A shattered survey of the Fractional Fourier Transform

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Report TW 337, April 2002

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In this survey paper we introduce the reader to the notion of the fractional Fourier transform, which may be considered as a fractional power of the classical Fourier transform. It has been intensely studied during the last decade, an attention it may have partially gained because of the vivid interest in time-frequency analysis methods of signal processing, like wavelets. Like the complex exponentials are the basic functions in Fourier analysis, the chirps (signals sweeping through all frequencies in a certain interval) are the building blocks in the fractional Fourier analysis. Part of its roots can be found in optics where the fractional Fourier transform can be physically realized. We give an introduction to the definition, the properties and computational aspects of both the continuous and discrete fractional Fourier transforms. We include some examples of applications and some possible generalizations.

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AMS(MOS) Classification : 42A38, 65T20

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Contents

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1 Introduction

The idea of fractional powers of the Fourier operator appears in the mathematical literature as early as 1929 [35, 11, 14]. It has been rediscovered in quantum mechanics [22, 18], optics [20, 24, 2] and signal processing [3]. The boom in publications started in the early years of the 1990's and it is still going on. A recent state of the art can be found in [25].

In fact [25] is a comprehensive survey of what was published up to 2001. It appeared while we were preparing this survey, and therefore most of what we include in this paper will also be found in a more complete and more exhaustive form in that book. However, we restrict ourselves to the minimum, discuss less of the optics, and pay relatively more attention to the discrete forms and the computational aspects. It is not our intention to be exhaustive, not in the contents of the paper and certainly not in the list of references. We see this paper as an appetizer for those who want to learn about a new and exciting subject that is still largely unexplored and that has many potential applications that are yet to be discovered.

The outline of the paper is as follows. Section 2 gives a motivating analysis of the classical Fourier transform which prepares the reader for several possible definitions of the fractional Fourier transform (FrFT) given in the next section. Some elementary properties are introduced in Section 4. The Wigner distribution is a function that essentially gives the distribution of the energy of the signal in a time-frequency plane. The effect of a FrFT can be effectively visualized with the help of this function. This is describer in Section 5 where we also include relations with the windowed or short time Fourier transform, with wavelets and chirplets, and with a general linear canonical transform characterized by a unimodular 2×2 matrix. The FrFT is a very special case where the matrix is a rotation matrix. In Section 6 we define the two-dimensional FrFT. The discrete form of the FrFT for finite (periodic) signals is discussed in Section 8. We do this with some detail because it is the basis of how to construct other fractional transforms that are only briefly described in Section 11. A selective set of applications is briefly mentioned in Section 10.

2 The classical Fourier Transform

We recall some of the definitions and properties that are related to the classical continuous Fourier transform (FT) so that we can motivate our definition of the fractional Fourier transform (FrFT) later.

2.1 The definition

Definition 2.1 *Let $\mathcal{L}$ be the Fréchet space of all smooth functions f (infinitely many times differentiable) such that*

$$\gamma_{m,n}(f) = \sup_{t \in \mathbb{R}} |t^m f^{(n)}(t)| < \infty \quad \forall m, n = 0, 1, 2, \ldots$$

Then the FT operator $\mathcal{F}$ is defined for $f \in \mathcal{L}$ as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \mathrm{d}t. \tag{1}$$

The inverse transform is also defined and is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \mathrm{d}\omega. \tag{2}$$

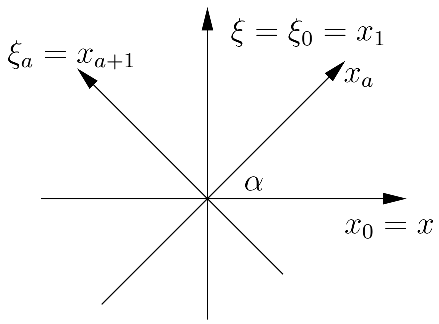
Since we do not want to keep the different notation for time (t) and frequency (ω), we shall use a "neutral" variable such as x or ξ to avoid confusion.

It will also be useful to introduce a notation for a variable along a rotated axis system. Let $x = x_0$ be the variable along the x-axis pointing to the East and $\xi = x_1$ is the variable along the ξ-axis pointing to the North, like for example the time and frequency variables in the time-frequency plane. If this coordinate system is rotated over an angle $\alpha = a\pi/2$, $a \in \mathbb{R}$ counter clockwise, then we denote the rotated variables as x_a and $\xi_a = x_{a+1}$ respectively. Thus

$$\begin{bmatrix} x_a \\ \xi_a \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}$$

which we also denote as $(x_a, \xi_a) = \mathcal{R}_a(x_0, \xi_0)$ or more generally for $\beta = b\pi/2$, $b \in \mathbb{R}$ we have $(x_b, \xi_b) = \mathcal{R}_{b-a}(x_a, \xi_a)$. It will always be assumed that $\xi_a = x_{a+1}$.

Figure 1: Notational convention for the variables and rotated versions. The angle is $\alpha = a\pi/2$.



Thus x_a and $\xi_a = x_{a+1}$ are always orthogonal and $x_{a+2} = \xi_{a+1} = -x_a$ and $x_{a-1} = -\xi_a$.

The notation $\mathcal{R}_a$ will also be used as an operator working on a function of two variables to mean $\mathcal{R}_a f(x, \xi) = f(\mathcal{R}_a(x, \xi)) = f(x_a, \xi_a)$.

It is well known [37, p.146-148] that the FT defines a homeomorphism.

Theorem 2.1 *$\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is a homeomorphism with an inverse:*

$$(\mathcal{F}^{-1} f)(x) = (\mathcal{F} f)(-x), \quad f \in \mathcal{L}, \quad x \in \mathbb{R}.$$

2.2 Interpretation in the time-frequency plane

Let $f(x)$ be a (time) signal, so it lives on the horizontal (time) axis. Its FT $(\mathcal{F} f)(\xi)$ is a function of frequency ξ and hence it lives on the vertical (frequency) axis.

Thus by the FT, the representation axis is changed from time to frequency, which corresponds to a counterclockwise rotation over an angle $\pi/2$.

Because applying (1) and then (2) leads to

$$(\mathcal{F}^2 f)(x) = (\mathcal{F}(\mathcal{F} f))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x} \mathrm{d}\xi = f(-x),$$

it is seen that

$$(\mathcal{F}^2 f)(x) = f(-x). \tag{3}$$

Therefore $\mathcal{F}^2$ is called the *parity operator*. Thus the representation axis is the reversed time axis, i.e., the time axis rotated over an angle π.

Similarly it follows that by using subsequently (3) and (1) we get

$$(\mathcal{F}^3 f)(\xi) = (\mathcal{F}(\mathcal{F}^2 f))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) e^{-i\xi x} \mathrm{d}x = F(-\xi).$$

Thus

$$(\mathcal{F}^3 f)(\xi) = (\mathcal{F} f)(-\xi) = F(-\xi), \tag{4}$$

which corresponds to a rotation of the representation axis over $3\pi/2$. It will now be clear that by another application of $\mathcal{F}$, we should get another rotation over $\pi/2$, which brings us back to the original time axis. Hence

$$\mathcal{F}^4(f) = f \quad \text{or} \quad \mathcal{F}^4 = \mathcal{I}. \tag{5}$$

Thus the FT operator corresponds to a rotation in the time-frequency plane of the axis of representation over an angle $\pi/2$. Thus all the representations that one can obtain by the classical FT correspond to representations on the (orthogonal) axes of time and frequency, possibly with a reversion of the orientation.

3 The fractional Fourier transform

In [25] the authors give 6 different possible definitions of the FrFT. It is probably a matter of taste, but the more intuitive way of defining the FrFT is by generalizing this concept of rotating over an angle that is $\pi/2$ in the classical FT situation. Like the classical FT corresponds to a rotation in the time frequency plane over an angle $\alpha = 1\pi/2$, the FrFT will correspond to a rotation over an arbitrary angle $\alpha = a\pi/2$ with $a \in \mathbb{R}$. This FrFT operator shall be denoted as $\mathcal{F}^a$ where $\mathcal{F}^1 = \mathcal{F}$ corresponds to the classical FT operator.

3.1 Derivation of the formulas

To arrive at a more formal definition, we can first define the FrFT on a basis for the space $\mathcal{L}$. For this basis we use a complete set of eigenvectors for the classical FT. Since $\mathcal{F}^4 = \mathcal{I}$, the eigenvalues are all in the set $\{1, -i, -1, i\}$ and thus there are only four different eigenvalues and thus only four different eigenspaces, all infinite dimensional. Consequently, the eigenvectors are not unique. The eigenvectors belonging to different eigenspaces will be automatically orthogonal because $\mathcal{F}$ is self adjoint, but within each eigenspace, the choice of an orthonormal system of eigenvectors can be arbitrary.

It is well known that a possible choice for eigenfunctions of the operator $\mathcal{F}$ are given by the normalized Hermite-Gauss functions:

$$\phi_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \quad \text{where} \quad H_n(x) = (-1)^n e^{x^2} \mathcal{D}^n e^{-x^2}, \quad \mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x},$$

is an Hermite polynomial of degree n. These eigenfunctions are normalized in the sense that $(2\pi)^{-1/2} \int_{-\infty}^{\infty} |\phi_n(t)|^2 \mathrm{d}t = 1$. That these are eigenfunctions means that there is an eigenvalue λ_n such that $\mathcal{F}(\phi_n(t)) = \lambda_n \phi_n(t)$. Since

$$\mathcal{F}\phi_n = e^{-in\pi/2} \phi_n, \tag{6}$$

we see that the eigenvalue for ϕ_n is given by $\lambda_n = e^{-in\pi/2} = \lambda^n$ with $\lambda = -i = e^{-i\pi/2}$ representing a rotation over an angle $\pi/2$. Because the Hermite-Gauss functions form a complete set in $\mathcal{L}$, it suffices to define the FrFT on this set of eigenfunctions ϕ_n. This is simple, since in view of our intuitive geometric definition, it seems natural to define the FrFT for an angle $\alpha = a\pi/2$ by

$$\mathcal{F}^a\phi_n = e^{-ina\pi/2}\phi_n = \lambda_n^a\phi_n = \lambda_a^n\phi_n,$$

with $\lambda_a = e^{-ia\pi/2} = e^{-i\alpha} = \lambda^a$ causing a rotation over an angle α. Thus the classical FT corresponds to the FrFT $\mathcal{F}^1$ and the FrFT corresponds to a fractional power of the FT operator $\mathcal{F}$.

Thus the Fourier kernel is

$$K_1(\xi, x) = \frac{e^{-i\xi x}}{\sqrt{2\pi}} = \sum_{n=0}^{\infty} \lambda_n\phi_n(\xi)\phi_n(x), \quad \lambda_n = e^{-i\frac{\pi}{2}n},$$

while the kernel of the FrFT is

$$K_a(\xi, x) = \sum_{n=0}^{\infty} \lambda_n^a\phi_n(\xi)\phi_n(x).$$

If we define the analysis operator $\mathcal{T}_\phi$, the synthesis operator $\mathcal{T}_\phi^*$ and the scaling operator $\mathcal{S}_\lambda$ as

$$\begin{aligned}
\mathcal{T}_\phi : &\quad f \mapsto \{c_n\}, \quad c_n = \int_{-\infty}^{\infty} f(x)\phi_n(x)\mathrm{d}x \\
\mathcal{S}_\lambda : &\quad \{c_n\} \mapsto \{\lambda_n c_n\}, \quad \lambda_n = e^{-i\frac{n\pi}{2}} \\
\mathcal{T}_\phi^* : &\quad \{d_n\} \mapsto \sum_{n=0}^{\infty} d_n\phi_n(x),
\end{aligned}$$

then it is clear that we may write

$$\mathcal{F} = \mathcal{T}_\phi^*\mathcal{S}_\lambda\mathcal{T}_\phi \quad \text{and} \quad \mathcal{F}^a = \mathcal{T}_\phi^*\mathcal{S}_\lambda^a\mathcal{T}_\phi. \tag{7}$$

Note that the operator $\mathcal{T}_\phi$ is unitary and that $\mathcal{T}_\phi^*$ is its adjoint.

This definition implies that $\mathcal{F}^a$ can be written as an operator exponential $\mathcal{F}^a = e^{-i\alpha\mathcal{H}} = e^{-ia\pi\mathcal{H}/2}$ where the Hamiltonian operator $\mathcal{H}$ is given by $\mathcal{H} = -\frac{1}{2}(\mathcal{D}^2 + \mathcal{U}^2 + \mathcal{I})$ with $\mathcal{D}$ the differentiation operator and $\mathcal{U}$ the multiplication or complex shift operator defined as $(\mathcal{U}f)(x) = ixf(x) = (\mathcal{F}\mathcal{D}\mathcal{F}^{-1})f(x)$ (see [18, 22, 25]). The form of the operator $\mathcal{H}$ can be readily checked by differentiating the relation

$$e^{-i\alpha\mathcal{H}}\left(e^{-x^2/2}H_n(x)\right) = e^{-in\alpha}\left(e^{-x^2/2}H_n(x)\right)$$

with respect to α, setting $\alpha = 0$ and then using the differential equation $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$, or equivalently $(\mathcal{D} + 2i\mathcal{U})\mathcal{D}H_n = -2nH_n$, the expression for $\mathcal{H}$ follows.

From this representation, we immediately derive a number of properties for the operator $\mathcal{F}^a$.

- For the classical FT we set $\alpha = \pi/2$, hence $a = 1$ and obtain $\mathcal{F}^1 = e^{-i\pi\mathcal{H}/2}$ with inverse $\mathcal{F}^{-1} = e^{i\pi\mathcal{H}/2}$.

- For $\alpha = a = 0$ we do get the identify operator: $\mathcal{F}^0 = e^0 = \mathcal{I}$
- For $\alpha = \pi$, hence $a = 2$ we get the parity operator $\mathcal{F}^2 = e^{-i\pi\mathcal{H}}$.
- Index additivity: $\mathcal{F}^a\mathcal{F}^b = e^{-ia\pi\mathcal{H}/2}e^{-ib\pi\mathcal{H}/2} = e^{-i(a+b)\pi\mathcal{H}/2} = \mathcal{F}^{a+b}$.
- As a special case we have $\mathcal{F}^{1/2}\mathcal{F}^{1/2} = \mathcal{F}$ and we call $\mathcal{F}^{1/2}$ the square root of $\mathcal{F}$. Also, it follows that the inverse of the FrFT $\mathcal{F}^a$ is $\mathcal{F}^{-a}$.
- Linearity: $\mathcal{F}^a[\sum_j \alpha_j f_j(u)] = \sum_j \alpha_j[\mathcal{F}^a f_j(u)]$
- Unitary: $(\mathcal{F}^a)^{-1} = (\mathcal{F}^a)^*$
- Commutativity: $\mathcal{F}^{a_1}\mathcal{F}^{a_2} = \mathcal{F}^{a_2}\mathcal{F}^{a_1}$
- Associativity: $\mathcal{F}^{a_3}(\mathcal{F}^{a_1}\mathcal{F}^{a_2}) = (\mathcal{F}^{a_3}\mathcal{F}^{a_2})\mathcal{F}^{a_1}$

3.2 Integral representations of the fractional Fourier transform

Recall that the Hermite-Gauss functions ϕ_n are eigenfunctions of the FrFT $\mathcal{F}^\alpha$ with eigenvalues $e^{-in\alpha} = e^{-ina\pi/2}$, i.e., $\mathcal{F}^a\phi_n = e^{-ina\pi/2}\phi_n$.

Any function $f \in L^2(-\infty, \infty)$ can be expanded in terms of these eigenfunctions $f = \sum_{n=0}^{\infty} a_n\phi_n$ with

$$a_n = \frac{1}{\sqrt{2^n n! \pi\sqrt{2}}} \int_{-\infty}^{\infty} H_n(x)e^{-x^2/2} f(x)\mathrm{d}x. \tag{8}$$

By applying the operator $\mathcal{F}^a$, we get

$$f_a := \mathcal{F}^a f = \mathcal{F}^a[\sum_{n=0}^{\infty} a_n\phi_n] = \sum_{n=0}^{\infty} a_n\mathcal{F}^a\phi_n = \sum_{n=0}^{\infty} a_n e^{-ina\pi/2}\phi_n.$$

Thus we have the definition of the FrFT in the form of a series, but this form is very impractical for computational purposes.

It will be much more practical to have an integral representation. This can be obtained by replacing the a_n in the series by their integral expression (8) (recall $\alpha = a\pi/2$):

$$\begin{aligned} f_a(\xi) &= \sum_{n=0}^{\infty} \left[\frac{1}{\sqrt{2^n n! \pi\sqrt{2}}} \int_{-\infty}^{\infty} \phi_n(x) f(x)\mathrm{d}x \right] e^{-ina\pi/2}\phi_n(\xi) \\ &= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-ina\pi/2}H_n(\xi)H_n(x)}{2^n n!\sqrt{\pi}} e^{-(x^2+\xi^2)/2} f(x)\mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}\sqrt{1-e^{-2i\alpha}}} \int_{-\infty}^{\infty} \exp\left\{ \frac{2x\xi e^{-i\alpha} - e^{-2i\alpha}(\xi^2+x^2)}{1-e^{-2i\alpha}} \right\} \exp\left\{ -\frac{\xi^2+x^2}{2} \right\} f(x)\mathrm{d}x \end{aligned}$$

where in the last step we used Mehler's formula ([22, p. 244] or [4, eq. (6.1.13)])

$$\sum_{n=0}^{\infty} \frac{e^{-in\alpha}H_n(\xi)H_n(x)}{2^n n!\sqrt{\pi}} = \frac{\exp\left\{ \frac{2x\xi e^{-i\alpha} - e^{-2i\alpha}(\xi^2+x^2)}{1-e^{-2i\alpha}} \right\}}{\sqrt{\pi(1-e^{-2i\alpha})}}.$$

To rewrite this expression, we observe that the following identities hold (they are easily checked)

$$\frac{2x\xi e^{-i\alpha}}{1-e^{-2i\alpha}} = -ix\xi\csc\alpha$$

$$\frac{1}{\sqrt{\pi}\sqrt{1-e^{-2i\alpha}}} = \frac{e^{-\frac{i}{2}(\frac{\pi}{2}\hat{\alpha}-\alpha)}}{\sqrt{2\pi|\sin\alpha|}}$$

$$\frac{e^{-2i\alpha}}{1-e^{-2i\alpha}} + \frac{1}{2} = -\frac{i}{2}\cot\alpha$$

where $\hat{\alpha} = \operatorname{sgn}(\sin\alpha)$. Obviously, such relations only make sense if $\sin\alpha \neq 0$, i.e., if $\alpha \notin \pi\mathbb{Z}$ or equivalently $a \notin 2\mathbb{Z}$. The branch of $(\sin\alpha)^{1/2}$ we are using for $\sin\alpha < 0$ is the one with $0 < |\alpha| < \pi$. With these expressions, we obtain a more tractable integral representation of $\mathcal{F}^a$ for $a \notin 2\mathbb{Z}$ viz.

$$f_a(\xi) := (\mathcal{F}^a f)(\xi) = \frac{e^{-\frac{i}{2}(\frac{\pi}{2}\hat{\alpha}-\alpha)}e^{\frac{i}{2}\xi^2\cot\alpha}}{\sqrt{2\pi|\sin\alpha|}}\int_{-\infty}^{\infty}\exp\left\{-i\frac{x\xi}{\sin\alpha}+\frac{i}{2}x^2\cot\alpha\right\}f(x)\mathrm{d}x, \qquad (9)$$

where $\hat{\alpha} = \operatorname{sgn}(\sin\alpha)$ and $0 < |\alpha| < \pi$.

Previously we defined $(\mathcal{F}^a f)(\xi) = f(\xi)$, if $\alpha = 0$, and $(\mathcal{F}^a f)(\xi) = f(-\xi)$, if $\alpha = \pm\pi$. That is consistent with this integral representation because for these special values, it holds that $\lim_{\epsilon\to 0} f_{a+\epsilon} = f_a$. Thus, with this limiting property, we can assume that the integral representation holds on the whole interval $|\alpha| \leq \pi$. Clearly, when $|\alpha| > 2\pi$, the definition is taken modulo 2π and reduced to the interval $[-\pi, \pi]$.

Defining the FrFT via this integral transform, we can say that the FrFT exists for $f \in L^1$ (and hence in L^2) or when it is a generalized function. Indeed, in that case, the integrand in (9) is also in L^1 (or L^2) or is a generalized function. Thus the FrFT exists in exactly the same conditions as in which the FT exists. Thus we have proved

Theorem 3.1 *Assume $\alpha = a\pi/2$ then the FrFT has an integral representation*

$$f_a(\xi) := (\mathcal{F}^a f)(\xi) = \int_{-\infty}^{\infty} K_a(\xi, x) f(x)\mathrm{d}x.$$

The kernel is defined as follows: For $a \notin 2\mathbb{Z}$, then with $\hat{\alpha} = \operatorname{sgn}(\sin\alpha)$,

$$K_a(\xi,x) = C_\alpha \exp\left\{-i\frac{x\xi}{\sin\alpha} + \frac{i}{2}(x^2+\xi^2)\cot\alpha\right\} \quad \textit{with} \quad C_\alpha = \frac{e^{-\frac{i}{2}(\frac{\pi}{2}\hat{\alpha}-\alpha)}}{\sqrt{2\pi|\sin\alpha|}} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}.$$

For $a \in 4\mathbb{Z}$ the FrFT becomes the identity, hence

$$K_{4n}(\xi, x) = \delta(\xi - x), \quad n \in \mathbb{Z}$$

and for $a \in 2 + 4\mathbb{Z}$, it is the parity operator:

$$K_{2+4n}(\xi, x) = \delta(\xi + x), \quad n \in \mathbb{Z}.$$

If we restrict a to the range $0 < |a| < 2$, then $\mathcal{F}^a$ is a homeomorphism of $\mathcal{L}$ (with inverse $\mathcal{F}^{-a}$).

The last statement is proved in [18, p. 162]. Some graphics representing the kernel are given in Figure 2.

Note that by $\mathcal{F}^b = \mathcal{F}^{b-a}\mathcal{F}^a$, we immediately have the more general formula

$$f_b(\xi) = \int_{-\infty}^{\infty} f_a(x) K_{b-a}(\xi, x) \mathrm{d}x.$$

Using the above expressions and the interpretation of the FrFT as a rotation, it is directly verified that the kernel K_a has the following properties.

Theorem 3.2 *IF $K_a(x,t)$ is the kernel of the FrFT as in Theorem 3.1, then*

1. $K_a(\xi, x) = K_a(x, \xi)$ *(diagonal symmetry)*
2. $K_{-a}(\xi, x) = \overline{K_a(\xi, x)}$ *(complex conjugate)*
3. $K_a(-\xi, x) = K_a(\xi, -x)$ *(point symmetry)*
4. $\int_{-\infty}^{\infty} K_a(\xi, t) K_b(t, x) \mathrm{d}t = K_{a+b}(\xi, x)$ *(additivity)*
5. $\int_{-\infty}^{\infty} K_a(t, \xi) \overline{K_a(t, x)} \mathrm{d}t = \delta(\xi - x)$ *(orthogonality)*

3.3 The chirp function

A chirp function (or chirp for short) is a signal that contains all frequencies in a certain interval and sweeps through it while it progresses in time. The interval can be swept in several ways (linear, quadratic, logarithmic,...), but we shall restrict us here to the case where the sweep is linear.

The complex exponential $e^{i\omega t}$ contains just one frequency: ω. This type of functions is essential in Fourier analysis. In fact, they form a basis for the space of functions treated by the FT. Indeed, the relation $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \mathrm{d}t$ can be seen as a decomposition of f into a (continuous) combination of the basis functions $\{e^{i\omega t}\}_{\omega \in \mathbb{R}}$.

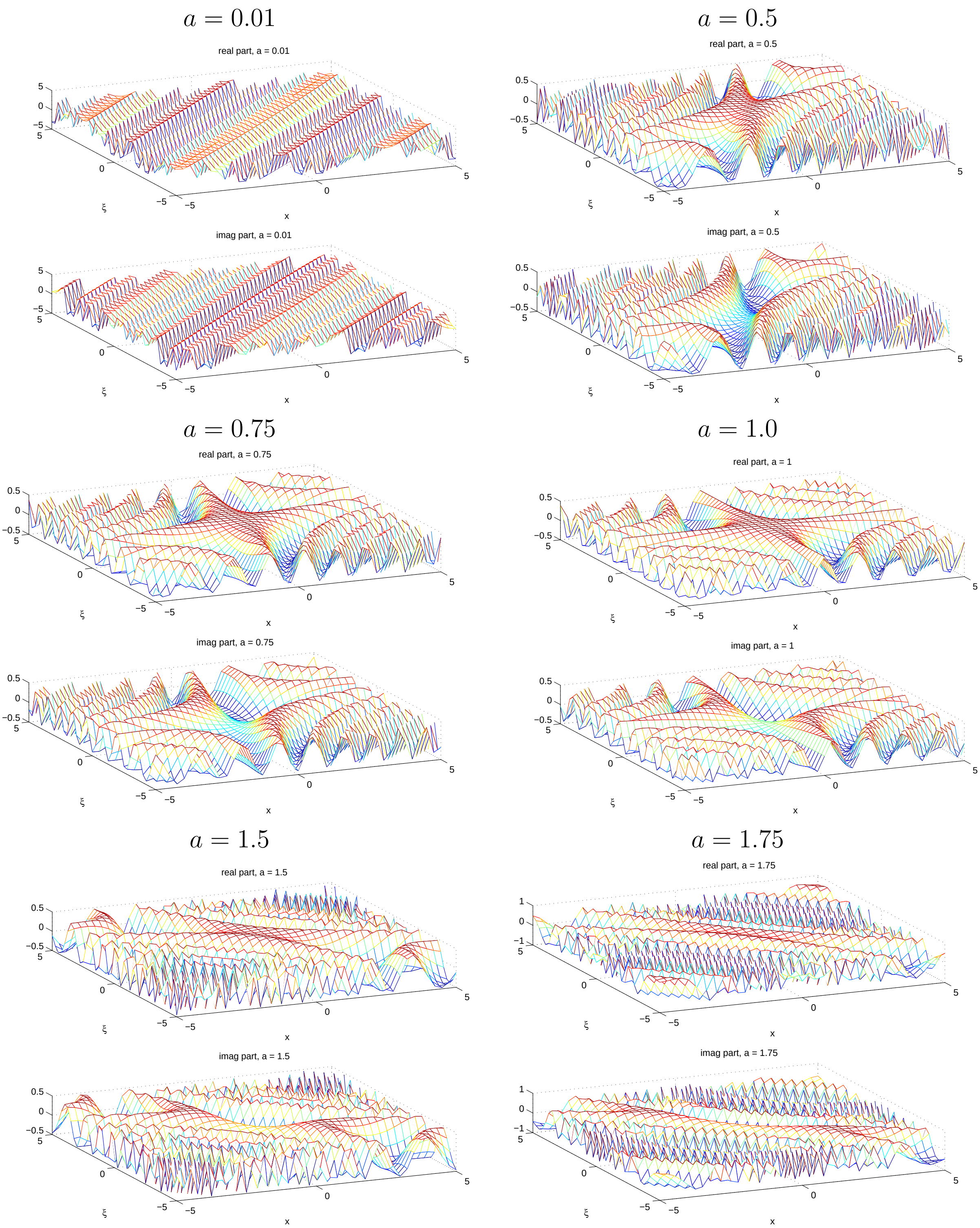
On the other hand, if the frequencies of the signal sweeps linearly through the frequency interval $[\omega_0, \omega_1]$ in the time interval $[t_0, t_1]$, then we should have $\omega = \omega_0 + \frac{\omega_1 - \omega_0}{t_1 - t_0}(t - t_0)$. Thus, such a function will look like $\exp\{i(\chi t + \gamma)t\}$. The parameter χ is called the sweep rate. Now consider the FrFT kernel $K_a(\xi, x)$, then, seen as a function of x and taking ξ as a parameter, this is a chirp with sweep rate $\frac{1}{2}\cot\alpha$. So, one way of describing a FrFT is

1. multiply by a chirp
2. do an ordinary FT
3. multiply by a chirp.

The inverse FrFT can be written as $f(x) = \int_{-\infty}^{\infty} f_a(\xi)\psi_\xi(x)\mathrm{d}\xi$ where $\psi_\xi(x) = K_{-a}(\xi, x)$ is a chirp parameterized in ξ with sweep rate $-\frac{1}{2}\cot\alpha$. Thus we see that the role played by the harmonics in classical FT, is now taken by chirps, and the latter relation is a decomposition of $f(x)$ into a linear combination of chirps with a fixed sweep rate determined by α. Note also that in this expansion in chirp series, the basis functions are orthogonal since

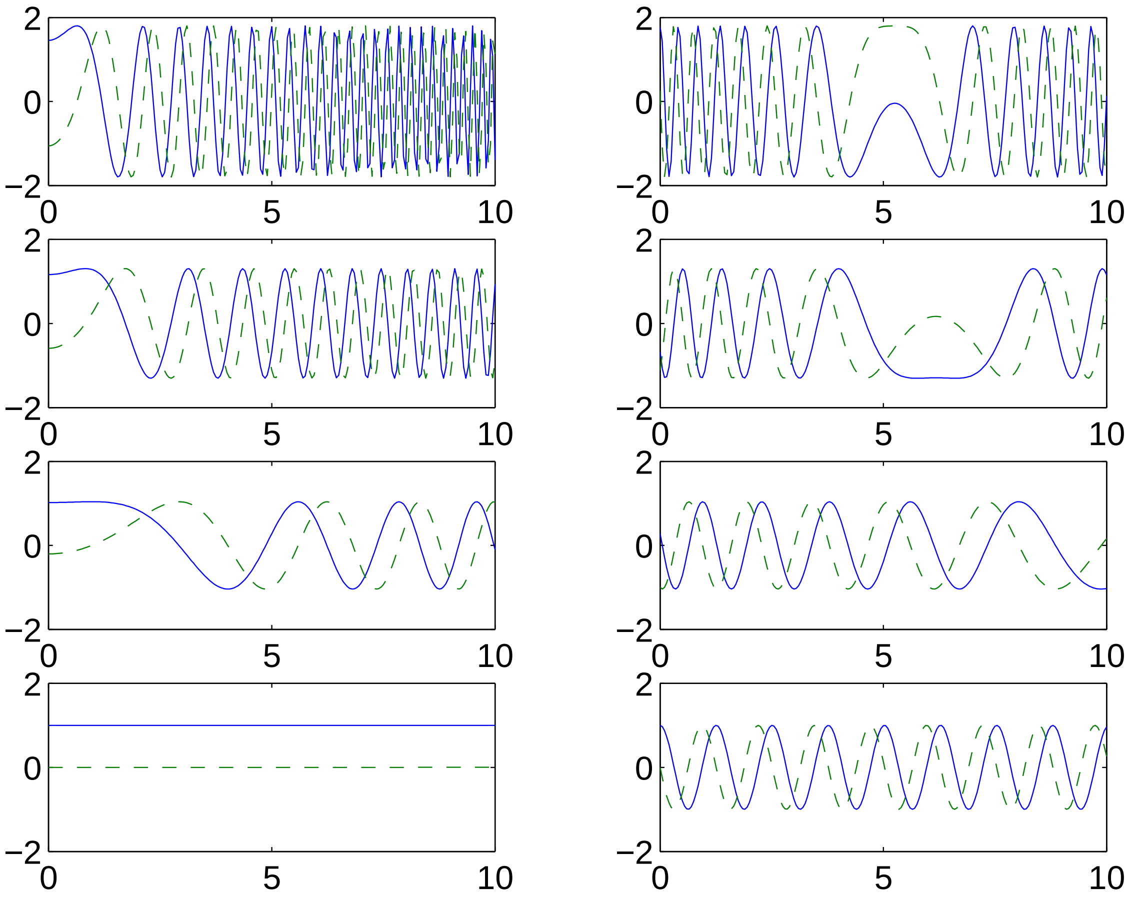
$$\int_{-\infty}^{\infty} K_a(\xi, x) K_a(\xi, y) \mathrm{d}\xi = \delta(x - y).$$

Figure 2: Real (top) and imaginary (bottom) parts of the FrFT kernel $K_a(\xi, x)$, $x, \xi \in [-5, 5]$ for $a = 0.01, 0.5, 0.75, 1, 1.5$, and 1.75. Note the highly oscillating character for a close to an even integer.



However, there is more. The chirps are in between harmonics and delta functions, which are basic for the classical FT. Indeed, up to a rotation in the time-frequency plane, the chirps *are* delta functions and harmonics. To see this, take the FrFT of a delta function $\delta(x-\gamma)$. That is $\mathcal{F}^a(\delta(\cdot - \gamma)) = K_a(\xi, \gamma)$, which is a chirp with sweep rate $\frac{1}{2}\cot\alpha$. Thus, given a (linear) chirp with sweep rate $\frac{1}{2}\cot\alpha$, we can by a FrFT $\mathcal{F}^{-a}$ transform it into a delta function and hence by taking the FT of the delta function, we can take the chirp by a FrFT $\mathcal{F}^{1-a}$ into an harmonic function.

Figure 3: $\mathcal{F}^a(\delta(\cdot-\gamma))$ for $a = 0.2, 0.4, 0.75, 1$ Left for $\gamma = 0$, right for $\gamma = 5$. Real part: solid line, imaginary part: dashed line



3.4 Fractional Fourier transforms of some common functions

Now that we have seen the definition and some essential properties, it is time to have a look at some of the actual transforms of some elementary functions.

Table 1: The fractional Fourier transform of some basic functions

	$f(u)$	$\mathcal{F}^a(u)$
1	$\delta(u-\gamma)$	$\sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{\frac{i}{2}(u^2\cot\alpha-2u\gamma\csc\alpha+\gamma^2\cot\alpha)}$, if $a \notin 2\mathbb{Z}$
2	1	$\sqrt{\frac{1+i\tan\alpha}{2\pi}}e^{-i\frac{u^2}{2}\tan\alpha}$, if $a \notin 2\mathbb{Z}+1$
3	$e^{\frac{i}{2}(\chi u^2+2\gamma u)}$	$\sqrt{\frac{1+i\tan\alpha}{1+\chi\tan\alpha}}e^{i\frac{u^2(\chi-\tan\alpha)+2u\gamma\sec\alpha-\gamma^2\tan\alpha}{2(1+\chi\tan\alpha)}}$, if $a-\frac{2}{\pi}\arctan\chi \notin 2\mathbb{Z}+1$
4	$e^{-\frac{i}{2}(\chi u^2+2\gamma u)}$	$\sqrt{\frac{1-i\cot\alpha}{\chi-i\cot\alpha}}e^{\frac{i}{2}\cot\alpha\frac{u^2(\chi^2-1)+2u\chi\gamma\sec\alpha+\xi^2}{\chi^2+\cot^2\alpha}}e^{-\frac{1}{2}\csc^2\alpha\frac{u^2\chi+2u\gamma\cos\alpha-\chi\gamma^2\sin^2\alpha}{\chi^2+\cot^2\alpha}}$, $\chi>0$
5	$\phi_l(u)$	$e^{-il\alpha}\phi_l(u)$
6	$e^{-\frac{u^2}{2}}$	$e^{-\frac{u^2}{2}}$

In (4), $\chi > 0$, is required for the convergence. In the table given above $\gamma, \chi \in \mathbb{R}$, and $\phi_l(u)$ are the Hermite-Gauss functions.

For a proof, we refer to the appendix.

For some graphics of several signals given in the table above, see Figure 4. Note that the figure near $\alpha = 0$ for the delta-function does not look much like a delta function, but behaves rather chaotic. The reason is purely numerical. Although, we have theoretically that a chirp can converge to a delta function. More precisely $\lim_{a\to 0} K_a(x,\gamma) = \delta(x-\gamma)$, this does not work well numerically because we use an approximation where terms of the form $0\cdot\infty$ will show up.

4 Properties of the FrFT

The previous relations imply several properties for the FrFT.

Theorem 4.1 *The FrFT satisfies the following properties.*

1. **conservation of symmetry:** *The FrFT of an even (odd) function is even (odd).*

2. **Parseval:**
$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x = \int_{-\infty}^{\infty} (\mathcal{F}^a f)(\xi)\overline{(\mathcal{F}^a g)(\xi)}\mathrm{d}\xi.$$
If $f = g$, this is the energy conservation property.

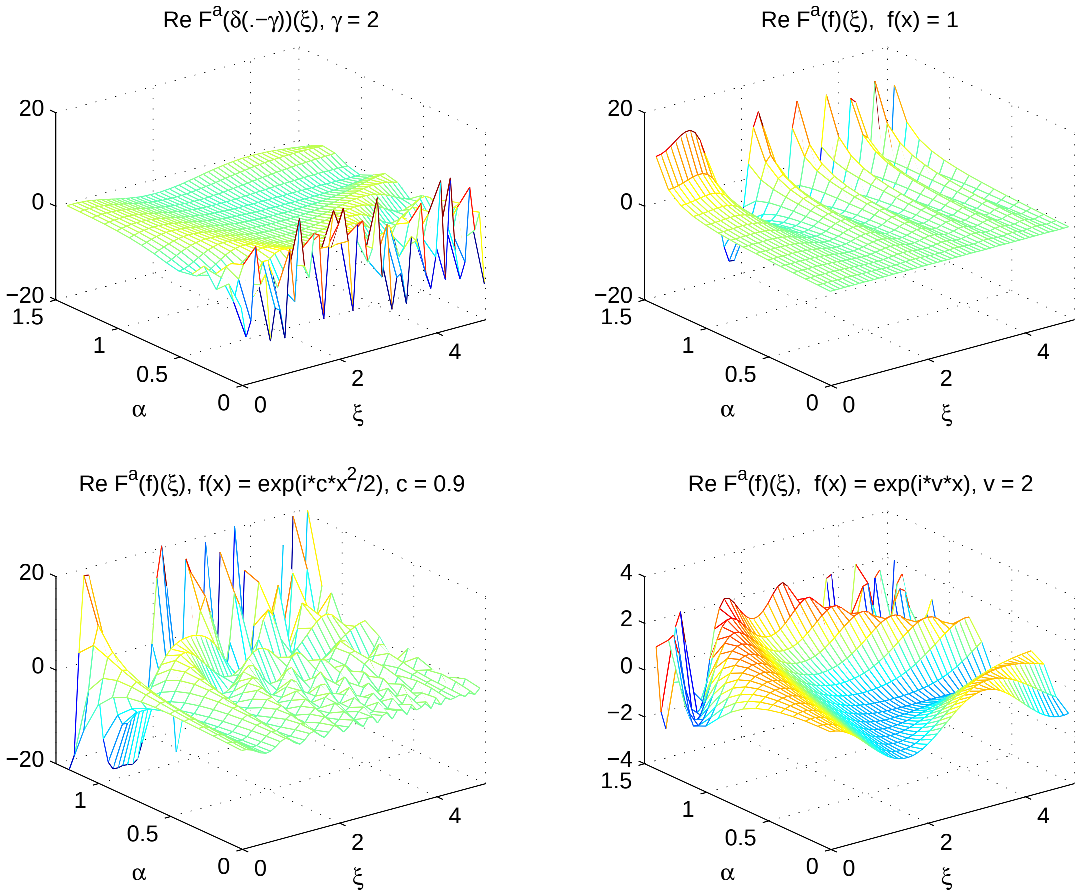
3. **multiplication rule:** $(\mathcal{F}^a\mathcal{U}^m f)(\xi) = [\ \ \cos\alpha\,\mathcal{U} + \sin\alpha\,\mathcal{D}]^m(\mathcal{F}^a f)(\xi)$

4. **differentiation rule:** $(\mathcal{F}^a\mathcal{D}^m f)(\xi) = [-\sin\alpha\,\mathcal{U} + \cos\alpha\,\mathcal{D}]^m(\mathcal{F}^a f)(\xi)$

Proof. (outline) The first one follows directly from the integral expression. The second is also obtained from the previous kernel properties and Fubini's theorem.

It requires much more technical manipulations to prove the latter two rules. They are proved by induction on m. For $m = 1$, the multiplication rule is obtained by differentiating

Figure 4: FrFT (real parts) of some signals



the integral representation of $f_a(\xi)$ with respect to ξ. The differentiation rule is obtained by using integration by parts for $\mathcal{F}^a\mathcal{D}f$ and using the multiplication rule. $\square$

The two properties (3) and (4) (with $m = 1$) can be written as

$$\mathcal{F}^a \begin{bmatrix} \mathcal{U} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{U}_a \\ \mathcal{D}_a \end{bmatrix} \mathcal{F}^a \quad \text{where} \quad \begin{bmatrix} \mathcal{U}_a \\ \mathcal{D}_a \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ \mathcal{D} \end{bmatrix}.$$

Thus $\mathcal{U}_a$ and $\mathcal{D}_a$ correspond to multiplication and differentiation in the variable of the FrFT domain. It are rotations of the usual $\mathcal{U}$ and $\mathcal{D}$: $(\mathcal{U}_a, \mathcal{D}_a) = \mathcal{R}_a(\mathcal{U}, \mathcal{D})$. The latter are in a sense orthogonal operations since for $a = 1$, i.e., $\alpha = \pi/2$, we recover the properties of the FT namely multiplication in the time domain corresponds to differentiation in the frequency domain, and differentiation in the time domain corresponds to minus the multiplication operation in the frequency domain, which is exactly what the previous relations say for $a = 1$, viz. $\mathcal{F}\mathcal{U} = \mathcal{U}_1\mathcal{F} = \mathcal{D}\mathcal{F}$ and $\mathcal{F}\mathcal{D} = \mathcal{D}_1\mathcal{F} = -\mathcal{U}\mathcal{F}$.

It is not difficult to derive from the previous relations the more general one:

$$\mathcal{F}^{b-a} \begin{bmatrix} \mathcal{U}_a^m \\ \mathcal{D}_a^m \end{bmatrix} = \begin{bmatrix} \mathcal{U}_b^m \\ \mathcal{D}_b^m \end{bmatrix} \mathcal{F}^{b-a};$$

$$\begin{bmatrix} \mathcal{U}_b \\ \mathcal{D}_b \end{bmatrix} = \begin{bmatrix} \cos(\beta-\alpha) & \sin(\beta-\alpha) \\ -\sin(\beta-\alpha) & \cos(\beta-\alpha) \end{bmatrix} \begin{bmatrix} \mathcal{U}_a \\ \mathcal{D}_a \end{bmatrix}; \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \pi/2.$$

Because the rotation is an orthogonal transformation, we also have $\mathcal{D}_a^2+\mathcal{U}_a^2 = \mathcal{D}^2+\mathcal{U}^2$, so that the Hamiltonian is rotation invariant: $\mathcal{H}_a = -\frac{1}{2}(\mathcal{D}_a^2+\mathcal{U}_a^2+\mathcal{I}) = -\frac{1}{2}(\mathcal{D}^2+\mathcal{U}^2+\mathcal{I}) = \mathcal{H}$.

Theorem 4.1 has some immediate consequences which we include for further reference.

Corollary 4.2 *For functions f and g we shall consistently use $f_a = \mathcal{F}^a f$ and $g_a = \mathcal{F}^a g$. For $f \in \mathcal{L}$ and $\alpha = a\pi/2 \in \mathbb{R}$ we have the following operational rules.*

1. **division rule:** *If $g(x) = (ix)^{-m} f(x)$, then*
$$e^{\frac{i\xi^2 \cot\alpha}{2}} g_a(\xi) = (\sin\alpha)^{-m} \int_{-\infty}^{\xi} e^{\frac{ix^2\cot\alpha}{2}} f_a(x)\mathrm{d}x.$$

2. **integration rule:** *If $g(x) = \int_c^x f(t)\mathrm{d}t$, then*
$$e^{-\frac{i\xi^2 \mathrm{tg}\alpha}{2}} g_a(\xi) = \sec\alpha \int_{c}^{\xi} e^{-\frac{ix^2\mathrm{tg}\alpha}{2}} f_a(x)\mathrm{d}x.$$

3. **mixed product rule:** *$\mathcal{F}^a(\mathcal{U}\mathcal{D})^m = (\mathcal{U}_a\mathcal{D}_a)^m\mathcal{F}^a$ and $\mathcal{F}^a(\mathcal{D}\mathcal{U})^m = (\mathcal{D}_a\mathcal{U}_a)^m\mathcal{F}^a$ while for $m = 1$: $\mathcal{F}^a\mathcal{U}\mathcal{D} = \mathcal{U}_a\mathcal{F}^a\mathcal{D}$ and $\mathcal{F}^a\mathcal{D}\mathcal{U} = \mathcal{D}_a\mathcal{F}^a\mathcal{U}$.*

For a proof we refer to the appendix.

Some other rules that transfer in a consistent way from the classical FT: It is a very useful property of the FT that it transforms a convolution into a product. In fact, this is the very reason why many signal processing problems become so simple when dealt with in the frequency domain. Of course, this property remains true when it concerns the convolution of two functions in the domain of the FrFT $\mathcal{F}^a$: if $g_a(x) = f_a(x) * h_a(x)$, then its FT becomes $g_{a+1}(\xi) = f_{a+1}(\xi)h_{a+1}(\xi)$. More generally, it is obvious that
$$f_a(x_a) * h_a(x_a) = f_{a\pm1}(x_{a\pm1})h_{a\pm1}(x_{a\pm1}) = f_{a\pm2}(x_{a\pm2}) * h_{a\pm2}(x_{a\pm2}).$$

And to conclude this section, two more rules that generalize the fact that for the classical FT of a shift in the time domain is transformed in an exponential multiplication in the frequency domain. For the FrFT the shift and exponential multiplication shall be mixed if a is not an integer. We have

Theorem 4.3 *As before, for functions f and g we shall consistently use $f_a = \mathcal{F}^a f$ and $g_a = \mathcal{F}^a g$. For $f \in \mathcal{L}$ and $\alpha = a\pi/2 \in \mathbb{R}$ we have*

1. **shift rule:** *If $g(x) = (\mathcal{T}_b f)(x) = f(x+b)$, then*
$$g_a(\xi) = e^{ib\sin\alpha(\xi+\frac{1}{2}b\cos\alpha)} f_a(\xi + b\cos\alpha).$$

2. **exponential rule:** *If $g(x) = e^{ibx} f(x)$, then*
$$g_a(\xi) = e^{ib\cos\alpha(\xi+\frac{1}{2}b\sin\alpha)} f_a(\xi + b\sin\alpha), \quad x \in \mathbb{R}.$$

For a proof we again refer to the appendix.

5 Relation with other transforms

There are several other time/frequency representations of a signal that are related to the FrFT. Some of them will be discussed in this section.

5.1 Wigner distribution

Let f be a signal, then its *Wigner distribution* or *Wigner transform* $\mathcal{W}f$ is defined as

$$(\mathcal{W}f)(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+u/2)\overline{f(x-u/2)}e^{-i\xi u}\mathrm{d}u.$$

Its meaning is roughly speaking one of energy distribution of the signal in the time-frequency plane. Indeed, setting $f_1 = \mathcal{F}f$, we have

$$\int_{-\infty}^{\infty} (\mathcal{W}f)(x,\xi)\mathrm{d}\xi = |f(x)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty} (\mathcal{W}f)(x,\xi)\mathrm{d}x = |f_1(\xi)|^2,$$

so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathcal{W}f)(x,\xi)\mathrm{d}\xi\mathrm{d}x = \|f\|^2 = \|f_1\|^2,$$

which is the energy of the signal f.

An important property of the FrFT is the following.

Theorem 5.1 *The Wigner distribution of a signal and its FrFT are related by a rotation over an angle* $-\alpha$*:*

$$(\mathcal{W}f_a)(x,\xi) = \mathcal{R}_{-a}(\mathcal{W}f)(x,\xi)$$

where $\alpha = a\pi/2$*,* $f_a = \mathcal{F}^a f$*, and* $\mathcal{R}_{-a}$ *represents a clockwise rotation of the variables* (x,ξ) *over the angle* α*. Equivalently*

$$\mathcal{R}_a(\mathcal{W}f_a)(x,\xi) = (\mathcal{W}f_a)(x_a,\xi_a) = (\mathcal{W}f)(x,\xi)$$

with $(x_a,\xi_a) = \mathcal{R}_a(x,\xi)$*.*

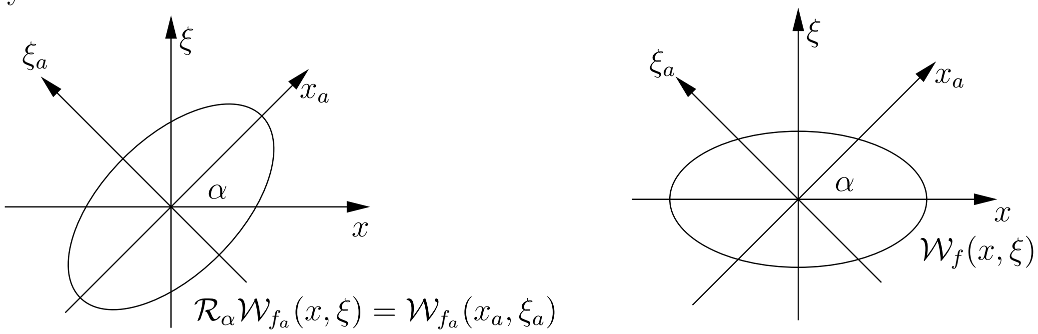
The proof is tedious but straightforward: replace in the definition of $(\mathcal{W}f_a)(x,\xi)$ the values of $f_a(x+u/2)$ and $f_a(x-u/2)$ by their integral representation, which leads to a triple integral. One of these integrals gives a delta function which allows to evaluate the second of these integrals. The remaining one can then be identified with the explicit formula for $\mathcal{R}_{-a}(\mathcal{W}f)(x,\xi)$. For detail see [3, p. 3087]. Looking at Figure 5, the result is in fact obvious since it just states that before and after a rotation of the coordinate axes, the Wigner distribution is computed in two different ways taking into account the new variables, and that should of course give the same result.

This implies for example

$$\int_{-\infty}^{\infty} (\mathcal{W}f_a)(x,\xi)\mathrm{d}\xi = |f_a(x)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathcal{W}f_a)(x,\xi)\mathrm{d}x\mathrm{d}\xi = \|f\|^2.$$

The first relation generalizes the above expressions given for $a = 0, 1$, the second gives the energy of the signal.

Figure 5: Wigner distribution of a signal f and the Wigner distribution of its FrFT are related by a rotation.



The *Radon transform* of a 2D function is the integral of this function along a straight line through the origin. The *Radon-Wigner transform* is the Radon transform of the Wigner distribution. If that line makes an angle $\alpha = a\pi/2$ with the x-axis, then it is given by

$$\int_{-\infty}^{\infty} (\mathcal{W}f)(r\cos\alpha, r\sin\alpha)\mathrm{d}r = \int_{-\infty}^{\infty} (\mathcal{W}f_a)(x,\xi)\mathrm{d}\xi = |f_a(x)|^2.$$

5.2 The ambiguity function

The ambiguity function is closely related to the Wigner distribution. Its definition is

$$(\mathcal{A}f)(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u + x/2)\overline{f(u - x/2)}e^{-iu\xi}\mathrm{d}u.$$

Thus it is like the Wigner distribution, but now the integral is over the other variable. The ambiguity function and the Wigner distribution are related by what is essentially a 2D Fourier transform. Whereas the Wigner distribution gives an idea about how the energy of the signal is distributed in the (x,ξ)-plane, the ambiguity function will have a correlative interpretation. Indeed $(\mathcal{A}f)(x,0)$ is the autocorrelation function of f and $(\mathcal{A}f)(0,\xi)$ is the autocorrelation function of $f_1 = \mathcal{F}f$. The Radon transform of the ambiguity function is

$$\int_{-\infty}^{\infty} (\mathcal{A}f)(r\cos\alpha, r\sin\alpha)\mathrm{d}r = \int_{-\infty}^{\infty} (\mathcal{A}f_a)(x,\xi)\mathrm{d}\xi = f_a(x/2)\overline{f_a(-x/2)}.$$

5.3 The linear canonical transform

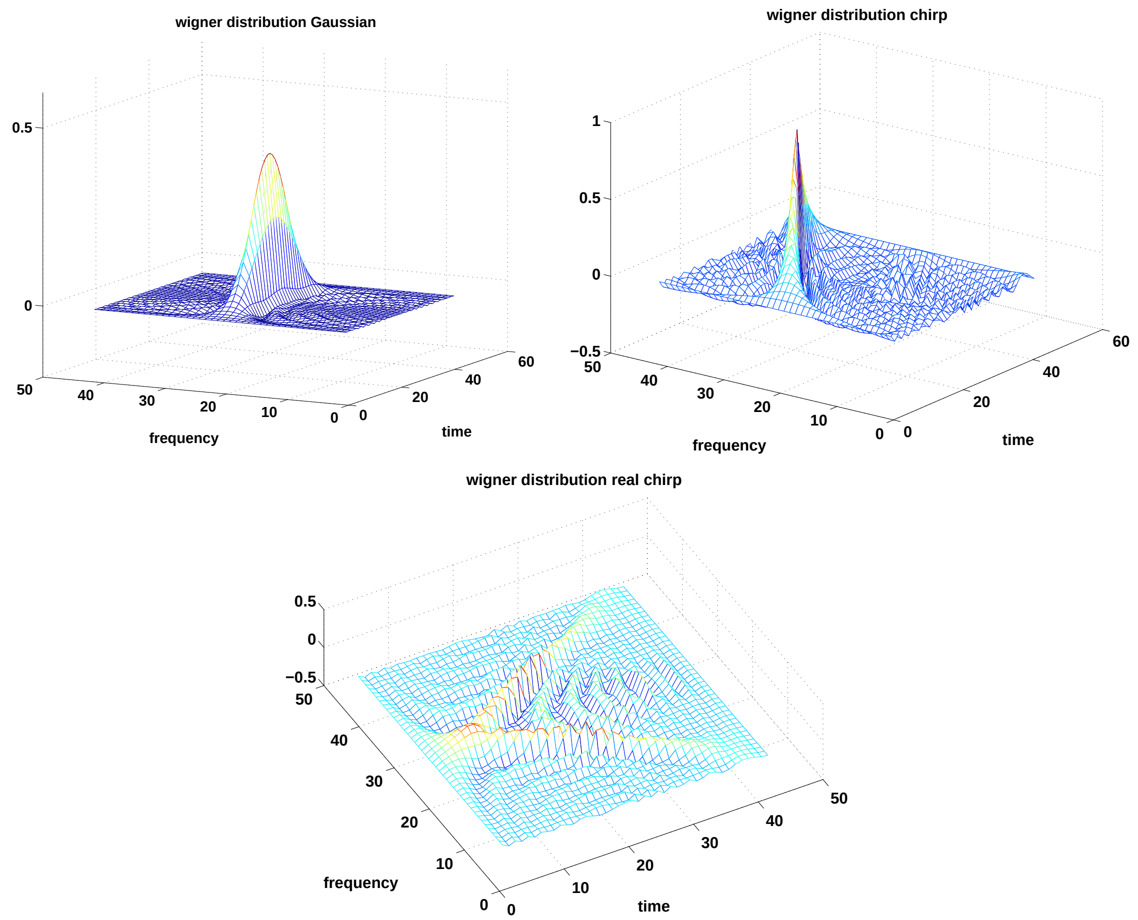
Consider a 2×2 unimodular matrix (i.e., whose determinant is 1). Such a matrix has 3 free parameters u, v, w which we shall arranged as follows

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{w}{v} & \frac{1}{v} \\ -v + \frac{uw}{v} & \frac{u}{v} \end{bmatrix} = \begin{bmatrix} \frac{u}{v} & -\frac{1}{v} \\ v + \frac{uw}{v} & \frac{w}{v} \end{bmatrix}^{-1} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}^{-1}.$$

The parameters can be recovered from the matrix by

$$u = \frac{D}{B} = \frac{1}{A}\left(\frac{1}{B} + C\right), \quad v = \frac{1}{B}, \quad w = \frac{A}{B} = \frac{1}{D}\left(\frac{1}{B} + C\right)$$

Figure 6: Some examples of Wigner distributions



A typical example is the rotation matrix associated with $\mathcal{R}_\alpha$ where $A = D = \cos\alpha$ and $B = -C = \sin\alpha$. Let us call this matrix R_α. The linear canonical transform $\mathcal{C}_M$ of a function f is an integral transform with kernel $C_M(\xi, x)$ defined by

$$C_M(\xi, x) = \sqrt{\frac{v}{2\pi}} e^{-i\frac{\pi}{4}} e^{\frac{i}{2}(u\xi^2 - 2v\xi x + wx^2)} = \frac{1}{\sqrt{2\pi B}} e^{-i\frac{\pi}{4}} e^{\frac{i}{2B}(D\xi^2 - 2\xi x + Ax^2)}$$

where u, v, and w are parameters (independent of ξ and x) arranged in the matrix M as above. Thus the transform is

$$f_M(\xi) = (\mathcal{C}_M f)(\xi) = \int_{-\infty}^{\infty} C_M(\xi, x) f(x) \mathrm{d}x.$$

Note that if M is the rotation matrix R_α, then the kernel C_M reduces almost to the FrFT kernel because $M = R_\alpha$ implies $u = w = \cot\alpha$ while $v = \csc\alpha$. Hence $\mathcal{C}_{R_\alpha} = e^{-i\alpha/2}\mathcal{F}^a$. If f denotes a signal, and f_M, its linear canonical transform, then the Wigner transform gives

$$(\mathcal{W} f_M)(Ax + B\xi, Cx + D\xi) = (\mathcal{W} f)(x, \xi). \tag{10}$$

The latter equation can be directly obtained from the definition of linear canonical transform and the definition of Wigner distribution. Thus if $\mathcal{M}_M$ is the operator defined by $\mathcal{M}_M f(\mathbf{x}) =$

$f(M\mathbf{x})$, then $\mathcal{W} = \mathcal{M}_M \mathcal{W} \mathcal{C}_M$. Note that this generalizes Theorem 5.1, since (up to a unimodular constant factor which does not influence the Wigner distribution) $\mathcal{C}_{R_\alpha}$ and $\mathcal{F}^a$ are the same. Again using the coordinate transform technique of Figure 5, the result is obvious since the result says that the Wigner distribution computed before and after a change of variables gives the same result.

The LCT has been introduced in [21],[36, Chap. 9]. Some of the properties like the relation with Wigner distribution, correlation and convolution, interpretation of the parameters etc. were discussed in [28]. The eigenfunctions for the LCT are analyzed in [29]. They can take quite different forms depending on $|A+B|$ being smaller, equal or larger than 2.

5.4 Short time Fourier transform

The short time Fourier transform or windowed Fourier transform (WFT) is defined as

$$(\mathcal{F}_w f)(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\overline{w(t-x)}e^{-i\xi t}\mathrm{d}t$$

where w is a window function. It is a *local* transform in the sense that the window function more or less selects an interval, centered at x to cut out some filtered information of the signal. So it gives information that is local in the time-frequency domain. We do not only find the frequencies in the signal but also the location of that frequency in time.

It can be shown that

$$(\mathcal{F}_w f)(x,\xi) = e^{-ix\xi}(\mathcal{F}_{w_1} f_1)(\xi,-x) = e^{-ix\xi}(\mathcal{F}_{w_1} f_1)(x_1,\xi_1)$$

where $w_1 = \mathcal{F}w$ and $f_1 = \mathcal{F}f$. Because of the asymmetric factor $e^{-ix\xi}$, it is more convenient to introduce a modified WFT defined by

$$(\tilde{\mathcal{F}}_w f)(x,\xi) = e^{ix\xi/2}(\mathcal{F}_w f)(x,\xi).$$

Then we have

Theorem 5.2 *The modified windowed Fourier transform satisfies*

$$(\tilde{\mathcal{F}}_w f)(x,\xi) = (\tilde{\mathcal{F}}_{w_a} f_a)(x_a,\xi_a)$$

A proof can be found in the appendix.

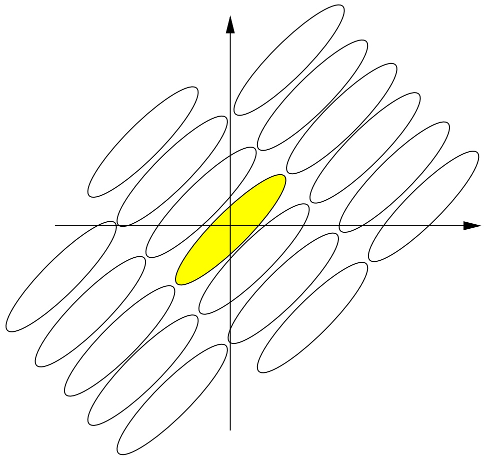
5.5 Wavelet transform

From its definition $f_a(\xi) = \int K_a(\xi,u)f(u)\mathrm{d}u$, we get by setting $x = \xi\sec\alpha$ and $g(x) = f_a(x/\sec\alpha)$

$$g(x) = C(\alpha)e^{-i4x^2\sin(2\alpha)} \int_{-\infty}^{\infty} \exp\left[\frac{i}{2}\left(\frac{x-u}{\tan^{1/2}\alpha}\right)^2\right] f(u)\mathrm{d}u.$$

$C(\alpha)$ is a constant that depends on α only. This can not be interpreted as a genuine wavelet transform. We do have a scaling parameter $\tan^{1/2}\alpha$ and a translation generated by the function $\psi(t) = e^{it^2}$ but since $\int_{-\infty}^{\infty}\psi(x)dx \neq 0$ and it has no compact support, this is not really a wavelet.

Figure 7: A discrete chirplet dictionary tiling the (x, ξ)-plane



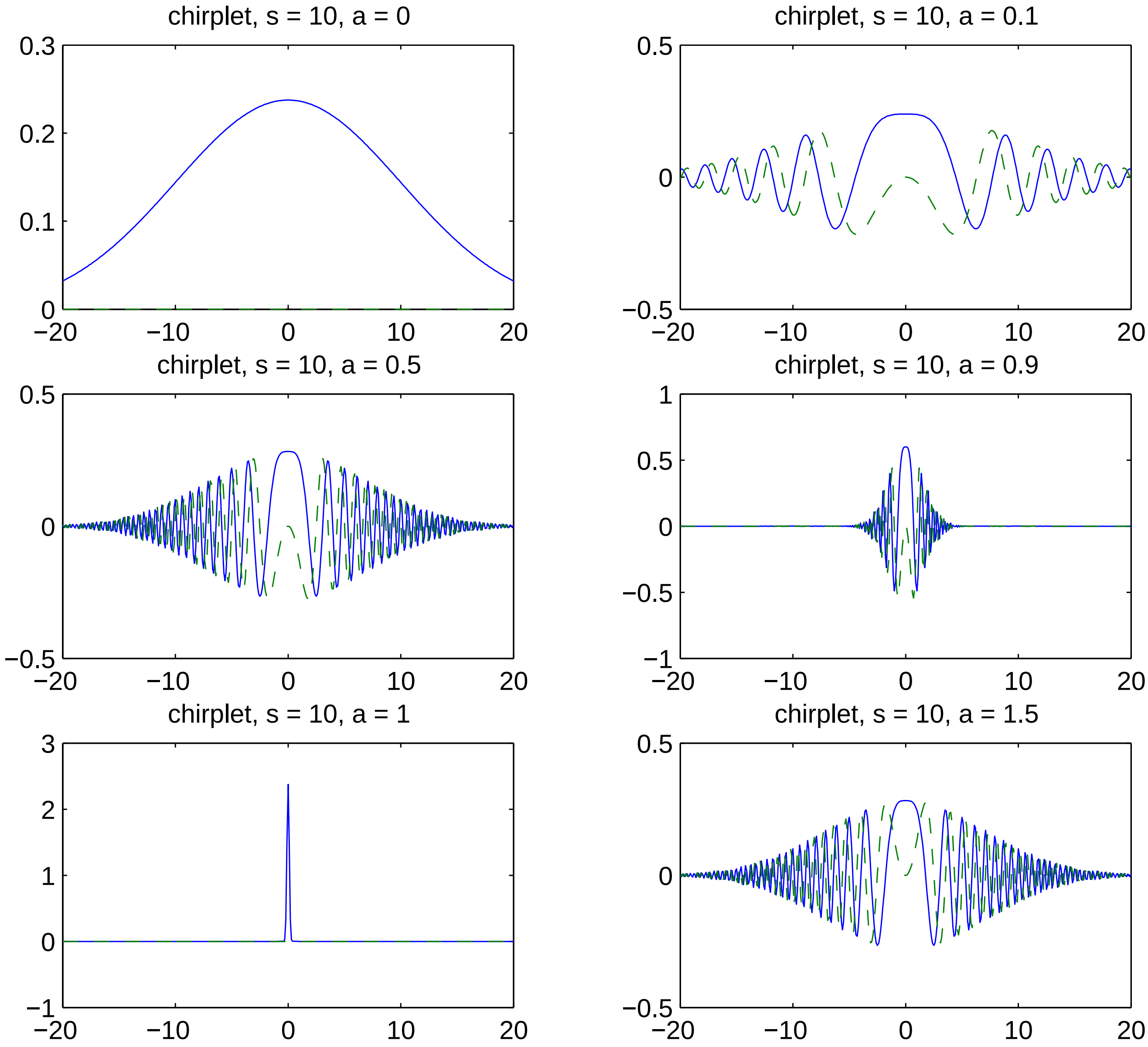
Multiscale chirp functions were introduced in [6, 17]. A. Bultan [7] has developed a so called chirplet decomposition which is related to wavelet package techniques. It is especially suited for the decomposition of signals that are chirps, i.e., whose Wigner distribution corresponds to straight lines in the (x, ξ)-plane.

The idea is that a dictionary of chirplets is obtained by scaling and translating an atom whose Wigner distribution is that of a Gaussian that has been stretched and rotated. So, we take a Gaussian $\tilde{g}(t) = \pi^{-1/4} e^{-x^2/2}$ with Wigner distribution $(\mathcal{W}\tilde{g})(x, \xi) = (2/\pi)^{1/2} \exp\{-(x^2 + \xi^2)\}$. Next we stretch it as $g(x) = s^{-1/2}\tilde{g}(x/s)$ giving $(\mathcal{W}g)(x, \xi) = (\mathcal{W}\tilde{g})(x/s, s\xi)$. Finally we rotate $(\mathcal{W}g)$ to give $(\mathcal{W}c)(x, \xi)$ with $c = \mathcal{F}^a g$. The chirplet c depends on two parameters s and a and its main support in the (x, ξ)-plane can be thought of as a stretched (by s) and rotated (by a) ellipse centered at the origin. To cover the whole (x, ξ)-plane, we have to tile it with shifted versions of this ellipse, i.e., we need the $(\mathcal{W}g)(x - u, \xi - \nu)$ corresponding to the functions $c(x - u)e^{i\nu x}$. With these four-parameters $\rho = (s, a, u, \nu)$ we have a redundant dictionary $\{c_\rho\}$. The next step is to find a discretization of these 4 parameters such that the dictionary is complete when restricted to that lattice. It has been shown [33] that such a system can be found for $a = 0$ that is indeed complete, and the rotation does not alter this fact.

If the discrete dictionary is $\{c_n\}$ with $c_n = c_{\rho_n}$, then a chirplet representation of the signal f has to be found of the form $f(x) = \sum_n a_n c_n(x)$. Such a discrete dictionary for a signal with N samples has a discrete chirplet dictionary with $O(N^2)$ elements. Therefore a matching pursuit algorithm [16] can be adapted from wavelet analysis. The main idea is that among all the atoms in the dictionary the one that matches best the data is retained. This gives a first term in the chirplet expansion. The approximation residual is then again approximated by the best chirplet from the dictionary, which gives a second term in the expansion etc. This algorithm has a complexity of the order $O(MN^2 \log N)$ to find M terms in the expansion. This is far too much to be practical. A faster $O(MN)$ algorithm based on local optimization has been published [13].

This approach somehow neglects the nice logarithmic and dyadic tiling of the plane that made more classical wavelets so attractive. So this kind of decomposition will be most appropriate when the signal is a composition of a number of chirplets. Such signals do exist

Figure 8: Chirplets for different rotation angles, $a = 0, 0.1, 0.5, 0.9, 1., 1.5$. Real part in solid line, imaginary part in dashed line.



like the example of a signal emitted by a bat which consists of 3 nearly parallel chirps in the (x,ξ)-plane. Other examples are found in seismic analysis. For more details we refer to [7]. An example in acoustic analysis was given in [13].

6 Two-dimensional fractional Fourier transforms

We know that the one-dimensional FrFT is defined by: $\mathcal{F}^a(f)(\xi) = \int_{-\infty}^{\infty} K_a(\xi,x)f(x)\mathrm{d}x$ where $K_a(\xi,x)$ is the kernel as in Theorem 3.1. The generalization of the FrFT to two dimension is given by:

$$(\mathcal{F}^{\mathbf{a}}f)(\boldsymbol{\xi}) = (\mathcal{F}^{a,b}f)(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{a,b}(\xi,\eta;x,y)f(x,y)\mathrm{d}x\mathrm{d}y$$

where $K_{a,b}(\xi,\eta;x,y) = K_a(\xi,x)K_b(\eta,y)$.

In the case of the two-dimensional FrFT we have to consider two angles of rotation $\alpha = a\pi/2$ and $\beta = b\pi/2$. If one of these angles is zero, the 2D transformation kernel reduces to the 1D transformation kernel. The FrFT can be extended for higher dimensions as:

$$(\mathcal{F}^{a_1,\ldots,a_n}f)(\xi_1,\ldots,\xi_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} K_{a_1,\ldots,a_n}(\xi_1,\ldots,\xi_n;x_1,\ldots,x_n)f(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n,$$

or shorter

$$(\mathcal{F}^{\mathbf{a}}f)(\boldsymbol{\xi}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} K_{\mathbf{a}}(\boldsymbol{\xi},\mathbf{x})f(\mathbf{x})\mathrm{d}\mathbf{x},$$

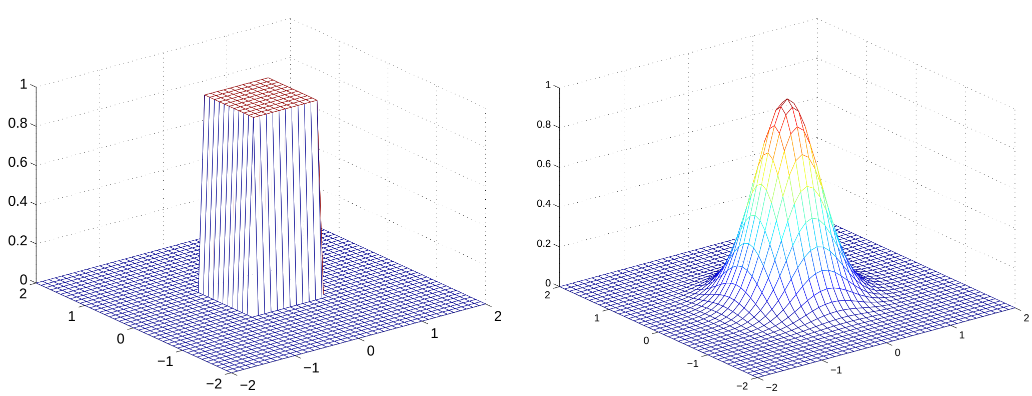
where

$$K_{\mathbf{a}}(\boldsymbol{\xi},\mathbf{x}) = K_{a_1,\ldots,a_n}(\xi_1,\ldots,\xi_n;x_1,\ldots,x_n) = K_{a_1}(\xi_1,x_1)K_{a_2}(\xi_2,x_2)\cdots K_{a_n}(\xi_n,x_n).$$

Moreover $\boldsymbol{\xi} = \xi_1\boldsymbol{\xi}_1 + \cdots + \xi_n\boldsymbol{\xi}_n$, $\mathbf{a} = a_1\boldsymbol{\xi}_1 + \cdots + a_n\boldsymbol{\xi}_n$, where $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n$ are units vectors in the $\xi_1,\ldots,\xi_n$ directions.

As an example consider the function $\mathrm{rect}(x)$, defined as the indicator function for the interval $[0,1)$. Figure 9 gives the magnitude of the 2D FrFT of the function $\mathrm{rect}(x,y) = \mathrm{rect}(x)\,\mathrm{rect}(y)$ with $a_x = a_y = 1$.

Figure 9: 2D function rect and the magnitude of its FrFT



The Wigner distribution of a 2D function $f(x, y)$, is defined as:

$$\mathcal{W}_f(x, y; \xi, \eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + \frac{u}{2}, y + \frac{v}{2}) \overline{f(x - \frac{u}{2}, y - \frac{v}{2})} e^{-i(\xi u + \eta v)} \mathrm{d}u\mathrm{d}v$$

The separable 2D linear canonical transform of $f(x, y)$, is defined as:

$$f_M(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_M(x, y; u, v) f(u, v) \mathrm{d}u\mathrm{d}v, \quad C_M(x, y; u, v) = C_{M_x}(x, u) \oplus C_{M_y}(y, v)$$

where

$$M = \begin{bmatrix} A_x & 0 & B_x & 0 \\ 0 & A_y & 0 & B_y \\ C_x & 0 & D_x & 0 \\ 0 & C_y & 0 & D_y \end{bmatrix}, \quad M_x = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix}, \quad M_y = \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix}.$$

Thus defined we may multiply the matrices associated with two transforms and find the matrix associated with their concatenation. The Wigner distribution of $f_M(x, y)$, is related to that of $f(x, y)$ according to:

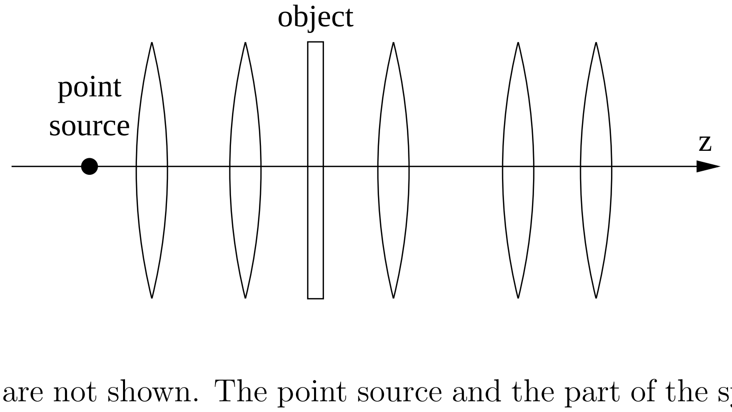
$$(\mathcal{W} f_M)(x, y; \xi, \eta) = (\mathcal{W} f)(D_x x - B_x \xi, D_y y - B_y \eta; -C_x x + A_x \xi, -C_y y + A_y \eta) \qquad (11)$$

Thus we have again as in the 1D case that $\mathcal{W} = \mathcal{M}_M \mathcal{W} \mathcal{C}_M$.

7 Optical systems and fractional Fourier transform

Just like Fourier analysis, the FrFT has some very close connections with optical systems. It is possible to build an optical system such that the output of the system is (under some ideal assumptions) exactly the Fourier transform of the input signal. For example, consider a system like depicted in Figure 10. It consists of several thin lenses and possibly other optical

Figure 10: Optical system



components which are not shown. The point source and the part of the system left from the object serves to illuminate the object. The part to the right of the object will generate a sequence of images. At a certain distance, one will see the inverted object. Before that, there is a certain place where one can observe the Fourier transform of the object, and to the right of the inverted image, one can observe an inverted Fourier transform, somewhat further an upright object etc. This corresponds to the fact that the system acts at certain distances as (integer) powers of the Fourier operator, so that we observe $\mathcal{F}^k$ for $k = 1, 2, 3, 4, \ldots$ at certain specific distances. However at the intermediate positions one may expect (and this is indeed the case) that we get some non-integer (i.e., fractional) powers of the Fourier operator. The places where the Fourier transform is observed are precisely those places where one would have observed the point source if there had not been an object in between.

Normally, one has to study optics in 2D planes and analyse these as one progresses along a z-axis. Under some ideal circumstances of monochromatic light, axial symmetry etc., it is sufficient to study the systems in one variable only. This is what we shall do in the sequel. For much more details on the optics we refer to the extensive treatment in [25].

In general, an optical system will be characterized by a certain kernel $h(x, x')$ and the input $f(x)$ will be transformed into the output $g(x)$ by a convolution: $g(x) = \int h(x, x')f(x')\mathrm{d}x'$. If we consider systems that consist only of thin lenses, free space, and quadratically graded-index media, then the transfer kernel has the special form

$$h(x, x') = \sqrt{b}e^{-i\pi/4} \exp[i\pi(ax^2 - 2bxx' + cx'^2)]. \tag{12}$$

These systems are called quadratic phase systems for obvious reasons and they are studied in Fourier optics. This formula resembles very much the kernel of the linear canonical transform described in Section 5.3. This illustrates that we do get a FrFT or a transform that is strongly related to it for appropriate values of the parameters.

A thin lens causes a phase-only transform. The kernel is in that case

$$h_{\text{lens}}(x, x') = \delta(x - x') \exp\left[-i\frac{\pi x^2}{\lambda f}\right]$$

where λ is the wavelength and f the focal length of the lens.

A transform caused by free space has a kernel of the form

$$h_{\text{space}}(x, x') = e^{i\pi\sigma d} e^{-i\pi/4} \frac{1}{\sqrt{\lambda d}} \exp\left[i\frac{\pi(x - x')^2}{\lambda d}\right].$$

In this case, d is the distance of free space the light travels through, and $\sigma = 1/\lambda$.

Both of the previous optical components form a special case of the general transform (12). However, the most pure form of the FrFT is realized by the quadratically-graded index media. This is a medium where the refractive index depends quadratically on x, namely $n^2(x) = n_0^2[1 - (x/\chi)^2]$. Here n_0 and χ are parameters of the medium. The transfer kernel is in this case

$$h_{\text{qgim}}(x, x') = e^{i2\pi\sigma d} \frac{e^{-id/2\chi}}{\sqrt{\lambda\chi}} A_\alpha \exp\left[\frac{i\pi}{\lambda\chi}(\cot\alpha\ x^2 - 2\csc\alpha\ xx' + \cot\alpha\ x'^2\right]$$

with $\sigma = 1/\lambda$, $\alpha = d/\chi$ and $A_\alpha = \sqrt{1 - i\cot\alpha}$. It is clear that up to some scaling and a multiplicative constant, this corresponds to a genuine FrFT with sweep rate $\cot\alpha$.

In general the system (12) is more general than the FrFT because it has 3 free parameters while the FrFT has only one. However, by allowing an extra scaling factor and some extra phase curvature parameter in the FrFT, one obtains precisely the more general linear canonical transform. Thus up to a rescaling of the variable and the phase factor, any quadratic phase system realizes a FrFT up to a constant. The interpretation as a linear canonical transform is most interesting because we can catch the free parameters in a matrix and the concatenation of several such optical elements corresponds to a matrix that is the product of the corresponding composing matrices.

8 The discrete FrFT

The purpose of this section to define a discrete FrFT (DFrFT), which is a discrete approximation of the continuous FrFT, just like the discrete FT (DFT) is a discrete version of the ordinary continuous FT. We also restrict ourselves to signals of finite duration. So we are working with vectors in $\mathbb{C}^N$. Of course this is of great importance for practical computations.

8.1 The DFT for finite signals and its eigenstructure

The DFT $\mathfrak{F}f$ of a vector $f = [f(0), \ldots, f(N-1)]^T$ is defined as the vector

$$(\mathfrak{F}f)(\nu) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-i2\pi\nu n/N} = \sum_{n=0}^{N-1} F(\nu, n) f(n) \quad \text{or} \quad \mathfrak{F}f = Ff.$$

Thus the DFT is obtained by multiplying the vector with the kernel matrix

$$F(\nu, n) = \frac{1}{\sqrt{N}} W^{\nu n}, \quad \nu, n = 0, \ldots, N-1, \quad W = e^{-i2\pi/N}.$$

The inverse is given by $(\mathfrak{F}^{-1} f_1) = F^* f_1$. This confirms that F is a unitary matrix ($F^{-1} = F^*$). Thus the matrix has unimodular eigenvalues and is diagonalizable. Since $\mathfrak{F}^4$ is the identity operator, $F^4 = I$, and therefore the eigenvalues are like in the continuous case $\lambda_n = \lambda^n$, with $\lambda = e^{-i\pi/2}$, i.e., $1, -i, -1, i$, each with a certain multiplicity depending on N modulo 4. However, they are not just λ^n with $n = 0, \ldots, N-1$. There is some anomaly for even N. Indeed, the eigenvalues turn out to be λ^n for $n \in \mathcal{N}$ where $\mathcal{N} = \{0, \ldots, N-1\}$ for N odd and $\mathcal{N} = \{0, \ldots, N-2, N\}$ for N even. Hence the table of multiplicities [19]

N	1	$-i$	-1	i
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	m	$m+1$	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

There must also exist a set of N independent eigenvectors (the discrete versions of the Hermite-Gauss functions). Like in the continuous case, the eigenvectors with respect to different eigenvalues will be orthogonal, but within one of the four eigenspaces there is some freedom to choose an orthonormal set.

Dickinson and Steiglitz [12] have found a procedure to construct a set of real orthogonal eigenvectors for F. They are constructed from the (real and orthogonal) eigenvectors of a real symmetric matrix H with distinct eigenvalues that commutes with F. If e is an eigenvector $He = \lambda e$, then from $HFe = FHe = \lambda Fe$, it follows that if Fe is also an eigenvector of H for the same eigenvalue, so that by simplicity of the eigenvalues of H, there must exist a constant β such that $Fe = \beta e$, i.e., e is also an eigenvector of F (with eigenvalue β). Thus the problem is reduced to the construction of such a matrix H and the construction of its eigenvectors. Such a matrix H turns out to have the form $H = D^2 + U^2$ where D^2 is a circulant matrix whose first row is $[-2, 1, 0, \ldots, 0, 1]$ (this is the matrix associated with the second difference operator and it is thus the discrete analog of the second derivative $\mathcal{D}^2$) and $U^2 = FD^2F^{-1}$ is a diagonal matrix $U^2 = 2\text{diag}(\Re(\lambda_n) - 1)$ (this is the discrete analog of $\mathcal{U}^2$). Recall that $\Re(\lambda_n) = \cos(\frac{2\pi}{N} n)$.

It is easily seen that H is real, symmetric and commutes with F. It is the discrete analog of the Hamiltonian $\mathcal{H}$. The matrix H has distinct eigenvalues, hence unique eigenvectors, for all N that are not a multiple of 4. If $N = 4m$, then there is a eigenvalue -4 with multiplicity 2, but an orthogonal system can still be constructed. Using the matrix V defined as

$$V = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccc|ccc} \sqrt{2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & 1 & & & & & 1 \\ \vdots & & \ddots & & & \iddots & \\ 0 & & & 1 & 1 & & \\ \hline 0 & & & 1 & -1 & & \\ \vdots & & \iddots & & & \ddots & \\ 0 & 1 & & & & & -1 \end{array} \right]$$

(sizes of blocks are 1, $\lfloor \frac{N}{2} \rfloor$, and $\lfloor \frac{N-1}{2} \rfloor$) we get after a similarity transform (note $V = V^T = V^{-1}$) a decoupled matrix that splits into two blocks

$$VHV = \left[\begin{array}{cc} E & 0 \\ 0 & O \end{array} \right]$$

with E and O symmetric tridiagonal matrices of size $\lfloor \frac{N}{2} \rfloor + 1$ and $\lfloor \frac{N-1}{2} \rfloor$ respectively. Thus the problem is reduced to finding the eigenvectors of E and O and multiply them (after appropriate zero padding) with V. The remaining problem is to find out which eigenvector of H should be associated with a particular eigenvalue of F. This is defined in terms of the number of zero crossings. This is like in the continuous case where ϕ_n corresponding to the eigenvalue $(-i)^n$ has n zeros. In the discrete case, the nth eigenvector corresponding to $(-i)^n$ should have n zero crossings where the eigenvector is cyclically extended and a zero crossing corresponds to the fact that two successive components differ in sign: $h(n)h(n + 1) < 0$ for $n = 0, \ldots, N - 1$ (with $h(N) = h(0)$). However, it turns out that ordering the eigenvectors with respect to the eigenvalues they have as eigenvectors of H is precisely the ordering we need. Note that the fact that we have now uniquely defined a set of orthogonal eigenvectors is not in contradiction with the fact that we said earlier that there are infinitely many sets of orthogonal eigenvectors. The condition of being even or odd actually fixes them uniquely (up to sign change).

So now we know the eigenvalue decomposition of F: $F = G\Lambda G^T$ with $G = [g_0, \ldots, g_{N-1}]$ the orthogonal matrix of eigenvectors and $\Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$, $(\lambda_n : n = 0, \ldots, N-1) = (e^{-in\pi/2} : n \in \mathcal{N})$, $\mathcal{N}$ as defined above.

8.2 Definition and properties of the DFrFT

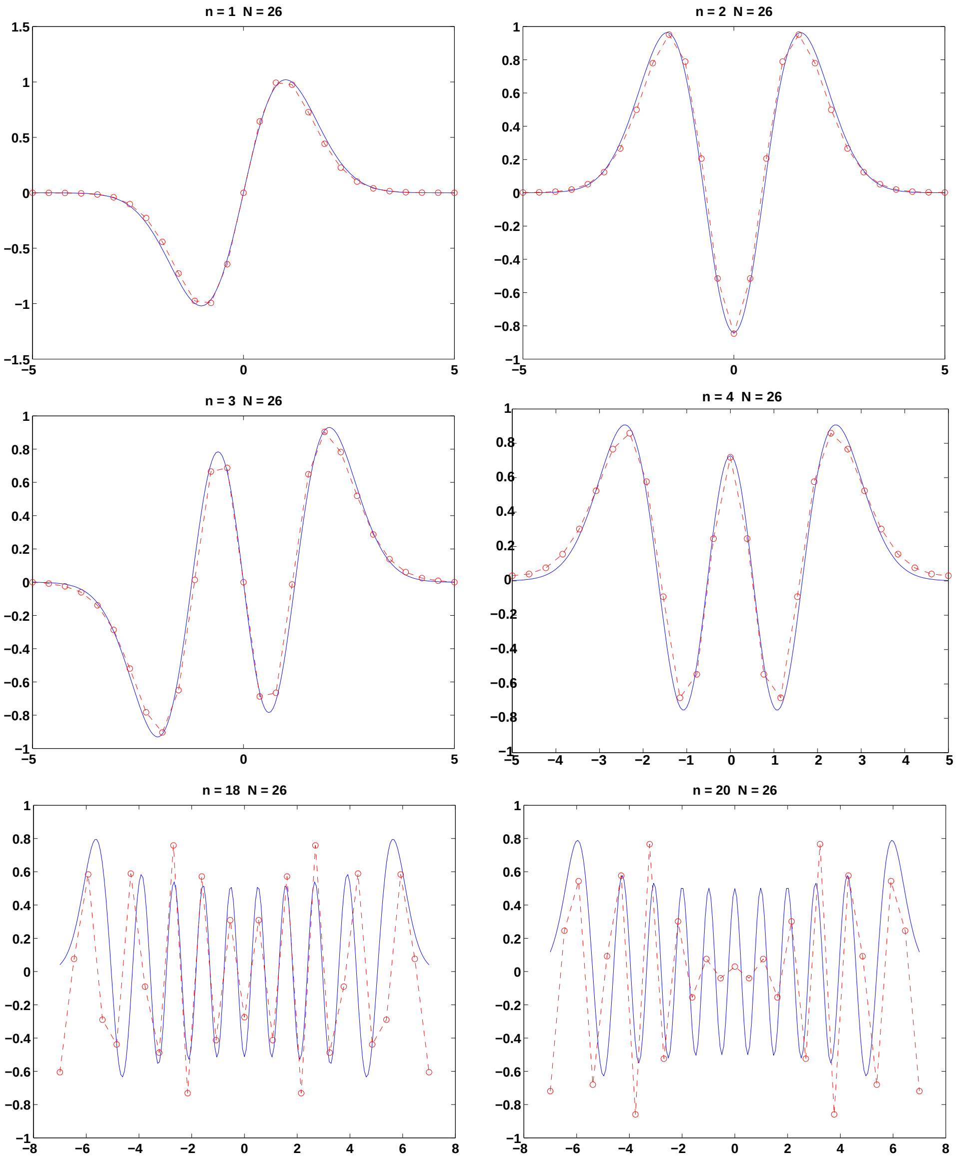
The definition of the DFrFT is now exactly like in the continuous case:

$$\mathfrak{F}^a f = F^a f = G\Lambda^a G^T f.$$

As before we shall denote by an index the DFrFT of order a: $f_a = \mathfrak{F}^a f_0$ or more generally $f_{a+b} = \mathfrak{F}^b f_a$. Also the index variable will be denoted as n_a for the a-domain. Thus for example

$$(\mathfrak{F}^{a+b} f)(n_{a+b}) = \sum_{n_a=0}^{N-1} F^b(n_b, n_a) f_a(n_a).$$

Figure 11: Some continuous Hermite-Gauss functions ϕ_n, $n = 1, 2, 3, 4, 18, 20$ and the corresponding eigenvectors (indicated with circles) for the matrix H of size $N = 26$. Note that for smaller n, the approximation is better.



It should be obvious that with this similarity in definition, it also follows that almost all the properties holding for the continuous FrFT also holds for the DFrFT. For example, all the properties listed at the end of Section 3.1 still hold for the DFrFT. So we have e.g.

1. **linearity:** $\mathfrak{F}^a(f+g) = \mathfrak{F}^a f + \mathfrak{F}^a g$.
2. **index additivity:** $\mathfrak{F}^{a+b} = \mathfrak{F}^b\mathfrak{F}^a = \mathfrak{F}^a\mathfrak{F}^b$.
3. **special cases:** $\mathfrak{F}^1$ is the ordinary DFT, $\mathfrak{F}^2$ is the parity operator.
4. **unitarity:** $(\mathfrak{F}^a)^{-1} = (\mathfrak{F}^a)^*$.
5. **conservation of symmetry:** an even (odd) vector is transformed into an even (odd) vector.
6. **Parseval:** $f^*g = f_a^*g_a$.

8.3 Relation between the FrFT and the DFrFT

As we have seen above, there are many properties that are transfered from the FrFT to the DFrFT. We can ask how good the DFrFT will be an approximation of the FrFT. This is closely related with how good the discrete versions of out Hermite-Gauss functions will approximate the continuous ones.

8.3.1 Discrete Mathieu functions

To start with, from what we have seen above, it is directly seen that the continuous Hermite-Gauss function ϕ, corresponding to the eigenvalue λ satisfies the differential equation $\mathcal{H}\phi = \lambda\phi$ or equivalently

$$(\mathcal{D}^2 + \mathcal{U}^2)\phi(x) = -(2\lambda+1)\phi(x), \tag{13}$$

with $\mathcal{D}$ and $\mathcal{U}$ as defined before.

The eigenvector g of H corresponding to the eigenvalue λ satisfies however the difference equation

$$\Delta^2 g(n) + [2\cos(n\tfrac{2\pi}{N}) - (\lambda-2)]g(n) = 0 \tag{14}$$

Where Δ is the difference operator $\Delta g(n) = g(n+\frac{1}{2}) - g(n-\frac{1}{2})$, and hence $\Delta^2 g(n) = g(n+1) - 2g(n) + g(n-1)$, where the indices are taken modulo N. Recall the circulant matrix D^2 which is the kernel matrix for Δ^2. Similarly $\mathfrak{U}^2 = \mathfrak{F}\Delta^2\mathfrak{F}^{-1}$ gives

$$\mathfrak{U}^2 f(n) = \left[e^{in\frac{2\pi}{N}} - 2 + e^{-in\frac{2\pi}{N}}\right] f(n) = 2\left[\cos(n\tfrac{2\pi}{N}) - 1\right] f(n).$$

It is easily seen that our previously introduced matrix U^2 is the kernel matrix for $\mathfrak{U}^2$. Thus $\mathfrak{U}$ is not a complex shift, but only an approximation to it because

$$\mathfrak{U}^2 f(n) = 2\left(\cos(n\tfrac{2\pi}{N}) - 1\right) f(n) = \left(-(\tfrac{2n\pi}{N})^2 + \cdots\right) f(n) \approx -(\tfrac{2n\pi}{N})^2 f(n).$$

The continuous counterpart of (14) is therefore

$$\mathcal{D}^2\psi(x) + 2[\cos(2\pi x) + 1]\psi(x) = (\lambda+4)\psi(x). \tag{15}$$

The periodic solutions of this equation are Mathieu functions [12]. In other words, one may not expect that the eigenvectors have components that are samples of the continuous Hermite-Gauss functions. However Figure 11 does show a close relationship between them. So the question can be raised how good the eigenvectors approximate the continuous Hermite-Gauss functions, since this is important for the numerical computation of the continuous FrFT.

8.3.2 DFrFT by orthogonal projection

This problem has been discussed in [27]. If we go from FT to DFT, then an integral over the whole real line is approximated by a summation over N points. First, the continuous Hermite-Gauss functions have an infinite support while the eigenvectors have only N components. However, the continuous Hermite-Gauss functions decay like $t^n e^{-t^2}$. So for practical purposes, they can be considered as zero outside a finite interval. However, for larger n, the decay is slower and hence the approximation is worse for the same interval. It turns out that a sampling frequency $T = \sqrt{2\pi/N}$ is precisely what one should use to obtain the same variance for continuous and discrete versions of the Hermite-Gauss functions. Secondly, there is the integral that has been replaced by a summation which is in fact a trapezoidal approximation. Both the truncation error for the interval and the integration error of the trapezoidal rule will go to zero when N tends to infinity. Hence it may not come as a surprise that the eigenvector in $Fg_n = \lambda_n g_n$ for the DFT can be approximated by samples of the continuous Hermite-Gauss function ϕ_n. The following theorem was shown in [27].

Theorem 8.1 *Let ϕ_n be the continuous Hermite-Gauss function and set $T = \sqrt{2\pi/N}$. Define*

$$\tilde{\phi}_n(k) = \begin{cases} \phi_n(kT), & \text{for } 0 \le k \le \lfloor \frac{N-1}{2} \rfloor \\ \phi_n((k-N)T), & \text{for } \lfloor \frac{N-1}{2} \rfloor + 1 \le k \le N-1, \end{cases}$$

then it holds that for N large $F\tilde{\phi}_n \approx \lambda_n \tilde{\phi}_n$, where F is the DFT kernel matrix.

That is why in [27] another DFrFT algorithm is proposed.

1. construct the vectors $\tilde{\phi}_n$ as in the previous theorem
2. construct the eigenvectors g_n of H as in the previous paragraph
3. for $k = 0, 1, 2, 3$, project the vectors $\tilde{\phi}_{4m+k}$, $k = 0, 1, \ldots$ onto the kth eigenspace
4. re-orthogonalize the projected vectors within each eigenspace

The orthogonalization across eigenspaces is not necessary because that orthogonality is automatic. One could also do the re-orthogonalization while at the same time minimizing the distance from the original $\tilde{\phi}_n$ vectors. This procedure requires a singular value decomposition, but for n close to $N-1$ the results are then better than without the projection.

8.3.3 Higher order approximants

Another refinement that could be interesting when it is the intention to approximate the FrFT accurately is to use better approximations for $\mathcal{D}^2$ and $\mathcal{U}^2$. So if $\tilde{D}^2$ is a matrix kernel corresponding to a higher order approximation $\tilde{\Delta}^2$ for $\mathcal{D}^2$ and if we set $\tilde{U}^2 = F\tilde{D}^2F^{-1}$, then the same analysis as the one done in the last two paragraphs goes true, but better approximations of the continuous Hermite-Gauss functions are obtained. For more details see [10, 25].

8.4 Discretization of the FrFT

If the previous approximations of the FrFT are not sufficient, one could directly discretize the integral transform. Because the highly oscillating character of the kernel $K_a(\xi, x)$ for values of a that are close to an even integer, (see Figure 2) it is clear that this should be avoided. Therefore it is best to evaluate the FrFT only for a in the interval $[0.5, 1.5]$ and to use the relation $\mathcal{F}^a = \mathcal{F}\mathcal{F}^{1-a}$ for $a \in [0, 0.5) \cup (1.5, 2]$.

Still, the direct evaluation of the integral or the spectral decomposition of the FrFT by computing its Hermite-Gauss functions is computationally expensive. So the problem could be formulated, independent from the DFrFT, as a problem of transforming samples of the signal into samples of its FrFT. This has been discussed in [23]. The method assumes that the signal is band limited within $[-B, B]$. The signal can then be sampled in the points $l/2B$ and the result is an $O(N \log N)$ procedure with $N = B^2$ which takes the form

$$f_a(k/2B) \approx Cc(k) \sum_{l=-N}^{N-1} r(k-l)s(l).$$

C is a constant, $c(k)$ are samples of a chirp, r is a vector depending on a, and s is a vector depending on a and samples of the given signal f. Thus we have to compute a convolution of the s and r vectors, which has an $O(N \log N)$ complexity, followed by a chirp multiplication.

A brief derivation is given in the appendix.

9 Other definitions of the (D)FrFT

Since there are many possible definitions of the FrFT possible, many of which are equivalent, there is also a variety of possible definitions of DFrFT which may or may not be equivalent.

An older definition given in [32] is based on the fact that there exist certain expressions $p_i(a)$ such that

$$\mathcal{F}^a = \sum_{k=0}^{3} p_k(a)\mathcal{F}^k, \quad p_k(a) = \frac{1}{4}\frac{1-e^{i2\pi a}}{1-e^{i\frac{\pi}{2}(a-k)}}. \tag{16}$$

This is known to be different form the definition given before. In fact, depending on the choice of the eigenvectors and the order of the eigenvalues, there is a multitude of possible definitions for the FrFT and these may not always be equivalent.

A unifying framework has been given recently [9] which takes care of these two ambiguities in the definition (7):

1. the eigenvalues can be reordered

2. the orthogonal basis ϕ_n is not unique.

So the unifying definition becomes

$$\mathcal{F} = \mathcal{T}_\psi^* \mathcal{S}_\mu^a \mathcal{T}_\psi,$$

where $\mu = (\mu_n = e^{-i\frac{\pi}{2}(n+4q_n)} : n = 0, 1, \ldots)$, with $q_n \in \mathbb{Z}$ an arbitrary sequence and

$$\psi_{4n+k} = \sum_{m=0}^{\infty} \alpha_{4m+k,4n+k} \phi_{4m+k},$$

i.e.,

$$[\psi_0, \psi_1, \psi_2, \ldots] = [\phi_0, \phi_1, \phi_2, \ldots]A, \quad A = [\alpha_{m,n}]_0^\infty.$$

The sequence $\gamma_n = n + 4q_n$ is called the generating sequence and the double sequence $\{\alpha_{m,n}\}$ is called a perturbing sequence.

The choice of generating sequence with $q_n = 0$ and using the standard basis ϕ_n leads to the definition used here, while for $q_n = \lfloor n/4 \rfloor$, we get the definition (16) whatever the basis is. We shall not go into the details here.

Of course the above alternatives can be adapted to the DFrFT case, which we shall not elaborate either.

10 Applications of the FrFT

The FrFT has many applications in signal processing (especially optics). It is of course impossible to cover all of them, we just give a selection.

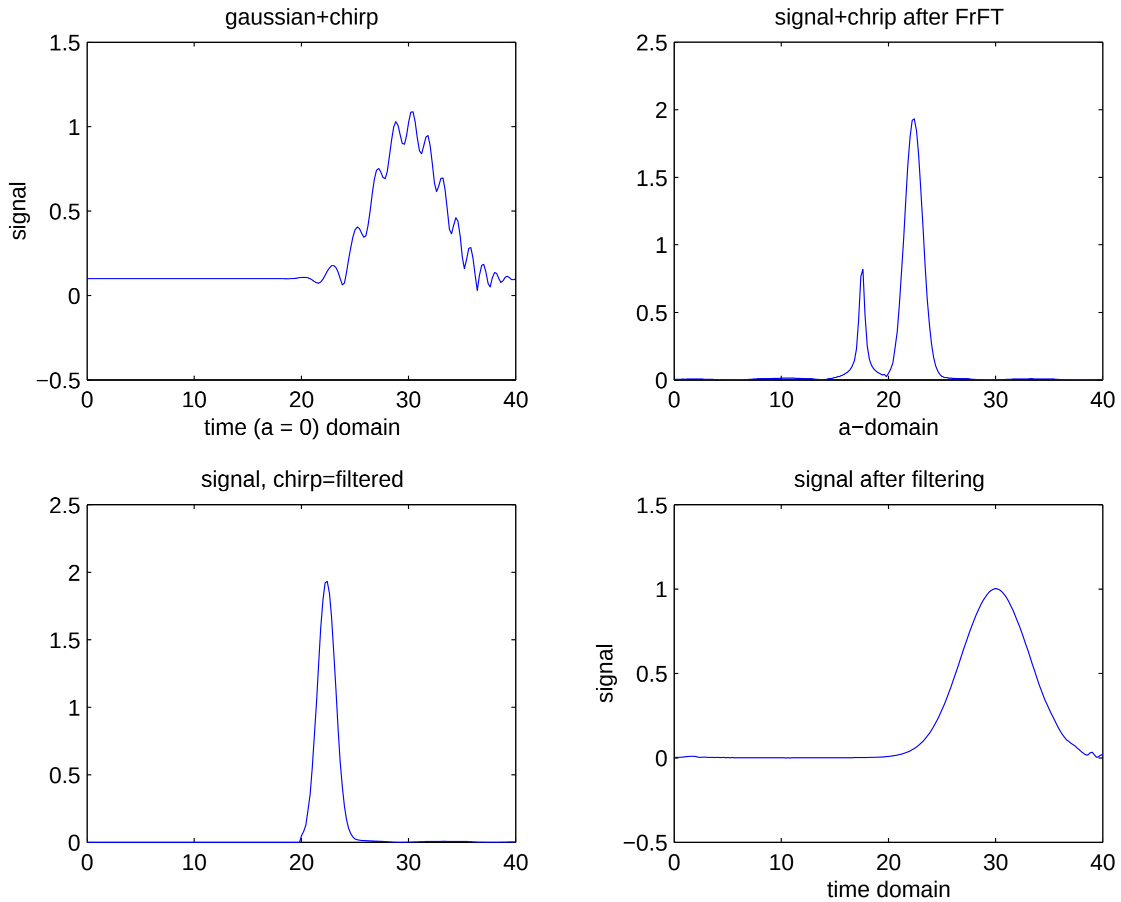
10.1 Filtering in the fractional Fourier domain

In some signal processing applications we have to identify a number of chirps that are mixed up in one signal, or a certain signal may be corrupted by chirps that we want to eliminate. Here we describe an algorithm to filter out a chirp with sweep rate, α.

1. Given a signal $f(t)$ plus chirp function with sweep rate α
2. calculate the $\mathcal{F}^a$ transform
3. multiply by a stop band filter
4. apply the inverse FrFT $\mathcal{F}^{-a}$

We recall that the FrFT maps a chirp function (signal), from the $a = 0$-domain to a delta function in the a-domain if a corresponds exactly to the sweep rate of the chirp. By the stop band filters we remove the delta function as well as possible, so that step 4 brings the filtered signal back to the original domain. A graphical example of this algorithm is given in Figure 12, where the chirp function is given by: $0.1 \exp\{i(t^2/10 - 2t)\}$ and the signal is a Gaussian $\exp\{-(t-30)^2/20\}$, with a time slot $(0, 40)$ and a sampling frequency 100 Hz. Figure 13 gives the Wigner distributions that correspond to the previous pictures. It is seen that the first FrFT rotates the distribution in such a way that the projection onto the horizontal axis (the a-domain) separates the signal from the chirp (top right). The chirp can now be removed (bottom left) and after back transformation (bottom right), only the signal remains.

Figure 12: Filtering of a signal. The signal is a Gaussian: $\exp(-(t-30)^2/20)$ corrupted by a chirp: $0.1\exp(i(t^2/10-2t))$. All figures contain absolute values.



If the noise is a real chirp like $0.2\cos(t^2/10-2t)$, then we have to filter in the a and the $-a$ domain. Indeed, the cosine consists of the sum of two complex chirps: one to be filtered out in the a-domain, the other one in the $-a$ domain. This is illustrated in Figure 14.

Of course this idea can be extended, and by several FrFT's and corresponding deletion of a part of the measured signal, one can isolate a signal that has a Wigner distribution in a convex polygon of the time-frequency plane. This is illustrated in Figure 16

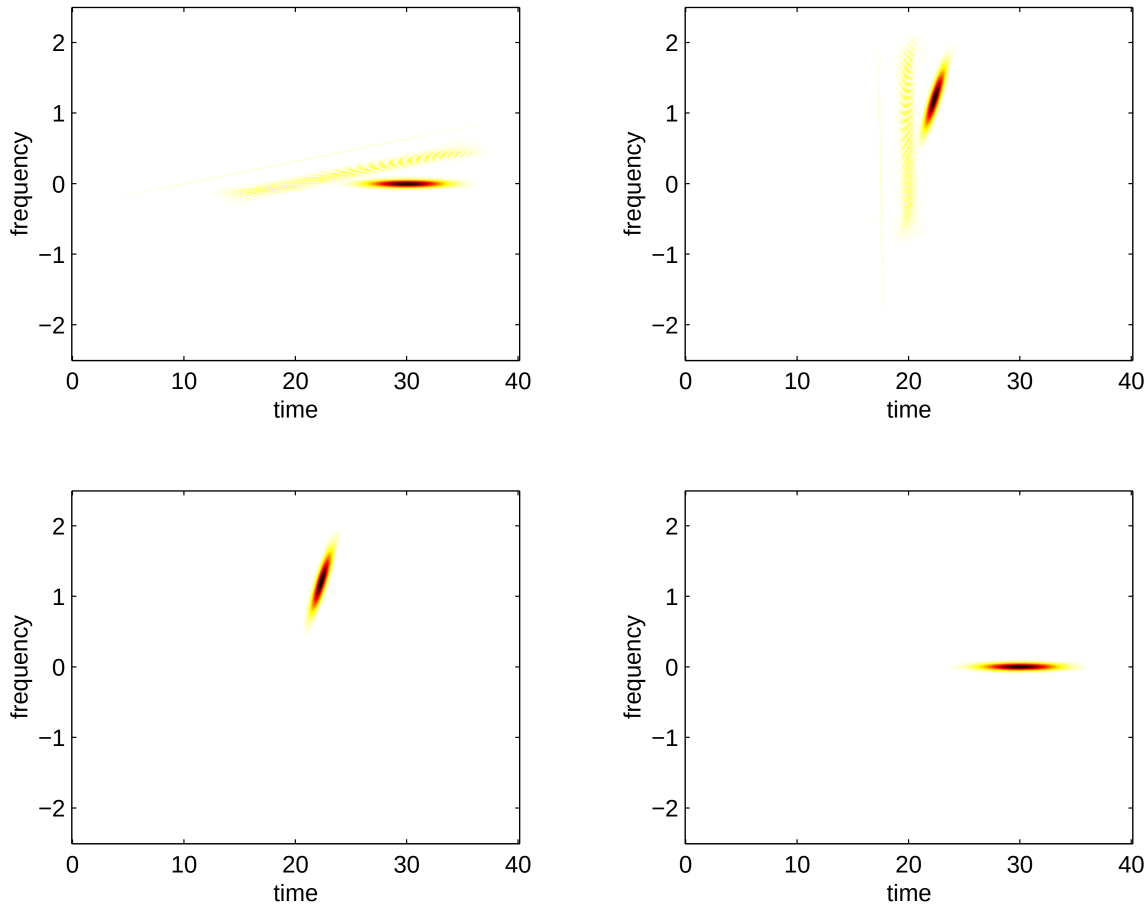
10.2 Signal recovery

Assume that a signal $f(x)$ is to be recovered from observations $g(x)$ which are given by

$$g(x) = \int D(x,x')f(x')\mathrm{d}x' + n(x)$$

where $D(x,x')$ is a distortion kernel and $n(x)$ is a noise signal. The idea is to apply the technique that we described before for the removal of a chirp component: First rotate ($\mathcal{F}^a$), then multiply with a filter function h (e.g. the rect function to isolate a particular part) and then back-rotate ($\mathcal{F}^{-a}$), so that the filtering operation becomes $\mathcal{F}^{-a}\mathcal{M}_h\mathcal{F}^a$ where $\mathcal{M}_h$ represents the multiplication with h. For example an ideal classical bandpass filter uses $a = 1$

Figure 13: Filtering of a signal, Wigner distributions. The signal is a Gaussian: $\exp(-(t-30)^2/20)$ corrupted by a chirp: $0.1\exp(i(t^2/10-2t))$.



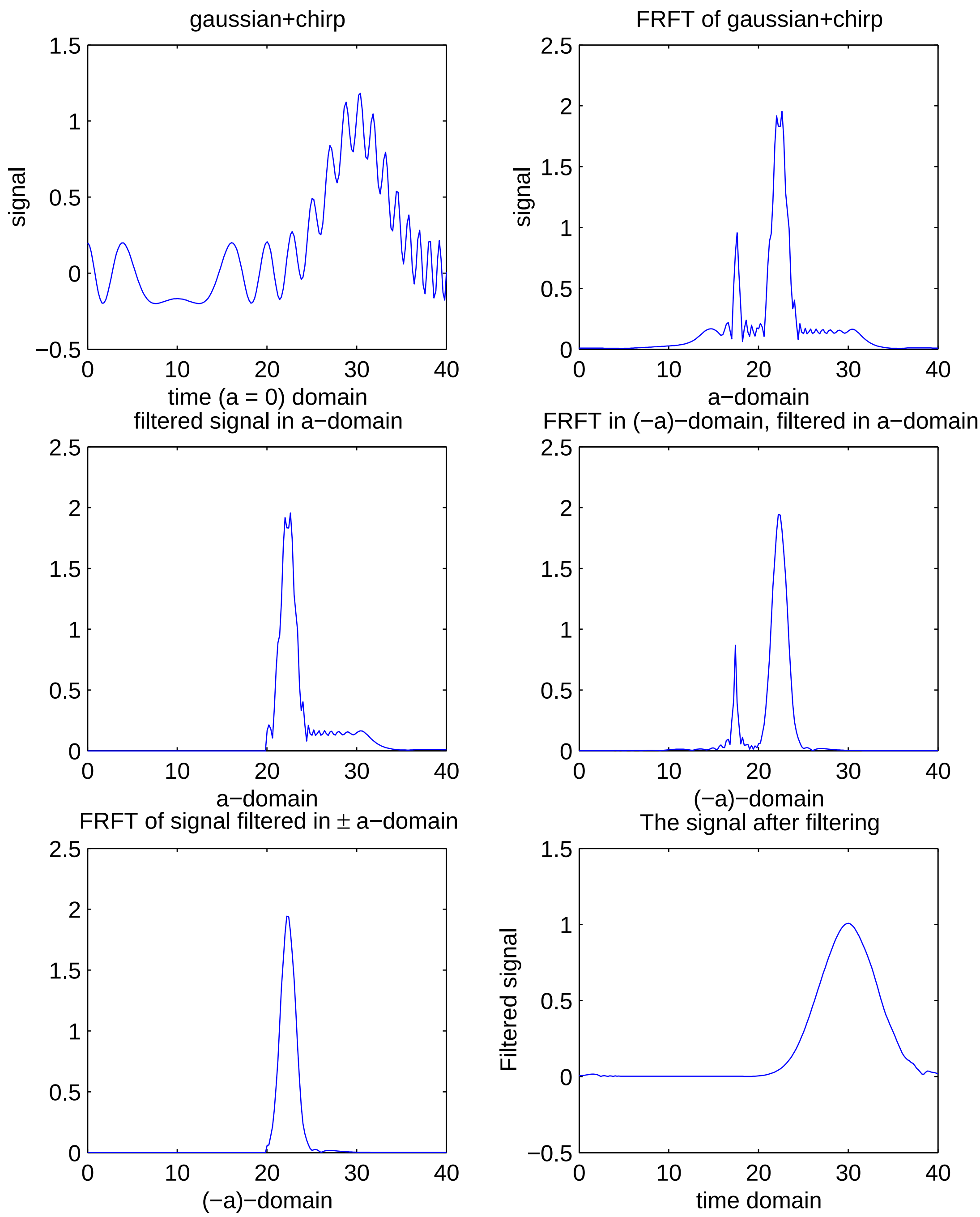
and a rect function for h. Whatever the rotation $\mathcal{F}^a$ is, the optimal multiplicative filtering function can be found from the autocorrelation function for f and n, which we assume to be known. This is based on classical Wiener filter theory. The remaining problem is to find the optimal rotation to be inserted, i.e., to find an optimal a. This is done by computing the filtered signal for several a-values and taking the one that gives the minimal error for the data given. Several variations are possible e.g., by doing a sequence of such transformations (like we did in the example for the real chirp removal) or we can do them in parallel and recombine them afterwards. For more details we refer to [15, 25] and the many references they contain.

10.3 Signal recognition

The convolution of two signals f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(t-x)\mathrm{d}t.$$

Figure 14: Filtering of the signal. The signal is a Gaussian: $\exp(-(t-30)^2/20)$, the noise is a real chirp: $0.2\cos((t^2/2-2t))$ All figures contain absolute values.

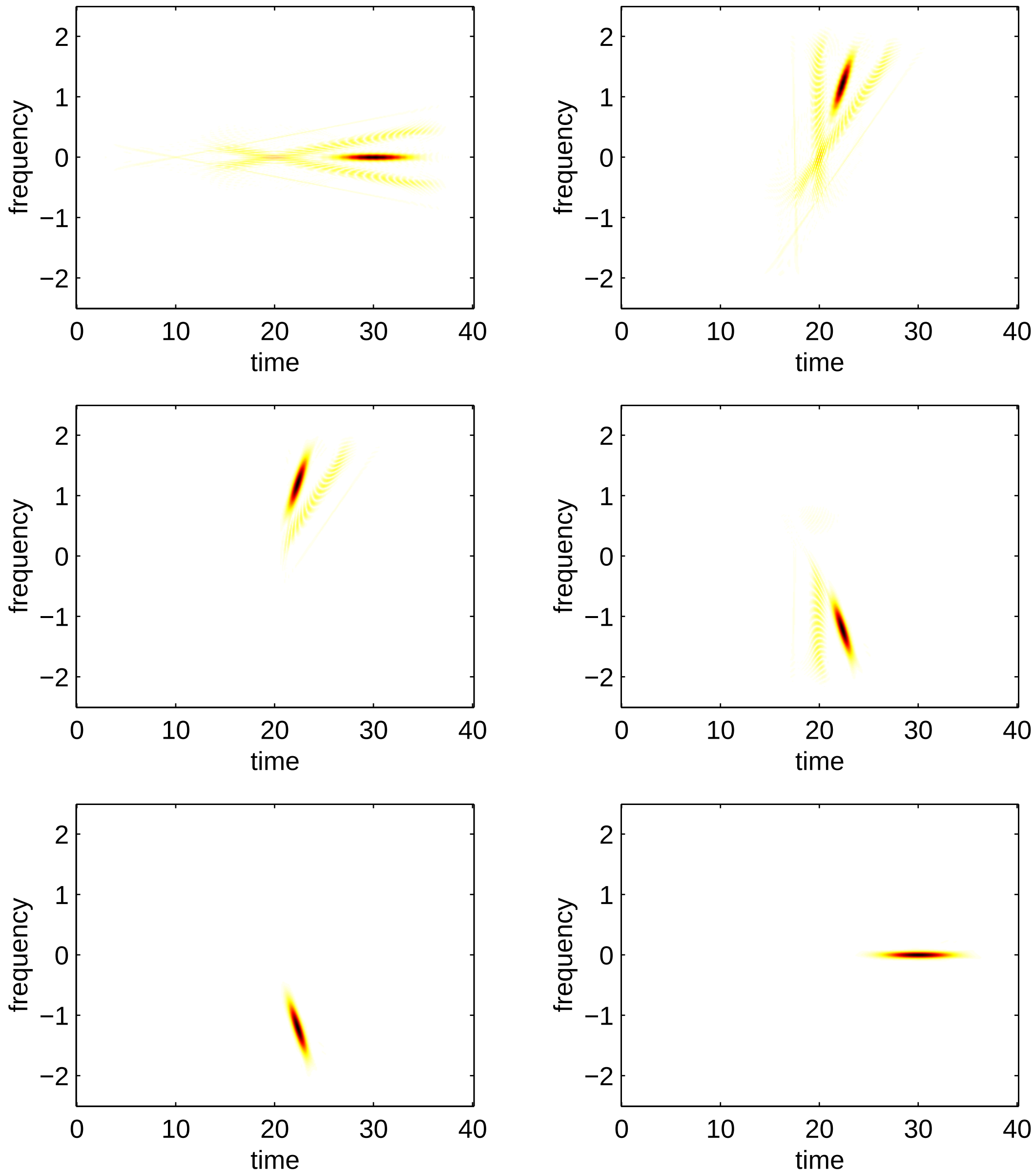


The cross correlation of these signals is the convolution of $f(x)$ and $\tilde{g}(x) = \overline{g(-x)} = \overline{(\mathcal{F}^2 g)(x)}$:

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t)\overline{g(x-t)}\mathrm{d}t.$$

Therefore $\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$ and $\mathcal{F}(f \star g) = (\mathcal{F}f)\overline{(\mathcal{F}g)}$. For the case that $f = g$, $f \star f$ is called the *autocorrelation function* and $\mathcal{F}(f \star f) = |(\mathcal{F}f)|^2$.

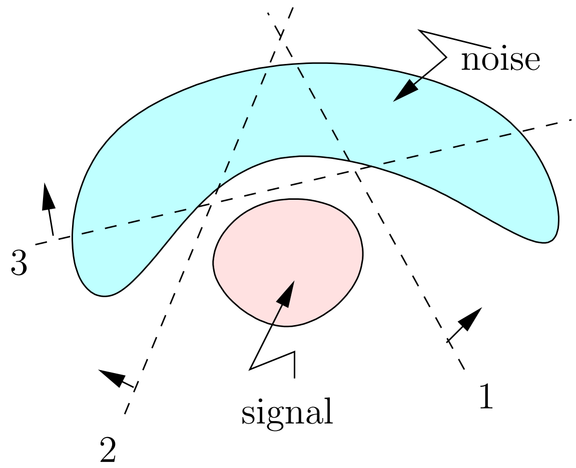
Figure 15: Wigner distributions corresponding to Figure 14



Thus, the autocorrelation function of a signal being the convolution of the signal with the complex conjugate of a shifted and x-inverted version of itself, it is obvious that the unshifted version will give a peak in the autocorrelation function at the origin.

To see if a signal f, matches another signal g, we compute again the correlation $h = f \star g$ and the signal f will more-or-less match the signal g if there is a distinct peak in the function h. This makes it possible to recognize a certain signal or image, when it matches a certain reference signal or image. The correlation operator is known to be shift invariant: if either f or g is shifted, then the correlation will be shifted over the same amount. Thus the signal will still be recognized if it is shifted. Thus shift invariance can be an advantage, but it can also be a disadvantage if one wants the detection to be local. In that case the fractional correlation can be useful because the parameter a will be a parameter to control

Figure 16: Filtering out noise by multiple masking in 3 FrFT domains. Do 3 FrFT $\mathcal{F}^a$ with a-domain orthogonal to the directions 1, 2 and 3 and rmove everything that is at one side of the dashed line indicated by the arrow.



the shift invariance. This motivates the definition of a fractional convolution and a fractional correlation operation.

A simple generalization of the convolution is the *fractional convolution* defined as $f *_a g = \mathcal{F}^{-a}[(\mathcal{F}^a f)(\mathcal{F}^a g)]$ and the *fractional correlation* is defined as $f \star_a g = \mathcal{F}^{-a}[(\mathcal{F}^a f)\overline{(\mathcal{F}^a g)}]$. For more details on fractional convolution and fractional correlation see [1].

As we mentioned above, the case $a = 1$ gives a shift invariant correlation. There will be a peak, giving a certain maximal value, and when the signal is shifted, we shall get the same peak, with the same magnitude at a shifted position. However, the fractional correlation will not be shift invariant in general. There will still be a shift of the peak corresponding to the shift of one of the signals, but the peak will also be fading out; the peak height will decrease and its variance will increase. Thus matching will only be recognized when the signals are unshifted or only slightly shifted. This is in fact a continuous process: no fade out for a an odd integer and infinite fade out for a an even integer.

We can even make this process adaptive. For example to match a fingerprint with an image from a database, we can imagine that the central part of the print will be precise, so that we want a good match, with only a bit of shift invariance ($a = 0.8$ say) while at the borders of the image, the print may be corrupted by a lot of distortions, so that there shift invariance is much more appropriate (so that we may take there $a = 1$). Note that in an image we can allow a different tolerance in shift invariance for the two directions. For example if one wants to recognize letters on a single line, but not the ones on the previous or the next lines. See [25].

In [1], it is also shown how the fractional autocorrelation function can be used to analyse a noise corrupted signal composed of several chirps represented in the (x, ξ)-plane as straight lines through the origin. Instead of computing a Radon transform of the Wigner distribution or the ambiguity function, one can more efficiently compute a fractional convolution or autocorrelation function in several directions. This will define a function of the direction that will show a clear peak when the direction coincides with the direction of one of the chirps in the signal.

10.4 Multiplexing

The chirplet application that we mentioned before shows that they provide some special tiling of the (x, ξ)-plane. If a complex signal, or several signals from several users, has to be transmitted over a communication channel, then there will be a limitation in time and frequency, and the messages have to be transmitted within this window in the (x, ξ)-plane. Multiplexing is a method by which that window is tiled and messages are assigned to each of these tiles. For example in TDMA (time division multiple access), the window is subdivided in vertical stripes, which means that each message gets the whole bandwidth for a certain amount of time. In FDMA (frequency division multiple access), the window is subdivided in horizontal stripes, so that each message has part of the bandwidth available during the whole time. Of course the window can be tiled in other ways like the wavelet tiling or the chirplet tiling and each message has such a tile available for transmission. Of course the basis functions should be orthogonal to each other so that their energy in the time-frequency plane is well separated to avoid interference of the different messages. The chirplet tiling can be appropriate for signals whose energy distribution has a well defined orientation in the time-frequency plane.

11 Other fractional transforms

Using the same idea as in our definition of the FrFT, there is a wide class of fractional transformations that can be obtained besides the FrFT. The general idea is that if we have a linear operator $\mathcal{T}$ in a complex separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and if there is a complete set of orthonormal eigenvectors ϕ_n with corresponding eigenvalues λ_n, then any element in the space can be represented as $f = \sum_{n=0}^{\infty} a_n \phi_n$, $a_n = \langle f, \phi_n \rangle$, so that $(\mathcal{T} f) = \sum_{n=0}^{\infty} a_n \lambda_n \phi_n$. The fractional transform can the be defined as

$$(\mathcal{T}^a f)(\xi) = \sum_{n=0}^{\infty} a_n \lambda_n^a \phi_n(\xi) = \sum_{n=0}^{\infty} \lambda_n^a \langle f, \phi_n \rangle \phi_n(\xi) = \langle f, K_a(\xi, \cdot) \rangle ,$$

where

$$K_a(\xi, x) = \sum_{n=0}^{\infty} \overline{\lambda}_n^a \overline{\phi_n(\xi)} \phi_n(x).$$

Of course some of these operations require some condition like for example if it concerns the Hilbert space $L^2_\mu(I)$ of square integrable functions on an interval I with respect to a measure μ, then we need $K_a(\xi, \cdot)$ to be in this space, which means that $\sum_{n=0}^{\infty} |\lambda_n|^{2a} |\phi_n(\xi)|^2 < \infty$ for all ξ.

In view of the general development for the construction of fractional transforms, it is clear that the main objective is to find a set on orthonormal eigenfunctions for the transform that one wants to "fractionalize". There were several papers that give eigenvalues and eigenvectors for transforms. Zayed [38] studied several of these transforms.

The Mellin transform defined as

$$F(\xi) = \int_0^\infty x^{\xi - 1} f(x) \mathrm{d}x.$$

The Hankel transform, also called Fourier-Bessel transform

$$F(\xi) = \int_0^\infty f(x) J_n(\sqrt{\xi x}) \mathrm{d}x$$

with J_n the Bessel function of the first kind. And two more transforms: one whose eigenfunctions are related to Jacobi polynomials, and another one called Riemann-Liouville integrals, which have complex exponentials as eigenfunctions.

Several other papers are dealing with discrete transforms: discrete sine and cosine transforms and thus also Hartley (sine + cosine) transforms [31, 26], generalized discrete Fourier, Hartley, sine and cosine transforms [31, 8, 34] discrete Hadamard transform [30], discrete Fourier-Kravchuk transform [5] and we probably missed some.

12 Conclusion

Now that the basic ideas have appeared in the literature, more applied researchers start to exploit the applicability of the FrFT in several type of applications. This route and the algorithmic an computational details are yet to be explored more deeply. But as always, these ideas spread to other integral transforms and there are even further generalizations like the linear canonical transforms. Also here the theory and applications are still open for intensive research.

We hope that with this survey we have interested some of the readers to have a closer look at the literature of fractional transforms. The younger ones may find a gold mine of beautiful results to be discovered, and for those who do not want to change a domain chosen, we hope that it has given an impression of what is going on in this quickly growing field.

Appendix A: Some of the proofs

A.1 Proof of the properties in Table 1 of Section 3.4

(1) This proof is trivial, using the integral representation and the property of the delta function.

(2) This follows from the previous result and the fact that $\mathcal{F}(\delta(\cdot)) = 1$:

$$\mathcal{F}^a(1) = \mathcal{F}^a(\mathcal{F}(\delta(\cdot))) = \mathcal{F}^{a+1}(\delta(\cdot)).$$

Hence using (1) for a replaced by $a+1$ and $\gamma = 0$ we immediately get the result.

(3) For this we need some trigonometric identities and the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{i}{2}[(\chi+\cot\alpha)x^2+(\gamma-\xi\csc\alpha)2x]} dx = \frac{e^{\frac{i\pi}{4}}}{\sqrt{\chi+\cot\alpha}} e^{-\frac{i}{2}\frac{(\gamma-\xi\csc\alpha)^2}{\chi+\cot\alpha}}$$

(see [25, p. 57]). It then easily follows that

$$\begin{aligned}
\mathcal{F}^a(e^{\frac{i}{2}(\chi x^2+2\gamma x)})(\xi) &= \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(\xi^2\cot\alpha-2\xi x\csc\alpha+x^2\cot\alpha)} e^{\frac{i}{2}(\chi x^2+2\gamma x)} \mathrm{d}x \\
&= \sqrt{1-i\cot\alpha}\, e^{\frac{i}{2}(\xi^2\cot\alpha)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{i}{2}[(\chi+\cot\alpha)x^2+(\gamma-\xi\csc\alpha)2x]} \mathrm{d}x \\
&= \sqrt{\frac{1+i\tan\alpha}{1+\chi\tan\alpha}} e^{\frac{i}{2}\left(\frac{\xi^2(\chi-\tan\alpha)+2\gamma\xi\sec\alpha-\gamma^2\tan\alpha}{1+\chi\tan\alpha}\right)}
\end{aligned}$$

(4) Again this can be used using trigonometric identities and the relation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{i}{2}[-(i\chi+\cot\alpha)x^2+(-i\gamma+\xi\csc\alpha)2x]}\mathrm{d}x = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{i\chi+\cot\alpha}}e^{\frac{i}{2}\left(\frac{(\xi-u\csc\alpha)^2}{\chi+\cot\alpha}\right)}$$

(see [25, p. 57]).

$$\begin{aligned}\mathcal{F}^a(e^{-\frac{1}{2}(\chi x^2+2\gamma x)}) &= \sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{i}{2}(-\xi^2\cot\alpha+2\xi x\csc\alpha-x^2\cot\alpha)}e^{\frac{i}{2}(i\chi x^2+2i\gamma x)}\mathrm{d}x\\ &= \sqrt{1-i\cot\alpha}e^{-\frac{i}{2}(-\xi^2\cot\alpha)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{i}{2}[-(i\chi+\cot\alpha)x^2+(-i\gamma+\xi\csc\alpha)2x]}\mathrm{d}x\\ &= \sqrt{\frac{1-i\cot\alpha}{\chi-i\cot\alpha}}e^{\frac{i}{2}\xi^2\cot\alpha\frac{(\chi^2-1)+2\xi\chi\gamma\sec\alpha+\gamma^2}{\chi^2+\cot^2\alpha}}e^{-\frac{1}{2}\csc^2\alpha\frac{\xi^2\chi+2\xi\gamma\cos\alpha-\chi\gamma^2\sin^2\alpha}{\chi^2+\cot^2\alpha}}\end{aligned}$$

(5) and (6) are trivial.

A.2 Proof of Corollary 4.2

(1) We give the proof for $m=1$. For $m>1$ the proof is by induction. Because $f(x)=ixg(x)$, $f=\mathcal{U}g$ and thus

$$f_a=\mathcal{F}^a\mathcal{U}g=\mathcal{U}_a g_a=[\cos\alpha\mathcal{U}+\sin\alpha\mathcal{D}]g_a.$$

Thus

$$x\cos\alpha\ g_a(x)+\sin\alpha\ g_a'(x)=if_a(x).$$

Solution of this differential equation for g_a gives the desired result.

(2) Its proof is as before. Because $\mathcal{D}g=f$, we get after the FrFT

$$f_a=\mathcal{F}^a f=\mathcal{F}^a\mathcal{D}g=\mathcal{D}_a\mathcal{F}^a g=[-\sin\alpha\,\mathcal{U}+\cos\alpha\mathcal{D}]g_a.$$

Hence

$$-i\sin\alpha\ g_a(x)+\cos\alpha\ g_a'(x)=f_a(x).$$

Solving this differential equation gives the result.

(3) This is an immediate consequence of Theorem 4.1 (3-4).

A.3 Proof of Theorem 4.3

(1) This is by a change of variables $x'=x+b$ in the definition of the FrFT.

(2) This is seen by taking the FT of the previous rule. A shift transforms in an exponential multiplication, giving the left-hand side. On the other hand, taking the $\mathcal{F}$ transform of the $\mathcal{F}^a$ transform corresponds to taking the $\mathcal{F}^{a+1}$ transform, hence we replace α in the previous line by $\alpha+\pi/2$, which gives the right-hand side.

A.4 Proof of theorem 5.2

First we plug in the inversion formula for the FrFT on the defining equation for the modified WFT, then we get

$$(\tilde{\mathcal{F}}_w f)(x,\xi) = \frac{e^{ix\xi/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_a(z) \overline{K_a(z,t) w(t-x)} e^{-i\xi t} \mathrm{d}t \, \mathrm{d}z.$$

Because

$$\int_{-\infty}^{\infty} K_a(z,t) w(t-x) e^{i\xi t} \mathrm{d}t = w_a(-z + x\cos\alpha + \xi \sin\alpha) e^A,$$

with $A = -i\frac{\xi^2 - x^2}{2} \sin\alpha\cos\alpha - i(x\sin\alpha - \xi\cos\alpha) + ix\xi\sin^2\alpha$, putting $x_a = x\cos\alpha + \xi\sin\alpha$ and $\xi_a = -x\sin\alpha + \xi\cos\alpha$, it is easily verified that

$$e^{ix\xi/2} e^A = e^{-izv} e^{ix_a \xi_a /2}.$$

Thus

$$(\tilde{\mathcal{F}}_w f)(x,\xi) = \frac{e^{ix_a\xi_a/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_a(z) \overline{w_z(x_a - z)} e^{-i\xi_a z} \mathrm{d}z.$$

This proves the result.

A.5 Proof of the discretization of the FrFT in Section 8.4

If we assume that the signal f is band limited to the interval $[-B,B]$, so that for a in the neighborhood, say $0.5 \le |a| \le 1.5$, then also $h(x) = e^{i\frac{\cot\alpha}{2}x^2} f(x)$ can be restricted to the interval $[-B,B]$ and, by the sampling theorem, we can interpolate the function $h(x)$ by

$$\tilde{h}(x) = \sum_{l=-N}^{N-1} h\left(\frac{l}{2B}\right) \operatorname{sinc}\left[2B\left(x - \frac{l}{2B}\right)\right].$$

We have a finite sum with $N = B^2$ because we assumed that h is bandlimited to $[-B,B]$. We replace in the integral of

$$f_a(\xi) = C_\alpha e^{\frac{i}{2}\xi^2 \cot\alpha} \int_{-\infty}^{\infty} \exp\{-ix\xi\csc\alpha\} \, h(x) \mathrm{d}x$$

the function $h(x)$ by the interpolant $\tilde{h}(x)$. This leads to

$$f_a(\xi) \approx C_\alpha e^{\frac{i}{2}\xi^2\cot\alpha} \sum_{l=-N}^{N-1} h\left(\frac{l}{2B}\right) \int_{-\infty}^{\infty} \exp\{-i\xi x \csc\alpha\} \operatorname{sinc}\left[2B\left(x - \frac{l}{2B}\right)\right] \mathrm{d}x.$$

The latter integral is $\frac{1}{2B}\exp\{-i\xi\csc\alpha(l/2B)\}\operatorname{rect}(x\csc\alpha/2B)$ with the $\operatorname{rect}(\cdot)$ in this expression equal to 1 in $[-B,B]$. If we plug this in the relation for $f_a(\xi)$ we get after rearrangement

$$f_a\left(\frac{k}{2B}\right) \approx \frac{C_\alpha}{2B} c\left(\frac{k}{2B}\right) \sum_{l=-N}^{N-1} r\left(\frac{k-l}{2B}\right) s\left(\frac{l}{2B}\right),$$

with a chirp $c(x)$ and two functions $r(x)$ and $s(x)$ defined as

$$\begin{aligned} c(x) &= \exp\left\{i\frac{\cot\alpha - \csc\alpha}{2}\left(\frac{k}{2B}\right)^2\right\} \\ r(x) &= \exp\left\{i\frac{\csc\alpha}{2}x^2\right\} \\ s(x) &= \exp\left\{i\frac{\cot\alpha - \csc\alpha}{2}x^2\right\} f(x). \end{aligned}$$

Thus we have to compute a convolution of the s and r vectors, which has an $O(N \log N)$ complexity, followed by a chirp multiplication. If a is not in the range $0.5 \leq |a| \leq 1.5$, we have to apply a trick like $\mathcal{F}^a = \mathcal{F}^{a+1}\mathcal{F}^{-1}$, where the extra FT needs another $O(N \log N)$ operations.

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