Orthogonal rational functions on an interval

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Rational functions with real poles and poles in the complex lower half plane, orthogonal on the real line, are well known. Quadrature formulas similar to the Gauss formulas for orthogonal polynomials have been studied. We generalize to the case of arbitrary complex poles (with complex conjugate poles as a special case) and study orthogonality on a finite interval. The zeros of the orthogonal rational functions are shown to satisfy a quadratic eigenvalue problem. In the case of real poles, these zeros are used as nodes in the quadrature formulas.

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Abstract

Rational functions with real poles and poles in the complex lower half plane, orthogonal on the real line, are well known. Quadrature formulas similar to the Gauss formulas for orthogonal polynomials have been studied. We generalize to the case of arbitrary complex poles (with complex conjugate poles as a special case) and study orthogonality on a finite interval. The zeros of the orthogonal rational functions are shown to satisfy a quadratic eigenvalue problem. In the case of real poles, these zeros are used as nodes in the quadrature formulas.

1 Introduction

Orthogonal rational functions on the circle and the real line have been studied extensively in several papers, leading to a monograph which provides a comprehensive survey of the main results [2]. In the present paper it is our aim to extend some of these results to the situation of a finite interval, as well as to present some new results applying to both the real line and the interval. Special attention will go to the use of orthogonal rational functions to construct quadrature formulas. We do not consider the circle case here.

Given a sequence of points (the poles) $\{\alpha_k\}_{k=1}^{\infty}$ we introduce the notation

$$\pi_n(z) = \prod_{k=1}^{n}(z - \alpha_k).$$

If we let Π_n denote the space of the polynomials of degree at most n, the space $\mathcal{L}_n$ of rational functions can be expressed as $\mathcal{L}_n = \{p_n/\pi_n : p_n \in \Pi_n\}$. A distinction must me made between real poles and complex poles in the lower half plane $\mathbb{L}$.

1.1 Real poles

We assume for technical reasons that there is an $\alpha \in \mathbb{R}$ such that $\alpha_k \neq \alpha$ for all k. Without loss of generality we can take this value to be the origin. So we assume in this case that $\alpha_k \neq 0, k = 0, 1, \ldots$ A basis for $\mathcal{L}_n$ is given by

$$b_0 = 1, \quad b_n = \prod_{k=1}^{n} Z_k(z), \quad n = 1, 2, \ldots, \quad \text{with} \quad Z_k(z) = \frac{z}{1 - z/\alpha_k}.$$

By convention $\alpha_0 = \infty$. Consider the inner product

$$\langle f, g \rangle = \int_{\hat{\mathbb{R}}} f(z)\overline{g(z)}d\mu(z)$$

where μ represents a positive measure on the real line and $\hat{\mathbb{R}}$ denotes the extended real line, i.e. $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. By orthogonalization of the sequence $\{b_0, b_1, \ldots\}$, one obtains the orthonormal rational functions $\{\phi_0, \phi_1, \ldots\}$. Note that if all the $\alpha_k = \infty$, then the rational situation reduces to the polynomial case. We shall call ϕ_n *singular* if $p_n(\alpha_{n-1}) = 0$, where p_n is the numerator of ϕ_n, and *regular* otherwise.

Furthermore we define quasi-orthogonal functions as

$$Q_n(z, \tau) = \phi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}(z), \quad \tau \in \mathbb{R} \cup \{\infty\},\ n \geq 1.$$

We set by definition

$$Q_n(z, \infty) = \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}(z).$$

1.2 Complex poles in $\mathbb{L}$

If $\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{L}$, then a basis for $\mathcal{L}_n$ is given by finite Blaschke products

$$B_0 = 1, \quad B_n = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots,$$

with

$$\zeta_k(z) = \frac{\alpha_k^2 + 1}{|\alpha_k^2 + 1|} \frac{z - \overline{\alpha}_k}{z - \alpha_k}.$$

By convention $\alpha_0 = i$. The inner product has the same form as before:

$$\langle f, g \rangle = \int_{\hat{\mathbb{R}}} f(z)\overline{g(z)}d\mu(z).$$

Again we obtain orthonormal rational functions $\{\phi_0, \phi_1, \ldots\}$ by orthogonalization of the sequence $\{B_0, B_1, \ldots\}$. The involution operation or substar conjugate of a function $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ is defined as

$$f_*(z) = \overline{f(\overline{z})}$$

and the superstar transformation as

$$f^*(z) = B_n(z) f_*(z), \quad \text{if} \quad f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}.$$

In this case ϕ_n is said to be *degenerate* if $\phi_n^*(\alpha_{n-1}) = 0$ and *exceptional* if $\phi_n(\alpha_{n-1}) = 0$.

Instead of quasi-orthogonal functions, we now consider para-orthogonal functions defined as

$$Q_n(z, \tau) = \phi_n(z) + \tau \phi_n^*(z), \quad \tau \in \mathbb{T},\ n \geq 1$$

where $\mathbb{T}$ denotes the unit circle in the complex plane.

In the next section we will derive a recurrence relation for the ϕ_n which will hold for arbitrary complex poles and which therefore encompasses both of the above cases. We derive explicit formulas for the recurrence coefficients. Next we will limit our attention to the case of orthogonality on a finite interval for both real and complex conjugate poles. We especially study the quadrature formulas in these cases. In the case of real poles the nodes in the quadrature formulas are the zeros of $\phi_n(z)$. In the last section we will show that they satisfy a quadratic eigenvalue problem.

2 A fundamental recurrence relation

2.1 Arbitrary complex poles

In the case of real poles it can be shown [2] that the orthonormal functions ϕ_n satisfy the following three term recurrence relation iff both ϕ_{n-1} and ϕ_n are regular:

$$\phi_n = \left(A_n Z_n + B_n \frac{Z_n}{Z_{n-2}}\right)\phi_{n-1} + C_n \frac{Z_n}{Z_{n-2}}\phi_{n-2}$$

with A_n, B_n, C_n constants satisfying

$$\begin{aligned} A_n + B_n/Z_{n-2}(\alpha_{n-1}) &\neq 0, \\ C_n &\neq 0. \end{aligned}$$

In the following theorem we prove that the same relation holds for functions with arbitrary complex poles, if we replace Z_{n-2} with its substar conjugate $Z_{(n-2)*}$ (or equivalently α_{n-2} with $\overline{\alpha}_{n-2}$). Note that for arbitrary complex poles we may have a problem using the basis $\{B_0, B_1, \ldots, B_n\}$ for $\mathcal{L}_n$: consider the case where $\alpha_{i+1} = \overline{\alpha_i}$, then we have $B_{i+1} = B_{i-1}$. It is therefore better to use the basis $\{b_0, b_1, \ldots, b_n\}$. Let $\alpha_0 = \infty$ and assume $\alpha_k \neq 0, k = 1, 2, \ldots$

Theorem 2.1.1. *For $n = 2, 3, \ldots$, let $\phi_k \in \mathcal{L}_k, k = n-2, n-1, n$ be three successive orthonormal rational functions associated with the pole sequence $\{\alpha_1, \alpha_2, \ldots,\} \subset \mathbb{C} \setminus \{0\}$. Then ϕ_{n-1} is nondegenerate and ϕ_n is nonexceptional if and only if there exists a recurrence relation of the form*

$$\phi_n(z) = \left(A_n Z_n(z) + B_n \frac{Z_n(z)}{Z_{(n-2)*}(z)}\right)\phi_{n-1}(z) + C_n \frac{Z_n(z)}{Z_{(n-2)*}(z)}\phi_{n-2}(z), \quad n \geq 2 \tag{1}$$

with constants A_n, B_n, C_n satisfying the conditions

$$E_n = A_n + B_n/Z_{(n-2)*}(\alpha_{n-1}) \neq 0, \tag{2}$$

$$C_n \neq 0. \tag{3}$$

Proof. First suppose that ϕ_n is nonexceptional and ϕ_{n-1} is nondegenerate. Choose A_n arbitrary and define

$$W_n(z) = \frac{Z_{(n-2)*}(z)}{Z_n(z)}\phi_n(z) - A_n Z_{(n-2)*}(z)\phi_{n-1}(z).$$

Let $\phi_n(z) = q_n(z)/[\pi_n(z)/\pi_n(0)]$. Then

$$W_n(z) = -\frac{\overline{\alpha}_{n-2}}{z - \overline{\alpha}_{n-2}} \frac{q_n(z) - A_n z q_{n-1}(z)}{\pi_{n-1}(z)/\pi_{n-1}(0)}.$$

Thus if we choose

$$A_n = \frac{q_n(\overline{\alpha}_{n-2})}{\overline{\alpha}_{n-2} q_{n-1}(\overline{\alpha}_{n-2})} \tag{4}$$

we obtain that $W_n \in \mathcal{L}_{n-1}$. Recall that ϕ_{n-1} is nondegenerate and $\alpha_{n-2} \neq 0$, so that A_n is well defined. This implies that W_n can be written as

$$W_n(z) = B_n \phi_{n-1}(z) + C_n \phi_{n-2}(z) + \sum_{k=0}^{n-3} D_k \phi_k(z).$$

For $n = 2$, the sum is empty and the result is obvious. For $n \geq 3$, it is easily checked that $W_n \perp \mathcal{L}_{n-3}$, hence that all $D_k = 0, k = 0, \ldots, n-3$. What then remains is equivalent with the formula (1).

Taking the numerator of this formula and putting $z = \alpha_{n-1}$ gives

$$q_n(\alpha_{n-1}) = \alpha_{n-1}[A_n + B_n / Z_{(n-2)*}(\alpha_{n-1})] q_{n-1}(\alpha_{n-1}). \tag{5}$$

Because $q_n(\alpha_{n-1}) \neq 0$, this gives (2).

Observe then that $b_{n-1}(z)/Z_{n*}(z)$ is an element of $\mathcal{L}_{n-1}$. Thus it is orthogonal to ϕ_n and we get

$$\begin{aligned} 0 &= \left\langle Z_n \left(A_n + \frac{B_n}{Z_{(n-2)*}} \right) \phi_{n-1}, \frac{b_{n-1}}{Z_{n*}} \right\rangle + C_n \left\langle \frac{Z_n}{Z_{(n-2)*}} \phi_{n-2}, \frac{b_{n-1}}{Z_{n*}} \right\rangle \\ &= \left\langle \phi_{n-1}, \overline{\left(A_n + \frac{B_n}{Z_{(n-2)*}} \right)} b_{n-1} \right\rangle + C_n \left\langle \phi_{n-2}, \frac{b_{n-1}}{Z_{n-2}} \right\rangle. \end{aligned}$$

The right factor in the first product is in $\mathcal{L}_{n-1}$ and its leading coefficient $\overline{A_n + B_n / Z_{(n-2)*}(\alpha_{n-1})}$ is nonzero. Thus the first inner product is nonzero and therefore also the second term will be nonzero. This implies that $C_n \neq 0$.

Conversely, suppose that (1)-(3) holds for some $n \geq 2$. Since $\phi_{n-1} \in \mathcal{L}_{n-1} \backslash \mathcal{L}_{n-2}$, it follows that $q_{n-1}(\alpha_{n-1}) \neq 0$. Therefore it follows by (2) and (5) that $q_n(\alpha_{n-1}) \neq 0$. Because A_n is well defined, it follows from (4) that $q_{n-1}(\overline{\alpha}_{n-2}) \neq 0$. □

Next we derive formulas for the recursion coefficients.

2.2 Explicit formulas for the recursion coefficients

To calculate the function ϕ_n using the recurrence relation, we need the coefficients A_n, B_n and C_n. Explicit expressions for these coefficients can be found as follows. Taking the inner product of (1) with ϕ_n, ϕ_{n-1} and ϕ_{n-2} successively, gives a system of equations

$$\begin{bmatrix} -\alpha_n c_n^{(n-1)} & (\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1) c_n^{(n-1)} & (\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1) c_n^{(n-2)} \\ -1 - \alpha_n c_{n-1}^{(n-1)} & \frac{1}{\overline{\alpha}_{n-2}} + (\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1) c_{n-1}^{(n-1)} & (\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1) c_{n-1}^{(n-2)} \\ -\alpha_n c_{n-2}^{(n-1)} & (\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1) c_{n-2}^{(n-1)} & \frac{1}{\overline{\alpha}_{n-2}} + (\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1) c_{n-2}^{(n-2)} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_n} \\ 0 \\ 0 \end{bmatrix}$$

with the inner products $c_i^{(j)} = \left\langle \frac{\phi_j}{z - \alpha_n}, \phi_i \right\rangle$. This system easily reduces to an upper triangular system, which can be solved by backward substitution. Some algebra thus leads to the following expressions for the recursion coefficients:

$$C_n = \frac{\overline{\alpha}_{n-2}}{\alpha_n} \frac{c_{n-2}^{(n-1)}}{(\alpha_n - \overline{\alpha}_{n-2})\left(c_n^{(n-2)} c_{n-2}^{(n-1)} - c_n^{(n-1)} c_{n-2}^{(n-2)}\right) - c_n^{(n-1)}}, \tag{6}$$

$$B_n = \frac{1}{c_n^{(n-1)}} \left[\left(\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1 \right) \left(c_n^{(n-2)} - \alpha_n \left(c_n^{(n-1)} c_{n-1}^{(n-2)} - c_n^{(n-2)} c_{n-1}^{(n-1)} \right) \right) C_n - \frac{1}{\alpha_n} - c_{n-1}^{(n-1)} \right], \tag{7}$$

$$A_n = \frac{1}{\alpha_n c_n^{(n-1)}} \left[\left(\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1 \right) \left(c_n^{(n-1)} B_n + c_n^{(n-2)} C_n \right) - \frac{1}{\alpha_n} \right]. \tag{8}$$

These expressions however are not very practical to compute ϕ_n since they use ϕ_n in the inner products $c_n^{(n-1)}$ and $c_n^{(n-2)}$. To solve this problem, define $\lambda_n = c_n^{(n-2)}/c_n^{(n-1)}$. First we prove the following lemma.

Lemma 2.2.1. *With* $\lambda_n = c_n^{(n-2)}/c_n^{(n-1)}$ *and* $c_i^{(j)} = \left\langle \frac{\phi_j}{z-\alpha_n}, \phi_i \right\rangle$ *we have*

$$\lambda_n = \frac{\phi_{n-2}(\alpha_n)}{\phi_{n-1}(\alpha_n)}.$$

Proof. Expanding $\frac{\phi_{n-1}}{z-\alpha_n}$ in the orthonormal basis $\{\phi_0, \phi_1, \ldots, \phi_n\}$ yields

$$\frac{\phi_{n-1}}{z-\alpha_n} = \sum_{i=0}^{n} c_i^{(n-1)} \phi_i = \sum_{i=0}^{n} \sum_{k=0}^{i} c_i^{(n-1)} \beta_k^{(i)} b_k$$

where $\beta_k^{(i)}$ are the coefficients of the expansion of ϕ_i in the basis $\{b_0, b_1, \ldots, b_i\}$. Multiplying by $z - \alpha_n$ and substituting α_n for z, using the fact that $b_n(z) = b_{n-1}(z)\frac{-\alpha_n z}{z-\alpha_n}$, we obtain (let $\beta_n^{(n)} = \kappa_n$)

$$\begin{aligned} \phi_{n-1}(\alpha_n) &= -c_n^{(n-1)} \kappa_n \alpha_n^2 b_{n-1}(\alpha_n), \\ c_n^{(n-1)} &= -\frac{1}{\alpha_n^2 \kappa_n} \frac{\phi_{n-1}(\alpha_n)}{b_{n-1}(\alpha_n)}. \end{aligned} \tag{9}$$

A similar argument gives

$$c_n^{(n-2)} = -\frac{1}{\alpha_n^2 \kappa_n} \frac{\phi_{n-2}(\alpha_n)}{b_{n-1}(\alpha_n)}. \tag{10}$$

Dividing (10) by (9) then proves the lemma. □

Thus λ_n can be readily calculated. Using this lemma we may rewrite expressions (6)-(8) as

$$\begin{aligned} C_n &= \frac{1}{c_n^{(n-1)}} \tilde{C}_n, \\ B_n &= \frac{1}{c_n^{(n-1)}} \tilde{B}_n, \\ A_n &= \frac{1}{c_n^{(n-1)}} \tilde{A}_n, \end{aligned}$$

with

$$\tilde{C}_n = \frac{\overline{\alpha}_{n-2}}{\alpha_n} \frac{c_{n-2}^{(n-1)}}{(\alpha_n - \overline{\alpha}_{n-2})\left(\lambda_n c_{n-2}^{(n-1)} - c_{n-2}^{(n-2)}\right) - 1}, \tag{11}$$

$$\tilde{B}_n = \left(\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1\right)\left(\lambda_n + \alpha_n\left(\lambda_n c_{n-1}^{(n-1)} - c_{n-1}^{(n-2)}\right)\right)\tilde{C}_n - \frac{1}{\alpha_n} - c_{n-1}^{(n-1)}, \tag{12}$$

$$\tilde{A}_n = \frac{1}{\alpha_n}\left[\left(\frac{\alpha_n}{\overline{\alpha}_{n-2}} - 1\right)(\tilde{B}_n + \lambda_n \tilde{C}_n) - \frac{1}{\alpha_n}\right]. \tag{13}$$

Substituting these expressions in the recurrence relation (1) gives the relation

$$\tilde{\phi}_n = \left(\tilde{A}_n Z_n + \tilde{B}_n \frac{Z_n}{Z_{(n-2)*}}\right)\phi_{n-1} + \tilde{C}_n \frac{Z_n}{Z_{(n-2)*}}\phi_{n-2}$$

with $\tilde{\phi}_n = c_n^{(n-1)}\phi_n$, which after normalization yields the desired orthonormal rational function ϕ_n.

3 Real poles

We now turn to the case of a finite interval $[a, b]$ and real poles. To avoid problems with singularities we assume that all poles are outside the interval. The inner product is given by

$$\langle f, g\rangle = \int_a^b f(z)\overline{g(z)}d\mu(z).$$

The recurrence relation (1) is still valid, since the proof of theorem 2.1.1 does not depend on the actual form of the inner product.

3.1 Properties of orthogonal rational functions on an interval

Certain properties of polynomials orthogonal on an interval can be generalized to orthogonal rational functions. We state some theorems without giving any proof, since this is usually very similar to the polynomial case, see e.g. [1].

Theorem 3.1.1. *Let ϕ_n be an orthogonal rational function on the interval $[a, b]$ with real poles outside the interval. Then the zeros of ϕ_n are all simple and contained in $[a, b]$.*

The fact that all zeros are simple has already been shown in [2]. Using the orthogonality condition it is not difficult to prove that they are all contained in the interval. This theorem shows that in the case of real poles outside an interval the recurrence relation will always hold, since the regularity condition is satisfied. The following theorem is based upon the existence of a Christoffel-Darboux relation, derived in [2].

Theorem 3.1.2. *Let z_{ni} and $z_{n,i+1}$ be two successive zeros of the function ϕ_n as in the previous theorem. Then there exists a $z_{n-1,j} \in (z_{ni}, z_{n,i+1})$ such that $\phi_{n-1}(z_{n-1,j}) = 0$.*

The previous two theorems are direct generalizations of the polynomial case. It is well known that on a symmetric interval, polynomials of even degree orthogonal with respect to an even weigth function are even and those of odd degree are odd. For rational functions however this is never the case, because of the poles in the denominator (even if we choose these symmetrically, i.e. $\alpha_i = -\alpha_{i+1}$, i odd, the ϕ_n are neither odd nor even).

Theorem 3.1.3. *Let ϕ_n be as in the first theorem. Then in general $\phi_n(z)$ is not an even function nor an odd function.*

3.2 Quasi-orthogonal functions and quadrature

The use of the quasi-orthogonal functions $Q_n(z,\tau)$ as defined above lies in the fact that their zeros are simple and real and can be used as nodes in quadrature formulas which are exact in certain spaces of rational functions, analogous to the Gauss quadrature formulas for polynomials. Define the space $\mathcal{R}_n$ as

$$\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_n = \left\{ \frac{p(z)}{[\pi_n(z)]^2} : p \in \Pi_{2n} \right\} = \left\{ \frac{p_n(z)q_n(z)}{\pi_n(z)\pi_n(z)} : p_n, q_n \in \Pi_n \right\}$$

and the weights λ_{nk} as

$$\lambda_{nk} = \left[\sum_{j=0}^{n-1} (\phi_j(\xi_k))^2 \right]^{-1}$$

with $\xi_k = \xi_{nk}(\tau), k = 1, \ldots, n$ the zeros of the quasi-orthogonal function $Q_n(z,\tau)$. Then the quadrature formula

$$\int_{\hat{\mathbb{R}}} f(z) d\mu(z) \approx \sum_{k=1}^{n} \lambda_{nk} f(\xi_{nk}(\tau))$$

has domain of validity $\mathcal{R}_{n-1}$, as proved in [2]. For $\tau = 0$ the domain of validity becomes $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$. Again the proof does not depend on the form of the inner product and therefore the quadrature formula remains valid in the case of a finite interval. It is however not certain that all the nodes, i.e. the zeros of $Q_n(z,\tau)$, are inside the interval. Obviously if $\tau = 0$ then $Q_n(z,0) = \phi_n(z)$ and according to theorem 3.1.1 all its zeros are in the interval. When $\tau = \infty$ one of the zeros is equal to α_{n-1} which is outside the interval. The following theorem shows that there are always at least $n-1$ zeros in the interval $[a,b]$.

Theorem 3.2.1. *The quasi-orthogonal rational function $Q_n(z,\tau)$ has at least $n-1$ real, simple zeros on the interval of integration $[a,b]$.*

Proof. It is shown in [2] that the zeros of $Q_n(z,\tau)$ are real and simple. Since all the poles are real as well, we may omit the complex conjugate bar in the inner product. From the definition of $Q_n(z,\tau)$ it follows that Q_n is orthogonal to the space $\mathcal{L}_{n-1}(\alpha_n) = \{f \in \mathcal{L}_{n-1} : f(\alpha_n) = 0\}$. Therefore $Q_n(z,\tau) \perp (z-\alpha_n)/\pi_{n-1}(z)$, which means that

$$\int_a^b Q_n(z,\tau) \frac{(z-\alpha_n)}{\pi_{n-1}(z)} d\mu(z) = 0.$$

This is only possible if Q_n has at least one zero on the interval. Now suppose there are only $m \leq n-2$ zeros $\xi_1, \ldots, \xi_m$ on $[a,b]$. Then the function $Q_n(z,\tau)(z-\xi_1)(z-\xi_2)\ldots(z-\xi_m)(z-\alpha_n)/\pi_{n-1}$ has a constant sign on the interval so the integral

$$\int_a^b Q_n(z,\tau) \frac{(z-\xi_1)(z-\xi_2)\ldots(z-\xi_m)(z-\alpha_n)}{\pi_{n-1}(z)} d\mu(z)$$

is nonzero. This is impossible because $(z-\xi_1)(z-\xi_2)\ldots(z-\xi_m)(z-\alpha_n)/\pi_{n-1}(z) \in \mathcal{L}_{n-1}(\alpha_n)$. $\square$

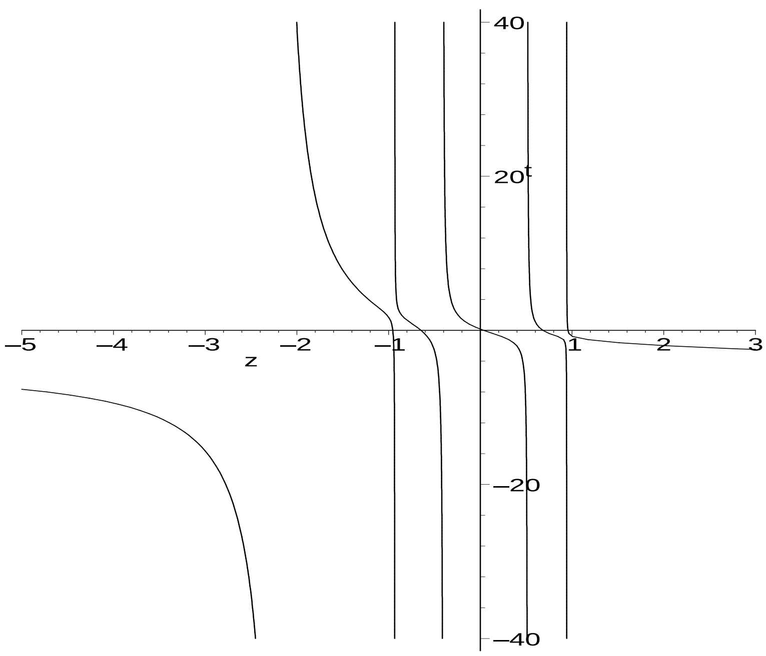

Figure 1: Zeros of $Q_5(z,\tau)$ as a function of τ. Poles at $\{\omega, -\omega, 2\omega, -2\omega, \ldots\}$ with $\omega = 1.1$ and weight $= 1$ in $[-1,1]$.

The zeros of $Q_n(z,\tau)$ as a function of τ (vertical axis) are shown in figure 1 for $[a,b]=[-1,1]$, poles $\{\omega, -\omega, 2\omega, -2\omega, \ldots\}$ with $\omega = 1.1$ and $d\mu(z) = dz$. For a certain value τ_c of τ the numerator polynomial of Q_n has degree $n-1$, which causes one of the zeros to tend to infinity as τ approaches τ_c. For a detailed discussion of these curves we refer to the appendix.

The interval (τ_m, τ_M) around $\tau = 0$ where $Q_n(z,\tau)$ has all its zeros on the interval $[a,b]$ can be found using the definition of Q_n. When constructing quadrature formulas on an interval it is customary to take all the nodes inside that interval, therefore only values of τ in the interval (τ_m, τ_M) are of interest. Suppose orthogonality is considered over $[a,b]$. Let τ_1 be such that $Q_n(a,\tau_1) = 0$, then

$$\tau_1 = -\frac{\phi_n(a)}{\phi_{n-1}(a)} \frac{Z_{n-1}(a)}{Z_n(a)}.$$

Similarly, denote by τ_2 the value for which one of the zeros of Q_n equals b, then it is clear that $\tau_m = \min\{\tau_1, \tau_2\}$ and $\tau_M = \max\{\tau_1, \tau_2\}$.

4 Complex conjugate poles

4.1 Para-orthogonal functions and quadrature

In the case of complex poles in the lower half plane, the para-orthogonal functions $Q_n(z,\tau) = \phi_n(z) + \tau\phi_n^*(z), \tau \in \mathbb{T}$ play the role of the quasi-orthogonal functions for the real poles. In [2] it is shown that their zeros are real and simple and can again be used to construct quadrature formulas which are exact in the space $\mathcal{R}_n$ defined as

$$\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_{n*} = \left\{ \frac{p_n(z)}{\pi_n(z)\pi_{n*}(z)} : p_n \in \Pi_{2n} \right\}.$$

The weights are given by

$$\lambda_{nk} = \left[\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2 \right]^{-1}$$

with $\xi_k = \xi_{nk}(\tau)$ the zeros of $Q_n(z,\tau)$.

Now we consider complex conjugate poles, i.e. $\alpha_{i+1} = \overline{\alpha}_i$ for $i = 1, 3, \ldots$ As mentioned above, the Blaschke products B_k no longer form a basis for $\mathcal{L}_n$, but the definition of $f^*(z) = B_n f_*(z)$ with $f(z) \in \mathcal{L}_n$ remains valid. It can be shown that also in this case the zeros of $Q_n(z,\tau)$ are real and simple, as stated in the following theorem. The proof is very similar to the one given in [2] for the case of complex poles in $\mathbb{L}$.

Theorem 4.1.1. *Let $\phi_n(z)$ be a orthonormal rational function on the real line or an interval with poles $\{\alpha_i\}_{i=1}^n$ such that $\alpha_{2k} = \overline{\alpha}_{2k-1}, k = 1, \ldots, \lfloor n/2 \rfloor$, $\alpha_i \notin \mathbb{R}$ and $Q_n(z,\tau) = \phi_n(z) + \tau\phi_n^*(z), \tau \in \mathbb{T}$ the associated para-orthogonal function, then all the zeros of Q_n are real and simple.*

The function $Q_n(z,\tau)$ is called para-orthogonal, because it is only orthogonal to a subspace of $\mathcal{L}_{n-1}$. It is easily checked that $Q_n \perp \mathcal{L}_{n-1} \cap \zeta_n \mathcal{L}_{n-1}$. This doesn't change if the poles are complex conjugate. It is this property and the fact that the zeros of Q_n are real and simple that are used in [2] to prove the validity of the quadrature formulas and therefore this still holds for complex conjugate poles.

Theorem 4.1.2. *The quadrature formula*

$$I_n\{f\} = \sum_{k=1}^{n} \lambda_{nk} f(\xi_k)$$

with nodes and weights as defined above for complex conjugate poles, has domain of validity $\mathcal{R}_{n-1}$.

4.2 Finite interval

If we restrict our attention to the case of a finite interval, the same comment as in Section 3.2 applies: it is not certain that all the nodes are inside this interval. By an argument similar to the one from Theorem 3.2.1 and using the para-orthogonality of Q_n it can be proved that there are always at least $n-1$ zeros inside the interval $[a,b]$, as stated in the following theorem.

Theorem 4.2.1. *The para-orthogonal function $Q_n(z,\tau)$ has at least $n-1$ real, simple zeros on the interval of orthogonality $[a,b]$.*

It is not immediately obvious though that there are values of τ for which all the zeros of Q_n are inside the interval. We do know that for a certain value τ_c the numerator polynomial of Q_n has degree $n-1$ so one of the zeros then tends to infinity. We will show by intuitive reasoning that for at least one value of τ the function $Q_n(z,\tau)$ has all its zeros on $[a,b]$. Suffice it to say that a more exact but cumbersome proof can be given, similar to the one in the appendix concerning the zeros of the quasi-orthogonal functions.

Since $\tau = e^{i\theta} \in \mathbb{T}$ we may as well consider $\arg(\tau) = \theta \in [0, 2\pi]$. Now suppose that for every value of θ only $n-1$ zeros are inside the interval. Let us denote them by $\xi_1(\theta), \ldots, \xi_{n-1}(\theta)$, in ascending order. For a certain value θ_1, one of the outer zeros, say $\xi_1(\theta)$, will leave the interval

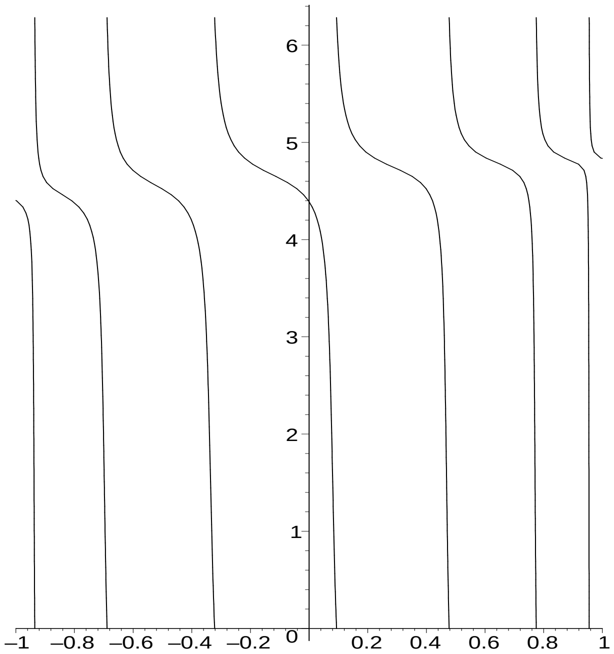

Figure 2: Zeros of $Q_7(z,\tau)$ as a function of θ. Poles at $\{1+i, 1-i, -1+i, -1-i, 2+i, \ldots\}$ and weight $= 1$ on the interval $[-1, 1]$.

either to approach the zero outside the interval, or to approach infinity. For another value θ_2 a zero $\xi_e(\theta)$ will enter the interval from the other side such that $\xi_e(2\pi) = \xi_{n-1}(0)$. If $\theta_2 > \theta_1$ then there are only $n-2$ zeros inside the interval for $\theta \in [\theta_1, \theta_2]$, which is impossible. It follows that $\theta_2 \leq \theta_1$. For $\theta = \theta_1$ we then have n zeros inside the interval.

Figure 2 shows the zeros of $Q_n(z,\tau)$ as a function of θ for $[a,b] = [-1,1]$, poles $\{1+i, 1-i, -1+i, -1-i, 2+i, \ldots\}$ and $d\mu(z) = dz$. Note that for most values of θ there are n zeros on the interval. The values of θ for which there are only $n-1$ zeros can again be found using the definition of Q_n. Let $\theta_c = \arg(\tau_c)$ and $\theta_1, \theta_2 \in [0, 2\pi]$ such that $Q_n(a, e^{i\theta_1}) = Q_n(b, e^{i\theta_2}) = 0$. Now suppose $\theta_1 = \arg(-\phi_n(a)/\phi_n^*(a)) < \theta_c < \theta_2 = \arg(-\phi_n(b)/\phi_n^*(b))$ (change the definition of θ_1 and θ_2 otherwise). Then for $\theta \in [\theta_1, \theta_2]$ there are only $n-1$ zeros on the interval. If both $\theta_1, \theta_2 < \theta_c$ then θ_c is very close to 2π and the interval becomes $[0, \min\{\theta_1, \theta_2\}] \cup [\max\{\theta_1, \theta_2\}, 2\pi]$. Similar expressions can be derived for θ_c very close to 0.

5 Zeros of orthogonal rational functions

5.1 A quadratic eigenvalue problem

It is a well known property of orthogonal polynomials that their zeros are the eigenvalues of a tridiagonal matrix, the Jacobi matrix, containing the recursion coefficients. In this section we derive a similar property for orthogonal rational functions, but here the zeros are eigenvalues of a quadratic eigenvalue problem.

Consider the recurrence relation

$$\phi_n(z) = \left(A_n Z_n(z) + B_n \frac{Z_n(z)}{Z_{(n-2)*}(z)}\right) \phi_{n-1}(z) + C_n \frac{Z_n(z)}{Z_{(n-2)*}(z)} \phi_{n-2}(z)$$

with $\phi_n(z) = p_n(z)/\pi_n(z)$. Taking the numerator and rearranging yields

$$-\alpha_{n-1}C_n p_{n-2}(z) + B_n p_{n-1}(z) + 1/\alpha_n p_n(z) =$$

$$[-(\alpha_{n-1}/\overline{\alpha}_{n-2}+1)C_n p_{n-2}(z) + (B_n/\overline{\alpha}_{n-2} - A_n)p_{n-1}(z)]z + C_n/\overline{\alpha}_{n-2}p_{n-2}(z)z^2.$$

This equation can be written down for every n and after collecting the coefficients in $n \times n$-matrices we obtain the following expression:

$$\begin{bmatrix} * & * & & & & & \\ -\alpha_1 C_2 & B_2 & \frac{1}{\alpha_2} & & & & \\ & -\alpha_2 C_3 & B_3 & \frac{1}{\alpha_3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\alpha_{n-2}C_{n-1} & B_{n-1} & \frac{1}{\alpha_{n-1}} \\ & & & & -\alpha_{n-1}C_n & B_n \end{bmatrix} \begin{bmatrix} p_0(z) \\ p_1(z) \\ p_2(z) \\ \vdots \\ p_{n-2}(z) \\ p_{n-1}(z) \end{bmatrix} =$$

$$\begin{bmatrix} * & & & & \\ -(\frac{\alpha_1}{\overline{\alpha}_0}+1)C_2 & \frac{B_2}{\overline{\alpha}_0} - A_2 & & & \\ & -(\frac{\alpha_2}{\overline{\alpha}_1}+1)C_3 & \frac{B_3}{\overline{\alpha}_1} - A_3 & & \\ & \ddots & \ddots & & \\ & & -(\frac{\alpha_{n-2}}{\overline{\alpha}_{n-3}}+1)C_{n-1} & \frac{B_{n-1}}{\overline{\alpha}_{n-3}} - A_{n-1} & \\ & & & -(\frac{\alpha_{n-1}}{\overline{\alpha}_{n-2}}+1)C_n & \frac{B_n}{\overline{\alpha}_{n-2}} - A_n \end{bmatrix} \begin{bmatrix} p_0(z) \\ p_1(z) \\ p_2(z) \\ \vdots \\ p_{n-2}(z) \\ p_{n-1}(z) \end{bmatrix} z +$$

$$\begin{bmatrix} * & & & & & \\ \frac{C_2}{\overline{\alpha}_0} & & & & & \\ & \frac{C_3}{\overline{\alpha}_1} & & & & \\ & & \ddots & & & \\ & & & \frac{C_{n-1}}{\overline{\alpha}_{n-3}} & & \\ & & & & \frac{C_n}{\overline{\alpha}_{n-2}} & 0 \end{bmatrix} \begin{bmatrix} p_0(z) \\ p_1(z) \\ p_2(z) \\ \vdots \\ p_{n-2}(z) \\ p_{n-1}(z) \end{bmatrix} z^2 - \frac{1}{\alpha_n} \begin{bmatrix} 0 \\ \\ \vdots \\ \\ 0 \\ p_n(z) \end{bmatrix} \tag{14}$$

The top row in each matrix can not be found using the recurrence relation but needs to be calculated with the Gram-Schmidt orthogonalization procedure. Let $c_0 = \langle 1, 1 \rangle$ then $\phi_0 = p_0 = 1/\sqrt{c_0}$ and some algebra yields the following expression for the first vector $\phi_1(z)$:

$$\phi_1 = \frac{\tilde{\phi}_1}{\|\tilde{\phi}_1\|}, \quad \tilde{\phi}_1 = \frac{1}{z-\alpha_1} - \frac{1}{c_0}\left\langle \frac{1}{z-\alpha_1}, 1 \right\rangle$$

which can be written as

$$-\frac{1}{\|\tilde{\phi}_1\|}\left(\frac{1}{\phi_0\alpha_1} + \phi_0\left\langle \frac{1}{z-\alpha_1}, 1 \right\rangle\right) p_0(z) + \frac{1}{\alpha_1}p_1(z) = -\frac{\phi_0}{\|\tilde{\phi}_1\|\alpha_1}\left\langle \frac{1}{z-\alpha_1}, 1 \right\rangle p_0(z)z.$$

If we define constants A_1, B_1 and α_{-1} as

$$\begin{aligned} A_1 &= \frac{c_0 c_1}{\alpha_1^2}, \\ B_1 &= c_1\left(\frac{c_0}{\alpha_1} + \left\langle \frac{1}{z-\alpha_1}, 1 \right\rangle\right), \\ \alpha_{-1} &= \overline{\alpha}_1, \end{aligned}$$

with $c_1 = -\phi_0/\|\tilde{\phi}_1\|$ then the previous equation reduces to

$$\begin{bmatrix} B_1 & 1/\alpha_1 \end{bmatrix} \begin{bmatrix} p_0(z) \\ p_1(z) \end{bmatrix} = \begin{bmatrix} B_1/\overline{\alpha}_{-1} - A_1 & 0 \end{bmatrix} \begin{bmatrix} p_0(z) \\ p_1(z) \end{bmatrix} z.$$

Comparison with (14) shows that this is consistent with the notation used above. Note that the term in z^2 is absent so the corresponding matrix contains a row of zeros.

We now have an equation of the form $-\mathcal{C}_n p(z) = \mathcal{B}_n p(z) z + \mathcal{A}_n p(z) z^2 - 1/\alpha_n q(z)$ with $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n \in \mathbb{C}^{n\times n}$ and $p(z), q(z) \in \mathbb{C}^n[z]$. Let λ be a zero of ϕ_n then this reduces to the following quadratic eigenvalue problem

$$(\mathcal{A}_n \lambda^2 + \mathcal{B}_n \lambda + \mathcal{C}_n) p(\lambda) = 0. \tag{15}$$

Thus we have proved that the zeros of ϕ_n satisfy a quadratic eigenvalue problem.

5.2 Only n regular eigenvalues

A quadratic eigenvalue problem of size n generally has $2n$ eigenvalues. However, $\phi_n(z)$ only has n zeros. We will show by an induction argument that due to the specific structure of the matrices $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$ there are only n finite eigenvalues, so the equation (15) does not introduce spurious solutions.

By defining

$$F = \begin{bmatrix} 0 & I \\ -\mathcal{C}_n & -\mathcal{B}_n \end{bmatrix}, \quad G = \begin{bmatrix} I_n & 0 \\ 0 & \mathcal{A}_n \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} x \\ \lambda x \end{bmatrix}, \tag{16}$$

equation (15) can be written as a generalized eigenvalue problem $Fy = \lambda G y$. It suffices to show that $\deg(\det(P(\lambda))) \leq n$ with $P(\lambda) = G\lambda - F$ to conclude that there are only n eigenvalues. The induction is on the dimension n, which will be shown as a subscript to the matrices. The element on position (i, j) in matrix X is denoted by x_{ij}. It is easily checked that the induction hypothesis holds for $n = 1$ and $n = 2$. Now assume that $\deg(\det(P_{n-1}(\lambda))) \leq n - 1$ and $\deg(\det(P_{n-2}(\lambda))) \leq n - 2$. Then we calculate the determinant of $P_n(\lambda)$ as

$$\det(P_n(\lambda)) = \det \begin{bmatrix} \lambda I_{n-1} & 0 & -I_{n-1} & 0 \\ 0 & \lambda & 0 & -1 \\ \mathcal{C}_{n-1} & v_{n-1} & \lambda \mathcal{A}_{n-1} + \mathcal{B}_{n-1} & 0 \\ w_{n-1} & c_{nn} & z_{n-1} & b_{nn} \end{bmatrix}$$

where $v_{n-1} = [\,0 \ \cdots \ 0 \ c_{n-1,n}\,]^T$, $w_{n-1} = [\,0 \ \cdots \ 0 \ c_{n,n-1}\,]$ and $z_{n-1} = [\,0 \ \cdots \ 0 \ \lambda a_{n,n-1} + b_{n,n-1}\,]$ are vectors of length n. Expanding along the n-th row yields

$$\det(P_n(\lambda)) = \lambda \det \begin{bmatrix} \lambda I_{n-1} & -I_{n-1} & 0 \\ \mathcal{C}_{n-1} & \lambda \mathcal{A}_{n-1} + \mathcal{B}_{n-1} & 0 \\ w_{n-1} & z_{n-1} & b_{nn} \end{bmatrix} +$$

$$(-1)^{n+1} \det \begin{bmatrix} \lambda I_{n-1} & 0 & -I_{n-1} \\ \mathcal{C}_{n-1} & v_{n-1} & \lambda \mathcal{A}_{n-1} + \mathcal{B}_{n-1} \\ w_{n-1} & c_{nn} & z_{n-1} \end{bmatrix}.$$

The first determinant equals $b_{nn}\det(P_{n-1}(\lambda))$ and the second one can be expanded along the n-th column, which only has 2 elements. Thus we find

$$\det(P_n(\lambda)) = \lambda b_{nn}\det(P_{n-1}(\lambda)) + c_{nn}\det(P_{n-1}(\lambda)) -$$

$$c_{n-1,n}\det\begin{bmatrix} \lambda I_{n-1} & -I_{n-1} \\ \mathcal{C}_{n-1}(1:n-2,:) & \lambda\mathcal{A}_{n-1}(1:n-2,:)+\mathcal{B}_{n-1}(1:n-2,:) \\ w_{n-1} & z_{n-1} \end{bmatrix},$$

where we have used the MATLAB-notation $X(1:n-2,:)$ to denote the first $n-2$ rows of X. Because of the induction hypothesis the first two terms have degree at most n. The last determinant can be written as

$$\det\begin{bmatrix} \lambda I_{n-2} & 0 & -I_{n-2} & 0 \\ 0 & \lambda & 0 & -1 \\ \mathcal{C}_{n-2} & v_{n-2} & \lambda\mathcal{A}_{n-2}+\mathcal{B}_{n-2} & 0 \\ 0 & c_{n,n-1} & 0 & \lambda a_{n,n-1}+b_{n,n-1} \end{bmatrix}.$$

Expanding along the last column gives

$$(-1)^n\det\begin{bmatrix} \lambda I_{n-2} & 0 & -I_{n-2} \\ \mathcal{C}_{n-2} & v_{n-2} & \lambda\mathcal{A}_{n-2}+\mathcal{B}_{n-2} \\ 0 & c_{n,n-1} & 0 \end{bmatrix} +$$

$$(\lambda a_{n,n-1}+b_{n,n-1})\det\begin{bmatrix} \lambda I_{n-2} & 0 & -I_{n-2} \\ 0 & \lambda & 0 \\ \mathcal{C}_{n-2} & v_{n-2} & \lambda\mathcal{A}_{n-2}+\mathcal{B}_{n-2} \end{bmatrix}$$

and this equals $c_{n,n-1}\det(P_{n-2}(\lambda)) + (\lambda a_{n,n-1}+b_{n,n-1})\lambda\det(P_{n-2}(\lambda))$, which has degree at most n according to the induction hypothesis.

5.3 Solving the QEP

The common way of solving a quadratic eigenvalue problem is through solving the aforementioned generalized eigenvalue problem $Fy = \lambda Gy$ using the QZ-algorithm, which is based on a generalized Schur decomposition (see e.g. [3]). This however does not take into account the fact that there are only n eigenvalues in our case, and it uses matrices of size $2n$ in stead of n. A recent article by N.J. Higham [5] presents another solution overcoming these problems. Consider the quadratic matrix equation

$$\mathcal{A}_n X^2 + \mathcal{B}_n X + \mathcal{C}_n = 0. \tag{17}$$

If S is a solution of (17) then

$$\mathcal{A}_n\lambda^2 + \mathcal{B}_n\lambda + \mathcal{C}_n = -(\mathcal{B}_n + \mathcal{A}_n S + \mathcal{A}_n\lambda)(S - \lambda I_n). \tag{18}$$

This means that every eigenvalue-eigenvector pair of S is also an eigenvalue-eigenvector pair for the quadratic eigenvalue problem. Since there are only n eigenvalues in our case, the existence of a solution S implies that its eigenvalues are the zeros of $\phi_n(z)$. The eigenvectors v_i corresponding to the zeros λ_i of $\phi_n(z)$ are

$$v_1 = \begin{bmatrix} p_0(\lambda_1) \\ p_1(\lambda_1) \\ p_2(\lambda_1) \\ \vdots \\ p_{n-2}(\lambda_1) \\ p_{n-1}(\lambda_1) \end{bmatrix}, v_2 = \begin{bmatrix} p_0(\lambda_2) \\ p_1(\lambda_2) \\ p_2(\lambda_2) \\ \vdots \\ p_{n-2}(\lambda_2) \\ p_{n-1}(\lambda_2) \end{bmatrix}, \cdots, v_n = \begin{bmatrix} p_0(\lambda_n) \\ p_1(\lambda_n) \\ p_2(\lambda_n) \\ \vdots \\ p_{n-2}(\lambda_n) \\ p_{n-1}(\lambda_n) \end{bmatrix}.$$

Since the polynomials $p_k(z)$ alle have degree exactly k, these vectors are linearly independent. It follows that $S = V\Lambda V^{-1}$ with $V = [\ v_1 \ \ \cdots \ \ v_n\ ]$ and $\Lambda = \text{diag}(\lambda_i)$ forms a solution to (17). We have thus proved the following theorem.

Theorem 5.3.1. *With $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$ as defined above, the zeros of the orthonormal rational function $\phi_n(z)$ are the solutions of the quadratic eigenvalue problem*

$$(\mathcal{A}_n\lambda^2 + \mathcal{B}_n\lambda + \mathcal{C}_n)p(\lambda) = 0$$

and the eigenvalues of any matrix X solving

$$\mathcal{A}_nX^2 + \mathcal{B}_nX + \mathcal{C}_n = 0.$$

One way of solving this quadratic matrix equation is using Newton's method as described in [4]. This is an iterative method which in the scalar case reduces to the Newton-Raphson procedure for solving nonlinear equations. Let $Q(X) = \mathcal{A}_nX^2 + \mathcal{B}_nX + \mathcal{C}_n$ then it follows that

$$\begin{aligned} Q(X+E) &= Q(X) + (\mathcal{A}_nEX + (\mathcal{A}_nX + \mathcal{B}_n)E) + \mathcal{A}_nE^2 \\ &= Q(X) + D_X(E) + \mathcal{A}_nE^2 \end{aligned}$$

where $D_X(E)$ is called the Fréchet derivative of Q at X in the direction E. Newton's method drops the second order term, defines E as the solution of $Q(X) + D_X(E) = 0$, and replaces X by $X + E$. Each step of Newton's method involves finding the solution E of

$$\mathcal{A}_nEX + (\mathcal{A}_nX + \mathcal{B}_n)E = -Q(X). \tag{19}$$

It is well known that this equation has a solution if and only if the pair $(-\mathcal{A}_n, \mathcal{A}_nX + \mathcal{B}_n)$ is regular and the eigenvalues of the pair are distinct from the eigenvalues of X. If X satisfies $Q(X) = 0$ then we know from (18) that the generalized eigenproblem $(\mathcal{A}_nX + \mathcal{B}_n)x = -\lambda\mathcal{A}_nx$ has exactly n eigenvalues at infinity and therefore the pair is regular. To solve (19) we can use the generalized Schur decomposition of $\mathcal{A}_n$ and $\mathcal{A}_nX + \mathcal{B}_n$,

$$W^*\mathcal{A}_nZ = T, \quad W^*(\mathcal{A}_nX + \mathcal{B}_n)Z = S \quad \text{en} \quad U^*XU = R,$$

where W,Z and U are unitary and T, S and U are upper triangular. Pre- and postmultiplying by W^* and U respectively then yields

$$TYR + SY = F, \quad F = -W^*Q(X)U, \quad Y = Z^*EU.$$

Equating kth columns and rearranging leads to

$$(S + r_{kk}T)y_k = f_k - \sum_{i=1}^{k-1} r_{ik}Ty_i, \quad Y = [\ y_1 \ \ y_2 \ \ \cdots \ \ y_n\ ].$$

Thus Y can be computed a column at a time for $k = 1, 2, \ldots, n$ by solving an upper triangular system. It remains an open problem to guarantee convergence for specific starting matrices, but if the starting matrix is close enough to a solvent, convergence will be quadratic.

5.4 Numerical results

Although it deserves further investigation how the quadratic eigenvalue should be solved efficiently taking into account the special structure of the matrices $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$, we can state some preliminary results already. Consider the space $\mathcal{L}_n$ with poles at the integer multiples of $\omega = 1.1$ and the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(z)\overline{g(z)}dz$$

as in the example of section 3.2. We have computed the zeros of $\phi_{10}(z)$ through the quadratic eigenvalue problem. The recursion coefficients were calculated in exact arithmetic, using a multiprecision package. To solve the quadratic eigenvalue problem, we used both the QZ-algorithm and Newton's method. Calculations were done in MATLAB (the QZ-algorithm is implemented in the MATLAB-function *polyeig* which solves a polynomial eigenvalue problem).

computed eigenvalues	zeros of $\phi_{10}(z)$
0.9817327	0.9843981
0.9074863	0.9090023
0.7529409	0.7510214
0.5041208	0.5033191
0.1805728	0.1841242
−0.1694727	−0.1643970
−0.4899013	−0.4885127
−0.7400332	−0.7429567
−0.9026296	−0.9061431
−0.9823931	−0.9839546

Table 1: Computed eigenvalues vs exact zeros of $\phi_{10}(z)$. Poles at $\{\omega, -\omega, 2\omega, -2\omega, \ldots\}$ with $\omega = 1.1$ and weight $= 1$ in $[-1, 1]$.

computed eigenvalues	zeros of $\phi_{10}(z)$
9.787736	9.787736
6.143134	6.143134
3.965084	3.965084
2.559230	2.559230
1.624316	1.624315
$9.940455\ 10^{-1}$	$9.940485\ 10^{-1}$
$5.687804\ 10^{-1}$	$5.687751\ 10^{-1}$
$2.872057\ 10^{-1}$	$2.872108\ 10^{-1}$
$1.118484\ 10^{-1}$	$1.118463\ 10^{-1}$
$2.072127\ 10^{-2}$	$2.072144\ 10^{-2}$

Table 2: Computed eigenvalues vs exact zeros of $\phi_{10}(z)$. There is a multiple pole $\alpha = -2$ and the weight function is $w(z) = e^{-z}$ in $[0, \infty)$.

Both methods yield approximately the same results, which are shown in table 1. Since the entries of matrices A, B and C were computed with all significant digits correct, comparison between the computed eigenvalues and the actual zeros indicates that the quadratic eigenvalue problem is ill-conditioned. Only three digits of the computed solution are correct. In the following example we computed the zeros of $\phi_{10}(z)$ with all poles equal to $\alpha = -2$, orthogonality over the interval $[0, \infty)$ and weight $w(z) = e^{-z}$. The results are shown in table 2. We find better results than in the previous example, especially for the larger zeros. However, if we increase the degree n of the function whose zeros we wish to compute, accuracy is lost. In the first example, if we compute the zeros of ϕ_{20} we only have two correct digits, for $n = 50$ none of the eigenvalues corresponds to a zero of ϕ_{50}. These experiments indicate that the quadratic eigenvalue problem is very ill-conditioned. Further research needs to be done to investigate this.

6 Conclusion

Our main goal in this paper was to present some new results about orthogonal rational functions on the real line and an interval of the real line. We derived a recurrence relation which holds for functions with arbitrary complex poles and studied some properties of the quasi- and para-orthogonal functions on a finite interval in relation with quadrature formulas. In the case of real poles the orthogonal rational functions are a special case of the quasi-orthogonal functions and their zeros are used as nodes in the quadrature formulas. We showed that they satisfy a quadratic eigenvalue problem which may be ill-conditioned.

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A Zeros of quasi-orthogonal functions on a finite interval

In this paragraph we investigate the behavior of the zeros of quasi-orthogonal functions as a function of the parameter τ. We will give results about the continuity of these zeros as a function of τ and we will show that these functions are monotonous increasing or decreasing.

Since we already established that, for a given value of τ, the zeros are simple, it follows that these functions cannot intersect. Further on we assume the interval of integration to be $[-1,1]$.

Let $z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}$ denote the zeros of $\phi_n(z)$ in ascending order and $\xi_1(\tau), \xi_2(\tau), \ldots, \xi_n(\tau)$ the zeros of $Q_n(z,\tau)$, also in ascending order. The numerator $q_n(z,\tau)$ of $Q_n(z,\tau)$ equals $p_n(z) + \tau\alpha_n/\alpha_{n-1}(z-\alpha_{n-1})p_{n-1}(z)$ with $p_n(z) = \sum_{i=0}^{n} c_i^{(n)} z^i$ the numerator of $\phi_n(z)$. The value τ_c for which q_n has degree $n-1$ equals

$$\tau_c = -\frac{\alpha_{n-1}}{\alpha_n}\frac{c_n^{(n)}}{c_{n-1}^{(n-1)}}.$$

For this value of τ the function $Q_n(z,\tau)$ has exactly $n-1$ zeros and therefore the coefficient of z^{n-1} in $q_n(z,\tau_c)$ is nonzero.

The implicit function theorem states that for every zero (ξ^*,τ^*) of $q_n(z,\tau)$ (as a function in two variables) there exists an interval around τ^* and a continuous differentiable function $\xi(\tau)$ such that $\xi(\tau^*) = \xi^*$ and $q_n(\xi(\tau),\tau) = 0$ for every τ in this interval, if

$$\left.\frac{\partial q_n}{\partial z}\right|_{(\xi(\tau),\tau)} \neq 0.$$

Since the zeros of $q_n(z,\tau)$ are simple, this condition is always satisfied. Now take $\tau = \tau_c$. Then $q_n(z,\tau)$ has exactly $n-1$ zeros which are continuous functions of τ on an interval around τ_c. Because there are at least $n-1$ zeros for every value of τ, this interval can be extended to the entire real line. Denote these zeros by $\xi_{1,c}(\tau), \xi_{2,c}(\tau), \ldots, \xi_{n-1,c}(\tau)$, again in ascending order. If $\tau \neq \tau_c$ then there is one more zero which is also a continuous function of τ. For $\tau > \tau_c$ we denote this zero by $\xi_{n,>}(\tau)$ and for $\tau < \tau_c$ by $\xi_{n,<}(\tau)$. Then it holds that $\xi_{n,>}(\tau)$ is a continuous function on the interval (τ_c,∞) and $\xi_{n,<}(\tau)$ is a continuous function on the interval $(-\infty,\tau_c)$.

Now let us consider what happens if $\tau \to \tau_c$. Let $a_i^{(n)}(\tau)$ denote the coefficient of z^i in $q_n(z,\tau)$, then for $\tau > \tau_c$ we have

$$\xi_{1,c}(\tau) + \xi_{2,c}(\tau) + \ldots + \xi_{n-1,c}(\tau) + \xi_{n,>}(\tau) = -\frac{a_{n-1}^{(n)}(\tau)}{a_n^{(n)}(\tau)}.$$

As mentioned above, the two coefficients in the right hand side cannot be both zero at the same time. Taking the limit as $\tau \to \tau_c$ then shows that

$$\lim_{\tau\to\tau_c} \xi_{n,>}(\tau) = \pm\infty$$

since the first $n-1$ functions on the left hand side are bounded. The sign of the limit depends on the position of the pole α_{n-1} and on the value of τ_c, as explained below. A similar argument holds for $\xi_{n,<}(\tau)$ and because τ_c is a simple zero of the leading coefficient $a_n^{(n)}(\tau)$ the sign of the limit will be the opposite. Define the function $\xi_{n,c}$ as follows:

$$\begin{aligned}\xi_{n,c}(\tau) &= \xi_{n,<}(\tau), \quad \tau < \tau_c,\\ &= \xi_{n,>}(\tau), \quad \tau > \tau_c.\end{aligned}$$

Now assume that the pole α_{n-1} is to the left of the interval of integration and that $\tau_c < 0$. A similar argument holds for the other cases. The correspondence between $\xi_{j,c}(\tau)$ and $\xi_i(\tau)$ as defined above is then as follows, as can easily be seen:

$$\begin{aligned}
\xi_i(\tau) &= \xi_{i,c}(\tau), \quad i = 2, \ldots, n-1 \\
\xi_1(\tau) &= \xi_{1,c}(\tau), \quad \tau > \tau_c \\
&= \xi_{n,kr}(\tau), \quad \tau < \tau_c \\
\xi_n(\tau) &= \xi_{n,c}(\tau), \quad \tau > \tau_c \\
&= \xi_{n-1,c}(\tau), \quad \tau < \tau_c
\end{aligned}$$

and we also have

$$\begin{aligned}
\xi_{i,c}(0) &= z_i^{(n)}, \quad i = 1, \ldots, n \\
\lim_{\tau \to \infty} \xi_{i,c}(\tau) &= z_{i-1}^{(n-1)}, \quad i = 2, \ldots, n \\
\lim_{\tau \to \infty} \xi_{1,c}(\tau) &= \alpha_{n-1} \\
\lim_{\tau \to -\infty} \xi_{i,c}(\tau) &= z_i^{(n-1)}, \quad i = 1, \ldots, n-1 \\
\lim_{\tau \to -\infty} \xi_{n,c}(\tau) &= \alpha_{n-1}, \\
\lim_{\tau \overset{>}{\to} \tau_c} \xi_{n,c}(\tau) &= \infty, \\
\lim_{\tau \overset{<}{\to} \tau_c} \xi_{n,c}(\tau) &= -\infty.
\end{aligned}$$

Finally we will show that in this case the functions $\xi_{i,c}(\tau)$ are monotonous increasing. Using the definition of $q_n(z, \tau)$ and the fact that $q_n(\xi_{i,c}(\tau), \tau) \equiv 0$, we may write

$$q_n'(\xi_{i,c}(\tau), \tau) \frac{d\xi_{i,c}(\tau)}{d\tau} + \frac{\alpha_n}{\alpha_{n-1}} (\xi_{i,c}(\tau) - \alpha_{n-1}) p_{n-1}(\xi_{i,c}(\tau)) = 0.$$

The derivative of $\xi_{i,c}(\tau)$ thus equals

$$\frac{d\xi_{i,c}(\tau)}{d\tau} = -\frac{\alpha_n}{\alpha_{n-1}} (\xi_{i,c}(\tau) - \alpha_{n-1}) \frac{p_{n-1}(\xi_{i,kr}(\tau))}{q_n'(\xi_{i,c}(\tau), \tau)}. \tag{20}$$

The functions $\xi_{1,c}(\tau), \xi_{2,c}(\tau), \ldots, \xi_{n-1,c}(\tau)$ are all continuous on the real line and for finite values of τ they differ from α_{n-1}. The derivative (20) can therefore only become zero if $\xi_{i,c}(\tau)$ is a zero of $p_{n-1}(z)$. According to the definition of $q_n(z, \tau)$ this would imply that also $p_n(\xi_{i,c}(\tau))$ equals zero. The zeros of p_n and p_{n-1} however interlace, and thus for $i = 1, \ldots, n-1$ the derivative (20) is nonzero which means that $\xi_{i,c}(\tau)$ is monotonous increasing or decreasing. In this case it is easily checked that the functions are increasing. The same holds for $\xi_{n,c}(\tau)$, which is monotonous increasing on both the intervals $(-\infty, \tau_c)$ and (τ_c, ∞).