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Abstract

We consider the quadrature method developed by Kravanja, Sakurai and Van Barel (BIT 39 (1999), no. 4, 646–682) for computing all the zeros of an analytic function that lie inside the unit circle. A new perturbation result for generalized eigenvalue problems allows us to obtain a detailed upper bound for the error between the zeros and their approximations. To the best of our knowledge, it is the first time that such a backward error estimate is presented for any quadrature method for computing zeros of analytic functions.

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An error analysis of two related quadrature methods for computing zeros of analytic functions, Part II

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Abstract. We consider the quadrature method developed by Kravanja, Sakurai and Van Barel (BIT 39 (1999), no. 4, 646–682) for computing all the zeros of an analytic function that lie inside the unit circle. A new perturbation result for generalized eigenvalue problems allows us to obtain a detailed upper bound for the error between the zeros and their approximations. To the best of our knowledge, it is the first time that such a backward error estimate is presented for any quadrature method for computing zeros of analytic functions.

Keywords: zeros of analytic functions, error analysis, quadrature methods

1 Introduction

Let the complex function f be analytic in a simply connected region W of the complex plane that includes the closed unit disk. Assume that f has no zeros on the unit circle $\mathbb{T}$. We consider the problem of computing *all* the zeros of f that lie inside $\mathbb{T}$, together with their respective multiplicities.

Methods for the determination of zeros of analytic functions that are based on the numerical evaluation of integrals are called *quadrature methods* [6]. Our approach to this problem can be seen as a continuation of the pioneering work by Delves and Lyness [2]. In recent years we have made a number of contributions, see in particular the papers [8, 9, 10, 12, 17, 16, 18, 19], the book [11] and the software package ZEAL [13]. In [15], our most recent paper, we presented an error analysis of the approaches in [2] (Delves & Lyness) and [9] (Kravanja, Sakurai & Van Barel). One of our conclusions was that in the latter approach, the quadrature error arising from the zeros located inside the unit circle does not affect the results of the algorithm. This is not true for the Delves-Lyness method.

This paper is a follow-up paper to [15], which was allowed only a limited number of pages. Whereas [15] contains only forward error estimates, i.e. theorems stating that the function evaluated at the approximate zeros is of a given small order of magnitude, this paper presents a backward error estimate: we obtain a detailed upper bound for the error between the zeros and their approximations as computed by the algorithm of Kravanja, Sakurai & Van Barel. To the best of our knowledge, it is the first time that such a backward error estimate is presented for any quadrature method for computing zeros of analytic functions.

Let us start by recalling our notations and the results already obtained in [15] that we will need later on.

1.1 Notations

Let N denote the total number of zeros of f that lie inside $\mathbb{T}$, i.e., the number of zeros where each zero is counted according to its multiplicity. Suppose that $N > 0$. The value of N can be calculated via numerical integration or by applying the principle of the argument [4, 20]. We may therefore assume that N is known.

Let n denote the number of mutually distinct zeros of f that lie inside $\mathbb{T}$. Let $z_1, \ldots, z_n$ be these zeros and $\nu_1, \ldots, \nu_n$ their respective multiplicities.

Define the *associated polynomial* P_N of degree N as

$$P_N(z) := \prod_{k=1}^{n} (z - z_k)^{\nu_k}.$$

Let the complex function $g : W \to \mathbb{C}$ be defined by $f = P_N g$. Then g is analytic in W and g has no zeros inside and on $\mathbb{T}$. The following holds:

$$\frac{f'(z)}{f(z)} = \frac{P_N'(z)}{P_N(z)} + \frac{g'(z)}{g(z)} = \sum_{k=1}^{n} \frac{\nu_k}{z - z_k} + \frac{g'(z)}{g(z)}.$$

The second term in the right-hand side, g'/g, is analytic inside and on $\mathbb{T}$. It follows that f'/f is meromorphic inside and on $\mathbb{T}$, with simple poles at the z_k and corresponding residues equal to ν_k.

Define the *moments* μ_p as

$$\mu_p := \frac{1}{2\pi i} \int_{\mathbb{T}} z^p \frac{f'(z)}{f(z)}\, dz, \qquad p = 0, 1, 2, \ldots.$$

The residue theorem implies that the μ_p's are equal to the *Newton sums* of the unknown zeros,

$$\mu_p = \sum_{k=1}^{n} \nu_k z_k^p, \qquad p = 0, 1, 2, \ldots. \tag{1}$$

The mutually distinct zeros are given by the eigenvalues of a generalized eigenvalue problem involving the following *Hankel matrices*:

$$H_n := \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ \mu_{n-1} & \cdots & \cdots & \mu_{2n-2} \end{bmatrix} \quad \text{and} \quad H_n^< := \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ \mu_n & \cdots & \cdots & \mu_{2n-1} \end{bmatrix}.$$

Theorem 1 *The eigenvalues of the pencil $H_n^< - \lambda H_n$ are given by $z_1, \ldots, z_n$.*

Note that the n mutually distinct zeros $z_1, \ldots, z_n$ are determined by the $2n$ moments $\mu_0, \mu_1, \ldots, \mu_{2n-1}$.

As explained in [9], the value of n is determined indirectly. Once n and $z_1, \ldots, z_n$ have been found, the problem becomes linear and the multiplicities $\nu_1, \ldots, \nu_n$ can be computed by solving a Vandermonde system, cf. Equation (1). As the multiplicities are known to be integers, this system does not need to be solved very accurately.

1.2 Quadrature error of the moments

The integral that defines μ_p is an integral along a closed curve and hence, once the curve is parametrized and the integral is written as a Riemann integral, it is the integral of a periodic function along one period. The trapezoidal rule is therefore an appropriate quadrature rule [1, 3, 5, 14].

We will write $\mu_p(f)$ instead of simply μ_p whenever we want to emphasize the dependence on f. With obvious definitions of $\mu_p(P_N)$ and $\mu_p(g)$, the following holds:

$$\mu_p(f) = \mu_p(P_N) + \mu_p(g), \qquad \mu_p(g) = 0.$$

In other words, only the contribution of P_N counts. Indeed, f and P_N have exactly the same zeros and corresponding multiplicities inside (and on) $\mathbb{T}$.

Let K be a positive integer. Then the Kth roots of unity are given by

$$\omega_j := e^{\frac{2\pi i}{K} j}, \qquad j = 0, 1, \ldots, K-1.$$

By approximating the integral that defines μ_p via the K-point trapezoidal rule (after having rewritten this integral as a Riemann integral over the interval $[0, 1]$), one obtains the following approximation for μ_p:

$$\hat{\mu}_p = \hat{\mu}_p(f) := \frac{1}{K} \sum_{j=0}^{K-1} \omega_j^{p+1} \frac{f'(\omega_j)}{f(\omega_j)}.$$

Note that $\hat{\mu}_p = \hat{\mu}_{p+K}$ for all p and hence only $\hat{\mu}_0, \ldots, \hat{\mu}_{K-1}$ are relevant.

Let us consider P_N'/P_N and g'/g in more detail. One can easily verify that the Laurent series at infinity of P_N'/P_N is given by

$$\frac{P_N'(z)}{P_N(z)} = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots.$$

The series converges for $|z| > \rho_I$ where

$$\rho_I := \max_{1 \le k \le n} |z_k| \ < \ 1.$$

(The subscript I stands for *interior.*) In other words, ρ_I is equal to the modulus of the zero(s) of f that lie(s) inside $\mathbb{T}$ and that is (are) closest to $\mathbb{T}$. As g'/g is analytic inside and on $\mathbb{T}$, it has a Taylor series expansion at the origin,

$$\frac{g'(z)}{g(z)} =: \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \cdots.$$

The series converges for $|z| < \rho_E$ where $\rho_E > 1$ is defined as the modulus of the zero(s) or the singularity of f that lie(s) outside $\mathbb{T}$ and that is (are) closest to $\mathbb{T}$. (The subscript E stands for *exterior*.) By combining these two series expansions, we obtain the following important equation:

$$\frac{f'(z)}{f(z)} = \cdots + \frac{\mu_2}{z^3} + \frac{\mu_1}{z^2} + \frac{\mu_0}{z} + \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \cdots$$

for $\rho_I < |z| < \rho_E$. The series converges in a ring around the unit circle. In particular, it converges on $\mathbb{T}$ itself, for example for z equal to one of the Kth roots of unity.

With obvious definitions of $\hat{\mu}_p(P_N)$ and $\hat{\mu}_p(g)$, the following holds:

$$\hat{\mu}_p(f) = \hat{\mu}_p(P_N) + \hat{\mu}_p(g).$$

(Note that $\hat{\mu}_p(g)$ is an approximation of zero.)

In [15] we have proved the following result.

Theorem 2 *For $p \in \{0, 1, \ldots, K-1\}$ the following holds:*

$$\hat{\mu}_p(P_N) = \mu_p + \sum_{r=1}^{+\infty} \mu_{p+rK} \qquad \text{and} \qquad \hat{\mu}_p(g) = \sum_{r=1}^{+\infty} \gamma_{rK-p-1}.$$

Since g'/g is analytic in the closed disk $\{z \in \mathbb{C} : |z| \le \rho\}$, $1 < \rho < \rho_E$, it follows that

$$|\gamma_j| \le \frac{M}{\rho^j}, \quad j = 0, 1, 2, \ldots,$$

where

$$M := \max_{|z|=\rho} \left| \frac{g'(z)}{g(z)} \right|.$$

It follows that

$$|\hat{\mu}_p(g)| \le \sum_{r=1}^{+\infty} \frac{M}{\rho^{rK-p-1}} = M \frac{(\frac{1}{\rho})^{K-p-1}}{1 - (\frac{1}{\rho})^K}. \tag{2}$$

Therefore

$$\hat{\mu}_p(g) = \hat{\mu}_p(g) - \mu_p(g) = \mathcal{O}(\rho^{p+1-K}).$$

Corollary 3 *For every $\rho \in \mathbb{R}$ such that $1 < \rho < \rho_E$, the following holds:*

$$\hat{\mu}_p(f) - \mu_p = \mathcal{O}(\rho_I^{p+K}) + \mathcal{O}(\rho^{p+1-K}).$$

Both P_N (via ρ_I) and g (via $\rho < \rho_E$) contribute to the approximation error. Note that, in a certain sense, these contributions work in opposite ways. More specifically, as far as the contribution of P_N is concerned, for fixed K, the *larger* p, the more accurate, while, as far as the contribution of g is concerned, again for fixed K, the *smaller* p, the more accurate. To obtain H_n and $H_n^<$ the moments $\mu_0, \mu_1, \ldots, \mu_{2n-1}$ are needed. Hence, the order of magnitude of μ_K and γ_{K-2n} determine the overall error.

2 Sensitivity of generalized eigenvalue problems

We will now briefly interrupt our discussion about zeros of analytic functions to derive a perturbation result for generalized eigenvalue problems. Although similar results exist, we have not come across this particular one in the literature. Readers who are familiar with our techniques for computing zeros of analytic functions will know why generalized eigenvalue problems are relevant to our approach. In any case, we will provide a summary in the next section. Right now, let us consider the pencil $A - \lambda B$ where A and B are square complex matrices. The matrix B is assumed to be nonsingular and the eigenvalue, which we also denote by λ, is assumed to be simple.

Let the vectors x and v be the corresponding right and left eigenvectors,

$$Ax = \lambda Bx \qquad \text{and} \qquad v^T A = \lambda v^T B.$$

We consider the following perturbed problem:

$$(A + \epsilon F)x(\epsilon) = \lambda(\epsilon)(B + \epsilon G)x(\epsilon)$$

where $\|F\| = \|G\| = 1$. As B is nonsingular and λ is a simple eigenvalue, standard results from function theory (see, e.g., Kato [7, p. 63]) imply that $x(\epsilon)$ and $\lambda(\epsilon)$ are differentiable in a neighbourhood of $\epsilon = 0$.

By differentiating with respect to ϵ we obtain:

$$\begin{gathered} A\dot{x}(\epsilon) + Fx(\epsilon) + \epsilon F\dot{x}(\epsilon) = \\ \lambda(\epsilon)B\dot{x}(\epsilon) + \dot{\lambda}(\epsilon)Bx(\epsilon) + \lambda(\epsilon)\epsilon G\dot{x}(\epsilon) + \lambda(\epsilon)Gx(\epsilon) + \dot{\lambda}(\epsilon)\epsilon Gx(\epsilon). \end{gathered}$$

By setting ϵ equal to zero, it follows that

$$A\dot{x}(0) + Fx(0) = \lambda(0)B\dot{x}(0) + \dot{\lambda}(0)Bx(0) + \lambda(0)Gx(0).$$

Since $x(0) = x$ and $\lambda(0) = \lambda$ we have that

$$(A - \lambda B)\dot{x}(0) + (F - \lambda G)x = \dot{\lambda}(0)Bx.$$

Multiplying with v^T leads to

$$v^T(A - \lambda B)\dot{x}(0) + v^T(F - \lambda G)x = \dot{\lambda}(0)v^T Bx.$$

The definition of v implies that the factor in front of $\dot{x}(0)$ is equal to zero. It follows that

$$\dot{\lambda}(0) = \frac{v^T(F - \lambda G)x}{v^T Bx}.$$

Since

$$\lambda(\epsilon) = \lambda(0) + \epsilon\dot{\lambda}(0) + \mathcal{O}(\epsilon^2),$$

we may conclude that

$$\lambda(\epsilon) - \lambda = \frac{v^T(\epsilon F - \lambda\,\epsilon G)x}{v^T Bx} + \mathcal{O}(\epsilon^2).$$

3 The Hankel case

Let us now use this perturbation result to derive a backward error estimate.

Define

$$\hat{H}_n(P_N) := \left[\ \hat{\mu}_{k+l}(P_N)\ \right]_{k,l=0}^{n-1} \quad \text{and} \quad \hat{H}_n^{<}(P_N) := \left[\ \hat{\mu}_{1+k+l}(P_N)\ \right]_{k,l=0}^{n-1}. \qquad (3)$$

In [15] we have proved the following result.

Theorem 4 *The eigenvalues of the pencil $\hat{H}_n^{<}(P_N) - \lambda \hat{H}_n(P_N)$ are given by $z_1, \ldots, z_n$. The corresponding multiplicities $\nu_1, \ldots, \nu_n$ are the solution of the linear system of equations*

$$\sum_{k=1}^{n} \left(\frac{z_k^p}{1 - z_k^K} \right) \nu_k = \hat{\mu}_p(P_N), \qquad p = 0, \ldots, n-1.$$

Define $\varphi_n(z)$ as the monic polynomial of degree n

$$\varphi_n(z) := \prod_{k=1}^{n} (z - z_k)$$

and let

$$q_l(z) := \frac{\varphi_n(z)}{\varphi_n'(z_l)(z - z_l)} =: \sum_{k=0}^{n-1} q_{k,l}\, z^k$$

for $l = 1, \ldots, n$. Note that $q_l(z)$ is a polynomial of degree $n-1$ and that $q_l(z_j) = \delta_{l,j}$ for $l, j = 1, \ldots, n$. The polynomials $q_1(z), \ldots, q_n(z)$ are linearly independent. Define the stacking vector $\vec{q_l}$ as

$$\vec{q_l} := \begin{bmatrix} q_{0,l} \\ q_{1,l} \\ \vdots \\ q_{n-1,l} \end{bmatrix}$$

for $l = 1, \ldots, n$.

Theorem 5 *The following holds:*

$$\hat{H}_n^{<}(P_N)\vec{q_l} = z_l \hat{H}_n(P_N)\vec{q_l} \qquad \textit{for } l = 1, \ldots, n.$$

In other words, $\vec{q_l}$ is the right eigenvector corresponding to the eigenvalue z_l.

Proof. Let $l \in \{1, \ldots, n\}$ and $k \in \{0, 1, \ldots, n-1\}$. The $(k+1)$st element of the matrix-vector product $\hat{H}_n(P_N)\vec{q_l}$ is given by

$$\sum_{j=0}^{n-1} \hat{\mu}_{k+j}(P_N) q_{j,l}.$$

Theorem 4 implies that this sum is equal to

$$\sum_{j=0}^{n-1} \sum_{r=1}^{n} \frac{\nu_r}{1 - z_r^K} z_r^{k+j} q_{j,l} = \sum_{r=1}^{n} \frac{\nu_r}{1 - z_r^K} z_r^k \underbrace{\sum_{j=0}^{n-1} q_{j,l}\, z_r^j}_{= q_l(z_r) = \delta_{l,r}} = \frac{\nu_l z_l^k}{1 - z_l^K}.$$

It follows that

$$\hat{H}_n(P_N)\vec{q_l} = \frac{\nu_l}{1-z_l^K}\begin{bmatrix} 1 \\ z_l \\ \vdots \\ z_l^{n-1} \end{bmatrix} =: \frac{\nu_l}{1-z_l^K}\vec{z_l}. \tag{4}$$

In an analogous way one can obtain that

$$\hat{H}_n^<(P_N)\vec{q_l} = \frac{\nu_l z_l}{1-z_l^K}\vec{z_l}.$$

The theorem immediately follows from these two equations. □

Note that Equation (4) implies that the matrix $\hat{H}_n(P_N)$ is nonsingular. Indeed, the following factorization is easily obtained:

$$\hat{H}_n^<(P_N) = \text{diag}\left(\frac{\nu_1}{1-z_1^K},\ldots,\frac{\nu_n}{1-z_n^K}\right)[\ \vec{z_1}\ \ \cdots\ \ \vec{z_n}\ ][\ \vec{q_1}\ \ \cdots\ \ \vec{q_n}\ ]^{-1}.$$

The matrix $[\ \vec{q_1}\ \ \cdots\ \ \vec{q_n}\ ]$ is nonsingular as the polynomials $q_1(z),\ldots,q_n(z)$ are linearly independent. The diagonal matrix is nonsingular as the multiplicities are different from zero and the zeros $z_1,\ldots,z_n$ do not lie on the unit circle. Finally, the Vandermonde matrix $[\ \vec{z_1}\ \ \cdots\ \ \vec{z_n}\ ]$ is nonsingular as the zeros are mutually distinct. It follows that $\hat{H}_n(P_N)$ is indeed nonsingular.

Let us now move on to the generalized eigenvalue problem involving f instead of (only) P_N. The matrices $\hat{H}_n(f)$, $\hat{H}_n^<(f)$, $\hat{H}_n(g)$ and $\hat{H}_n^<(g)$ are defined in a similar way as in (3). Let $\hat{z}_1,\ldots,\hat{z}_n$ denote the eigenvalues of the pencil $\hat{H}_n^<(f)-\lambda\hat{H}_n(f)$. Since

$$\hat{H}_n^<(f) = \hat{H}_n^<(P_N)+\hat{H}_n^<(g) \quad \text{and} \quad \hat{H}_n(f) = \hat{H}_n(P_N)+\hat{H}_n(g)$$

we can apply the perturbation result of Section 2. It follows that

$$\hat{z}_l - z_l = \frac{\vec{q_l}^T(\,\hat{H}_n^<(g)-z_l\hat{H}_n(g)\,)\vec{q_l}}{\vec{q_l}^T\hat{H}_n(P_N)\vec{q_l}} + \mathcal{O}(\epsilon^2)$$

for $l=1,\ldots,n$, where ϵ is given by

$$\epsilon = \max_{0\le k\le 2n-1}|\hat{\mu}_k(g)|.$$

Let us analyse this expression in more detail. We start by considering the denominator. Equation (4) immediately implies that

$$\vec{q_l}^T\hat{H}_n(P_N)\vec{q_l} = \frac{\nu_l}{1-z_l^K}\vec{q_l}^T\vec{z_l} = \frac{\nu_l}{1-z_l^K}q_l(z_l) = \frac{\nu_l}{1-z_l^K}.$$

Next we turn our attention to the numerator. The following holds:

$$\begin{aligned}\vec{q_l}^T\hat{H}_n(g)\vec{q_l} &= \sum_{i,j=0}^{n-1} q_{i,l}\,\hat{\mu}_{i+j}(g)\,q_{j,l} \\ &= \sum_{k=0}^{2n-2}\hat{\mu}_k(g)\sum_{\substack{i,j=0 \\ i+j=k}}^{n-1} q_{i,l}q_{j,l} \\ &= \sum_{k=0}^{2n-2} q''_{k,l}\,\hat{\mu}_k(g)\end{aligned}$$

where the coefficients $q''_{k,l}$ are defined via

$$[q_l(z)]^2 =: \sum_{k=0}^{2n-2} q''_{k,l}\, z^k.$$

Equation (2) implies that

$$\begin{aligned} |\vec{q_l}^T \hat{H}_n(g)\vec{q_l}| &\leq \frac{M}{1-\rho^{-K}} \sum_{k=0}^{2n-2} |q''_{k,l}|\rho^{-K+k+1} \\ &= \frac{M\rho}{\rho^K - 1} \sum_{k=0}^{2n-2} |q''_{k,l}|\rho^k. \end{aligned}$$

In a similar way one can obtain that

$$\vec{q_l}^T \hat{H}_n^{<}(g)\vec{q_l} = \sum_{k=0}^{2n-2} q''_{k,l}\, \hat{\mu}_{k+1}(g)$$

and that

$$|\vec{q_l}^T \hat{H}_n^{<}(g)\vec{q_l}| \leq \frac{M\rho^2}{\rho^K - 1} \sum_{k=0}^{2n-2} |q''_{k,l}|\rho^k.$$

By combining these results we finally obtain that

$$|\hat{z}_l - z_l| \leq \frac{M}{\nu_l}|1 - z_l^K| \frac{\rho(\rho + |z_l|)}{\rho^K - 1} \sum_{k=0}^{2n-2} |q''_{k,l}|\rho^k + \mathcal{O}(\epsilon^2)$$

for $l = 1, \ldots, n$, where $\epsilon = \rho^{2n-K}$. Note that the first term in the right-hand side is (indeed) $\mathcal{O}(\rho^{2-K+2n-2} = \rho^{2n-K})$.

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References

[1] P. J. Davis and P. Rabinowitz, *Methods of numerical integration*, Academic Press, New York, 1984.

[2] L. M. Delves and J. N. Lyness, *A numerical method for locating the zeros of an analytic function*, Math. Comput. **21** (1967), 543–560.

[3] R. J. Fornaro, *Numerical evaluation of integrals around simple closed curves*, SIAM J. Numer. Anal. **10** (1973), no. 4, 623–634.

[4] P. Henrici, *Applied and computational complex analysis: I. Power series—integration—conformal mapping—location of zeros*, Wiley, 1974.

[5] D. B. Hunter, *The evaluation of integrals of periodic analytic functions*, BIT **11** (1971), 175–180.

[6] N. I. Ioakimidis, *Quadrature methods for the determination of zeros of transcendental functions—a review*, Numerical Integration: Recent Developments, Software and Applications (P. Keast and G. Fairweather, eds.), Reidel, Dordrecht, The Netherlands, 1987, pp. 61–82.

[7] T. Kato, *Perturbation theory for linear operators*, Springer, 1976.

[8] P. Kravanja, M. Van Barel, and A. Haegemans, *Logarithmic residue based methods for computing zeros of analytic functions and related problems*, HERCMA '98: Proceedings of the Fourth Hellenic-European Conference on Computer Mathematics and its Applications (E. A. Lipitakis, ed.), Athens (Greece), September 24–26, 1998, LEA, 1999, pp. 201–208.

[9] P. Kravanja, T. Sakurai, and M. Van Barel, *On locating clusters of zeros of analytic functions*, BIT **39** (1999), no. 4, 646–682.

[10] P. Kravanja and M. Van Barel, *A derivative-free algorithm for computing zeros of analytic functions*, Computing **63** (1999), no. 1, 69–91.

[11] ———, *Computing the zeros of analytic functions*, Lecture Notes in Mathematics, vol. 1727, Springer, 2000.

[12] P. Kravanja, M. Van Barel, and A. Haegemans, *On computing zeros and poles of meromorphic functions*, Computational Methods and Function Theory 1997 (N. Papamichael, St. Ruscheweyh, and E. B. Saff, eds.), Series in Approximations and Decompositions, vol. 11, World Scientific, 1999, Proceedings of the third CMFT conference, Nicosia (Cyprus), October 13–17, 1997., pp. 359–369.

[13] P. Kravanja, M. Van Barel, O. Ragos, M. N. Vrahatis, and F. A. Zafiropoulos, *ZEAL: A mathematical software package for computing zeros of analytic functions*, Comput. Phys. Commun. **124** (2000), no. 2–3, 212–232.

[14] J. N. Lyness and L. M. Delves, *On numerical contour integration round a closed contour*, Math. Comput. **21** (1967), 561–577.

[15] T. Sakurai, P. Kravanja, H. Sugiura, and M. Van Barel, *An error analysis of two related quadrature methods for computing zeros of analytic functions*, To appear in J. Comput. Appl. Math.

[16] T. Sakurai and H. Sugiura, *On factorization of analytic functions and its verification*, Reliable Computing **6** (2000), no. 4, 459–470.

[17] ———, *Improvement of convergence of an iterative method for finding polynomial factors of analytic functions*, J. Comput. Appl. Math. **140** (2002), no. 1–2, 713–725.

[18] T. Sakurai, T. Torii, N. Ohsako, and H. Sugiura, *A method for finding clusters of zeros of analytic function*, Special Issues of Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM). Issue 1: Numerical Analysis, Scientific Computing, Computer Science, 1996, Proceedings of the International Congress on Industrial and Applied Mathematics (ICIAM/GAMM 95), Hamburg, July 3–7, 1995, pp. 515–516.

[19] T. Torii and T. Sakurai, *Global method for the poles of analytic function by rational interpolant on the unit circle*, World Sci. Ser. Appl. Anal. **2** (1993), 389–398.

[20] X. Ying and I. N. Katz, *A reliable argument principle algorithm to find the number of zeros of an analytic function in a bounded domain*, Numer. Math. **53** (1988), 143–163.