_0, \dots, b_n\}$ with respect to μ we obtain orthogonal rational functions $\{\phi_0, \dots, \phi_n\}$ where we choose the leading coefficient κ_n in the expansion $\phi_n(z) = \kappa_n b_n(z) + \dots$ to be real. The ϕ_n will be uniquely determined once the sign of κ_n is fixed. We will get back to this later on. The following lemma from [2] will be useful.

Lemma 2.1. *The orthonormal functions ϕ_n have real coefficients with respect to the basis $\{b_k\}$.*

It follows in particular that $\phi_n(z)$ is real for real z.

The orthogonal rational function ϕ_n is called *regular* if its numerator polynomial satisfies $p_n(\alpha_{n-1}) \neq 0$. The system $\{\phi_n\}$ is regular if ϕ_n is regular for every n. We now mention the most important theorem for the computation of orthogonal rational functions on the real line, which states that they satisfy a three term recurrence relation, analogous to the one for the polynomial case. For the proof of the theorem we refer to [2].

Theorem 2.2. *Put by convention $\alpha_{-1} = \alpha_0 = \infty$. Then for $n = 1, 2, \ldots$ the orthonormal rational functions ϕ_n satisfy the following three term recurrence relation if and only if ϕ_n and ϕ_{n-1} are regular:*

$$\phi_n(z) = \left(E_n Z_n(z) + B_n \frac{Z_n(z)}{Z_{n-1}(z)} \right) \phi_{n-1}(z) - \frac{E_n}{E_{n-1}} \frac{Z_n(z)}{Z_{n-2}(z)} \phi_{n-2}(z). \quad (2)$$

The initial conditions are $\phi_{-1}(z) \equiv 0$, $\phi_0(z) \equiv 1$ and the coefficients E_n are nonzero.

Note that the coefficient E_0 is never used and can be arbitrarily chosen. We take it equal to $E_0 = 1$. If we take the coefficient E_n to be positive, then the functions ϕ_n will be uniquely determined. This amounts to fixing the sign of κ_n.

If we take all poles outside the convex hull of $\mathrm{supp}(\mu)$, then the system $\{\phi_n\}$ will be regular and thus the recurrence relation will hold for every n. This follows from the fact that in this case the zeros of ϕ_n are inside the convex hull of $\mathrm{supp}(\mu)$. Therefore, if $\mathrm{supp}(\mu)$ is connected then $\{\phi_n\}$ will be regular (because of the assumptions we made on the location of the poles).

3 Main result

Suppose we are given a measure μ on the halfline $[0, \infty]$ and a set of poles $A = \{\alpha_1, \alpha_2, \ldots\} \subset (-\infty, 0)$. With this measure and poles we associate a set of orthonormal rational functions as explained in the previous section. Since $\mathrm{supp}(\mu)$ is connected, the recurrence relation holds for every n. We will denote the recurrence coefficients by $\{E_n, B_n\}_{n=1}^{\infty}$. According to the conventions from the previous section, we cannot have poles at infinity, because this is in the support of the measure.

We map the halfline to the interval $[-1,1]$ using the transformation

$$\tau(x) = \frac{1-x}{1+x}, \quad x \in [0,\infty]. \tag{3}$$

Note that $y = \tau(x)$ implies $x = \tau(y)$. Then associate to μ and A a measure $\tilde{\mu}$ on $[-1,1]$ and a set of poles $\tilde{A} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots\} \subset \hat{\mathbb{R}} \setminus [-1,1]$ in the following way. For every Borel measurable set $E \subset [-1,1]$ set

$$\tilde{\mu}(E) = \mu(\{\tau(y), y \in E\}) \tag{4}$$

and for the poles $\tilde{A}$ set

$$\tilde{\alpha}_n = \tau(\alpha_n), \quad n = 1, 2, \dots \tag{5}$$

From (4) we have $\tilde{\mu}' = |\tau'|(\mu' \circ \tau)$, where the prime means derivative (in case of a measure this is of course the Radon-Nikodym derivative with respect to the Lebesgue measure). More explicitly this yields

$$\tilde{\mu}'(y) = \frac{2}{(1+y)^2}\mu'\left(\frac{1-y}{1+y}\right) .$$

We will need several simple lemmas before we can prove the main theorem. Let $\tilde{\mathcal{L}}_n$ denote the space of rational functions of degree n with poles in $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}$ and $\tilde{b}_n(z) = \prod_{k=1}^n \tilde{Z}_k(z)$ the corresponding basis functions. Then we have the following lemma. The proof is a matter of straightforward computation and we omit it.

Lemma 3.1. *Put $y = \tau(x)$ then we have*

$$\begin{aligned} Z_n(x) &= -\frac{1}{2}\left(1 - \frac{1}{\tilde{\alpha}_n}\right)\left[\left(1 - \frac{1}{\tilde{\alpha}_n}\right)\tilde{Z}_n(y) - 1\right], \quad n \geq 1 \\ \frac{Z_n(x)}{Z_m(x)} &= \frac{1 - 1/\tilde{\alpha}_n}{1 - 1/\tilde{\alpha}_m}\frac{\tilde{Z}_n(y)}{\tilde{Z}_m(y)}, \quad n, m \geq 1 \end{aligned}$$

and for every pair of complex numbers (a,b) we have

$$a\tilde{Z}_n(y) + b = \left(a - \frac{b}{\tilde{Z}_{n-1}(\tilde{\alpha}_n)}\right)\tilde{Z}_n(y) + b\frac{\tilde{Z}_n(y)}{\tilde{Z}_{n-1}(y)}, \quad n \geq 1$$

Next we derive a relation between the rational functions in $\mathcal{L}_n$ orthonormal with respect to μ and those in $\tilde{\mathcal{L}}_n$ orthonormal with respect to $\tilde{\mu}$.

Lemma 3.2. *Let $\{\phi_n\}_{n=1}^{\infty}$ denote the set of orthonormal rational functions associated with (A, μ) and $\{\varphi_n\}_{n=1}^{\infty}$ those associated with $(\tilde{A}, \tilde{\mu})$. Then we have*

$$\phi_n \circ \tau = \pm\varphi_n, \quad n \geq 1$$

where the sign is determined by the normalization $E_n > 0$ for ϕ_n and $\tilde{E}_n > 0$ for φ_n.

Proof. It is clear that $\phi_n \circ \tau \in \tilde{\mathcal{L}}_n$. Because $\alpha_k \neq 0$ for all k we have $\tilde{\alpha}_k \neq 1$ and then it follows from lemma 3.1 that the basis functions b_k and $\tilde{b}_k$ satisfy the following relation,

$$b_k = \sum_{j=0}^{k} c_j^{(k)}(\tilde{b}_j \circ \tau), \quad c_k^{(k)} \neq 0$$

for some constants $\{c_j^{(k)}\}$. Next expand ϕ_n in the basis $\{b_0, \ldots b_n\}$,

$$\phi_n = \sum_{k=0}^{n} d_k^{(n)} b_k$$

where $d_n^{(n)} \neq 0$ because $\phi_n \in \mathcal{L}_n$ and $\phi_n \perp \mathcal{L}_{n-1}$. Then with the previous relation and the fact that $b_k \circ \tau \circ \tau = b_k$ it follows that

$$\begin{aligned} \phi_n \circ \tau &= \sum_{k=0}^{n} d_k^{(n)} (b_k \circ \tau) \\ &= \sum_{k=0}^{n} d_k^{(n)} \sum_{j=0}^{k} c_j^{(k)} \tilde{b}_j \\ &= \sum_{k=0}^{n} \tilde{d}_k^{(n)} \tilde{b}_k \end{aligned}$$

and $\tilde{d}_n^{(n)} = d_n^{(n)} c_n^{(n)} \neq 0$. This shows that $\phi_n \circ \tau \in \tilde{\mathcal{L}}_n \setminus \tilde{\mathcal{L}}_{n-1}$ and thus $\{\phi_0 \circ \tau, \ldots, \phi_n \circ \tau\}$ forms a basis for $\tilde{\mathcal{L}}_n$.

Because of orthogonality we have

$$\int \phi_n \phi_m d\mu = \delta_{nm}, \quad n, m \geq 0$$

where δ_{nm} is the Kronecker symbol. Using the definition of τ this becomes

$$\int (\phi_n \circ \tau)(\phi_m \circ \tau) d\tilde{\mu} = \delta_{nm}, \quad n, m \geq 0.$$

This means that $\phi_n \circ \tau \perp \tilde{\mathcal{L}}_{n-1}$ and $\|\phi_n \circ \tau\| = 1$. But we also have $\varphi_n \perp \tilde{\mathcal{L}}_{n-1}$ and $\|\varphi_n\| = 1$ so it follows that $\phi_n \circ \tau = \gamma_n \varphi_n$ with $|\gamma_n| = 1$. Using lemma 2.1 we then get

$$\phi_n \circ \tau = \pm \varphi_n$$

which proves the lemma. □

Using the last two lemmas we can express the recurrence coefficients $\{E_n, B_n\}$ for $\{\phi_n\}$ in terms of the recurrence coefficients $\{\tilde{E}_n, \tilde{B}_n\}$ for $\{\varphi_n\}$. We will need one more lemma before we can prove our main theorem.

Lemma 3.3. *If $n = 1$ then the recurrence coefficients for φ_1 satisfy*

$$\tilde{E}_1 > \tilde{B}_1.$$

Proof. From the recurrence relation (2), taking the inner product with $\phi_0 = 1$ on both sides, we obtain,

$$-\tilde{B}_1 = \tilde{E}_1 \frac{\int_{-1}^{1} \tilde{Z}_1(x) d\tilde{\mu}(x)}{\int_{-1}^{1} \frac{\tilde{Z}_1(x)}{x} d\tilde{\mu}(x)}$$

or writing $\tilde{Z}_1$ explicitly

$$\tilde{E}_1 - \tilde{B}_1 = \tilde{E}_1 \frac{\int_{-1}^{1} \frac{x+1}{1-x/\tilde{\alpha}_1} d\tilde{\mu}(x)}{\int_{-1}^{1} \frac{1}{1-x/\tilde{\alpha}_1} d\tilde{\mu}(x)}.$$

Both integrands are positive on $[-1, 1]$ and so is $\tilde{E}_1$. Therefore also $\tilde{E}_1 - \tilde{B}_1 > 0$, proving the lemma. □

We now state and prove our main theorem.

Theorem 3.4. *With the definitions of this section we obtain the following relations between* $\{E_n, B_n\}$ *and* $\{\tilde{E}_n, \tilde{B}_n\}$ *for* $n \geq 1$*,*

$$\begin{aligned} E_n &= [2\tilde{E}_n - \delta_{n1}(\tilde{E}_1 + \tilde{B}_1)]\left(1 - \frac{1}{\tilde{\alpha}_{n-1}}\right)^{-1}\left(1 - \frac{1}{\tilde{\alpha}_n}\right)^{-1}, \\ B_n &= -\left(1 - \frac{1}{\tilde{\alpha}_n}\right)^{-1}\left[\tilde{B}_n\left(1 - \frac{1}{\tilde{\alpha}_{n-1}}\right) + \tilde{E}_n - \delta_{n2}\frac{\tilde{E}_2}{\tilde{E}_1}\frac{1 - 1/\tilde{\alpha}_1}{\tilde{B}_1 - \tilde{E}_1}\right]. \end{aligned}$$

Note that the formulas simplify for $n > 2$ *because of the Kronecker symbols.*

Proof. First we will prove the theorem for the cases $n = 1$ and $n = 2$ and then for general $n > 2$. The first two cases are special because by convention we have $\alpha_{-1} = \alpha_0 = \infty$ but also $\tilde{\alpha}_{-1} = \tilde{\alpha}_0 = \infty$ which shows that $\tilde{\alpha}_k \neq \tau(\alpha_k)$ for $k = -1, 0$.

For $n = 1$ use lemma 3.2 to write $(\phi_1 \circ \tau)(x) = c\varphi_1(x)$ where $c = \pm 1$. Then write down the recurrence relations for ϕ_1 and φ_1, use the definition of τ and equate the coefficients of like powers of x to obtain (recall that $\phi_0 = \varphi_0 = 1$)

$$\begin{aligned} \left(1 - \frac{1}{\tilde{\alpha}_1}\right) E_1 &= c(\tilde{B}_1 - \tilde{E}_1), \\ \left(1 - \frac{1}{\tilde{\alpha}_1}\right) B_1 &= c(\tilde{B}_1 + \tilde{E}_1), \end{aligned}$$

Now it follows from $|\tilde{\alpha}_1| > 1$, $E_1 > 0$ and lemma 3.3 that $c = -1$, proving the theorem for $n = 1$.

For $n = 2$ we proceed in the same way. Write down the recurrence relation for $\phi_2 \circ \tau$, using the fact that we already know that $\phi_1 \circ \tau = -\varphi_1$ and of course $\phi_0 \circ \tau = \varphi_0$. Then use the relation

$$\tilde{E}_2 = \left.\frac{\varphi_2(x)}{\varphi_1(x)\tilde{Z}_2(x)}\right|_{x=\tilde{\alpha}_1}$$

and $\tilde{E}_2 > 0$ to find

$$\tilde{E}_2 = \frac{E_2}{2}\left(1 - \frac{1}{\tilde{\alpha}_2}\right)\left(1 - \frac{1}{\tilde{\alpha}_1}\right).$$

To find the relation for B_2 use $\varphi_2(0) = \tilde{B}_2\varphi_1(0) - \tilde{E}_2/\tilde{E}_1$ and $\varphi_1(0) = \tilde{B}_1$, again comparing the recurrence relations for $\phi_2 \circ \tau$ and φ_2 and using all the previous results.

The general case $n > 2$ is the easiest to prove. Write down the recurrence relation for φ_n and for $\phi_n \circ \tau$ using $\phi_k \circ \tau = c_k \varphi_k$ for $k = n, n-1, n-2$ and $c_k = \pm 1$ and the formulas from lemma 3.1. Comparing the factors in front of φ_{n-1} and φ_{n-2} and using $\tilde{E}_n > 0$ immediately yields the result, thus proving the theorem. □

4 Some remarks

If we take all poles $\alpha_k = -1$ then transforming to the interval we obtain the orthogonal polynomials on $[-1, 1]$ with respect to $\tilde{\mu}$. For the case of Legendre polynomials we would have to take μ absolutely continuous with weight $\mu'(x) = (1+x)^{-2}$. See [1] for an application of this special case.

Regarding computational aspects of orthogonal rational functions on a subset of the real line we refer to [4] and [3]. It is worth mentioning that the accurate computation of the recurrence coefficients in the case of poles very close to the boundary of the interval is still problematic (especially for high degrees).

References

[1] J. Shen B. Guo and Z. Wang. A rational approximation and its applications to differential equations on the half line. *Journal of Scientific Computing*, Vol.15, No.2:117–147, 2000.

[2] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal Rational Functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[3] J. Van Deun and A. Bultheel. The computation of orthogonal rational functions on an interval. 2002. to appear.

[4] J. Van Deun and A. Bultheel. An interpolation algorithm for orthogonal rational functions. *J. Comput. Appl. Math.*, 2002. submitted.