NURPS for quadrics and special effects

Joris Windmolders
Paul Dierckx

Report TW 317, December 2000

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

NURPS for quadrics and special effects

Joris Windmolders
Paul Dierckx

Report TW 317, December 2000

Department of Computer Science, K.U.Leuven

Abstract

NURPS are the rational extension of Powell–Sabin splines in their normalized B–spline representation. The weights associated with the control points allow to represent (parts of) some particular quadric surfaces. By choosing coincident control points, special effects on the surface can be generated.

Keywords : PS-spline, control point, quadric.

NURPS for special effects and quadrics

Joris Windmolders and Paul Dierckx

Abstract. NURPS are the rational extension of Powell–Sabin splines in their normalized B–spline representation. The weights associated with the control points allow to represent (parts of) some particular quadric surfaces. By choosing coincident control points, special effects on the surface can be generated.

§1. Preliminary Concepts

Given a simply connected subset $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\delta\Omega$, and a conforming triangulation Δ of Ω having n vertices V_i with coordinates (u_i, v_i), $i = 1, \ldots, n$, a Powell–Sabin refinement Δ^* which divides each triangle $\rho \in \Delta$ up into 6 nondegenerate subtriangles can easily be found (see, e.g., [8]). A Powell–Sabin (PS) spline is a piecewise quadratic polynomial with C^1 continuity on Ω, and has a quadratic Bézier representation on each PS–subtriangle $\tau \in \Delta^*$:

$$\boldsymbol{b}(t) = \sum_{|i|=2} \boldsymbol{b}_i B_i^2(t), \quad t = (t_1, t_2, t_3) \in \tau, \quad t_1 + t_2 + t_3 = 1.$$

Definition 1. *A PS–spline surface has a normalized B–spline representation*

$$s(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{3} \boldsymbol{c_{i,j}} B_i^j(u,v), \qquad (u,v) \in \Omega, \tag{1}$$

where $\boldsymbol{c_{i,j}} = (c_{i,j}^x, c_{i,j}^y, c_{i,j}^z)$ *are the B–spline control points and* $B_i^j(u,v)$ *are the normalized B–splines.*

The remainder of this section summarizes the relevant properties of this representation. For details we refer to [3]. The normalized B–splines are locally nonzero and constitute a convex partition of unity on Ω, hence this representation is affine invariant and has the local control and convex hull properties. Furthermore, the PS–spline surface is C^1–continuous regardless the choice of the control points. A control triangle $T_l(\boldsymbol{c_{l,1}}, \boldsymbol{c_{l,2}}, \boldsymbol{c_{l,3}})$ can be associated with each vertex V_l, $l = 1, \ldots, n$ which is tangent to the surface at $\boldsymbol{s}(V_l)$. The projection of each T_l onto the domain plane yields the

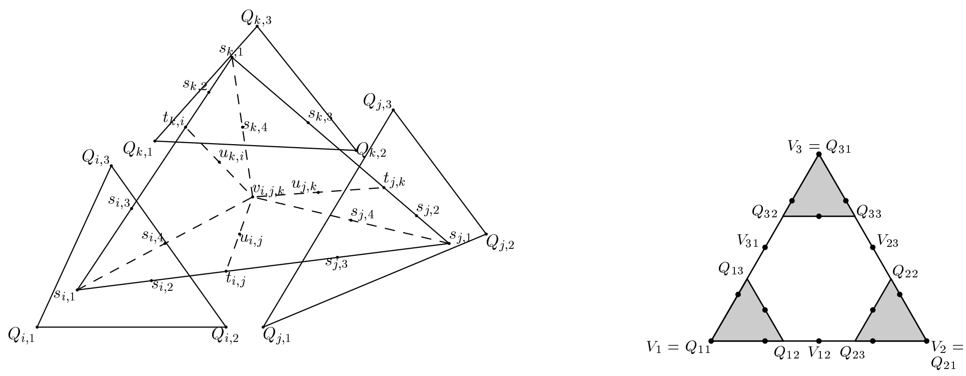

Fig. 1. A domain triangle.

PS–triangles $t_l(Q_{l,1}, Q_{l,2}, Q_{l,3})$, $l = 1, \ldots, n$, having the B–spline ordinates $Q_{l,j}(U_{l,j}, V_{l,j})$, $j = 1, 2, 3$ as its vertices.

Figure 1, left shows a domain triangle with its PS–subdivision and PS–triangles t_l. The Bézier ordinates are denoted $s_{v,l}$, $v = i, j, k, l = 1, 2, 3, 4$; $t_{l,m}, u_{l,m}$, $(l, m) \in \{(i, j), (j, k), (k, i)\}$ and $v_{i,j,k}$. Given a PS–subdivision, any PS–triangle t_l must contain the PS–points $s_{l,j}, j = 1, 2, 3, 4$ in order for the B–splines to constitute a convex partition of unity. In that case, The Bézier ordinates can be written as unique convex barycentric combinations of the B–spline ordinates:

$$\begin{aligned} s_{v,l} &= \alpha_{v,l}\ Q_{v,1} + \beta_{v,l}\ Q_{v,2} + \gamma_{v,l}\ Q_{v,3} \\ t_{l,m} &= \delta_{l,m}\ s_{l,2} + \epsilon_{l,m}\ s_{m,3} \\ u_{l,m} &= \delta_{l,m}\ s_{l,4} + \epsilon_{l,m}\ s_{m,4} \\ v_{i,j,k} &= \lambda_{i,j,k}\ s_{i,4} + \mu_{i,j,k}\ s_{j,4} + \nu_{i,j,k}\ s_{k,4}. \end{aligned} \tag{2}$$

The same combinations are valid for the Bézier control points, which can in turn be found as convex barycentric combinations of the PS–control points.

Definition 2. *A Non Uniform Rational Powell–Sabin (NURPS) spline surface has the form*

$$s(u, v) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{3} \boldsymbol{c_{i,j}} w_{i,j} B_i^j(u, v)}{\sum_{i=1}^{n} \sum_{j=1}^{3} w_{i,j} B_i^j(u, v)}, \qquad (u, v) \in \Omega \tag{3}$$

where $w_{i,j} > 0$ in order for $s(u, v)$ to be defined anywhere on Ω.

For the evaluation of this rational extension, calculating the corresponding rational Bézier representation, the geometric interpretation of the weights and the modelling of planar effects using relatively large weights we refer to [10].

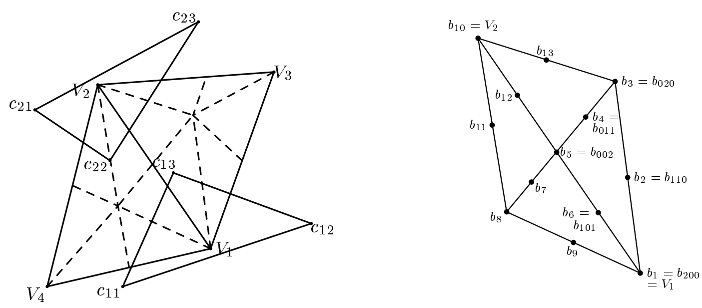

Fig. 2. Degenerate control triangles.

§2. Degenerate Control Triangles

In spite of the global C^1–continuity of (1), modeling in practice often requires the representation of C^0–continuous effects. Here we show how corners can be represented by choosing coincident control points.

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be the barycentric coordinates of two distinct domain points with respect to a triangle ρ. The difference $d = u - v = (d_1, d_2, d_3)$ defines a vector $(d_1 + d_2 + d_3 = 0)$. Be given a rational Bézier surface on ρ:

$$\boldsymbol{b}(t) = \frac{\sum_{|i|=2} w_i \boldsymbol{b}_i B_i^2(t)}{\sum_{|i|=2} w_i B_i^2(t)}, \quad t = (t_1, t_2, t_3) \in \tau. \tag{4}$$

Definition 3. *The directional derivative of a surface (4) at* $\boldsymbol{b}(u)$ *with respect to* d *is given by*

$$D_d \boldsymbol{b}(u) = d_1 \boldsymbol{b}_{t_1}(u) + d_2 \boldsymbol{b}_{t_2}(u) + d_3 \boldsymbol{b}_{t_3}(u) \tag{5}$$

where $\boldsymbol{b}_{t_i}(u)$ *is the partial derivative of* $\boldsymbol{b}(t)$ *with respect to* $t_i, i = 1, 2, 3,$ *evaluated at* u. *Note that these are not tangent vectors to the surface at* $\boldsymbol{b}(u)$.

Consider the domain triangles $\rho_1, \rho_2 \in \Delta$ with common boundary $V_1 V_2$. The PS–refinement and PS–triangles are shown in figure 2, left. The Bézier subtriangles along $V_1 - V_2$ are shown in figure 2, right. Be given the barycentric coordinates of a direction d and a point $u = (u_1, 0, u_3)$ with respect to $\tau(b_{200}, b_{110}, b_{101})$. The directional derivative of $\boldsymbol{b}(u)$ with respect to d is given by

$$\begin{aligned} D_d \boldsymbol{b}(u) = 2(&d_1(u_1 \boldsymbol{b}_{200} + u_3 \boldsymbol{b}_{101}) \\ &+ d_2(u_1 \boldsymbol{b}_{110} + u_3 \boldsymbol{b}_{011}) \\ &+ d_3(u_1 \boldsymbol{b}_{101} + u_3 \boldsymbol{b}_{002})) \end{aligned} \tag{6}$$

First, consider the case where one of the control triangles is degenerated, e.g., $\boldsymbol{c}_{1,1} = \boldsymbol{c}_{1,2} = \boldsymbol{c}_{1,3} = \boldsymbol{C}$. It follows from (2) that

$$D_d \boldsymbol{b}(u) = 2u_3(d_1 \boldsymbol{C} + d_2 \boldsymbol{b}_{011} + d_3 \boldsymbol{b}_{011})$$

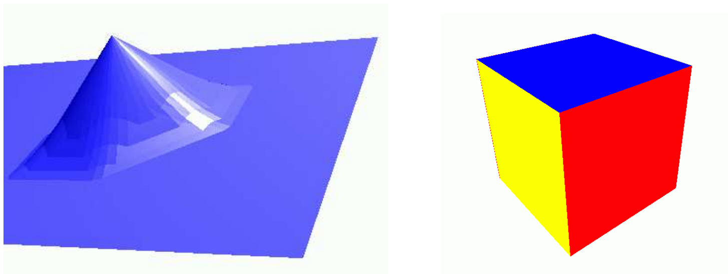

Fig. 3. Degenerate control triangles.

The directional derivate is seen to degenerate to the zero vector at $u_3 = 0$, hence the surface has a singulartity at V_1. Figure 3, left shows an example.

Next, consider the case where both control triangles at V_1 and V_2 are degenerated: $\boldsymbol{c_{1,1}} = \boldsymbol{c_{1,2}} = \boldsymbol{c_{1,3}} = \boldsymbol{C}$ and $\boldsymbol{c_{2,1}} = \boldsymbol{c_{2,2}} = \boldsymbol{c_{2,3}} = \boldsymbol{D}$. We then have

$$D_d\boldsymbol{b}(u) = 2\alpha u_3 d_1(\boldsymbol{C} - \boldsymbol{D})$$

for some $\alpha \in (0, 1)$. This vector degenerates to the zero vector for $u_3 = 0$ and $d_1 = 0$ and coincides with $\boldsymbol{d_c} = \frac{\boldsymbol{C}-\boldsymbol{D}}{||\boldsymbol{C}-\boldsymbol{D}||}$ elsewhere. As an application, a corner along $V_1 - V_2$ in the direction of $\boldsymbol{d_c}$ can be represented. Figure 3, right shows a cube represented as a NURPS surface.

§3. An optimal Conversion

In [10] a general formula is provided for calculating the NURPS representation of a given quadratic rational Bézier surface on one triangle. Following Dierckx [3], this representation is not *optimal* because the PS–triangles are not the smallest triangles containing the PS–points. The advantage of optimal PS–triangles is that the control triangles are closer to the surface than in the nonoptimal case. Suppose we are given a quadratic Bézier control net. We are free to choose the domain triangle to be equilateral. The optimal PS–triangles for this case can be found by solving a quadratic programming problem (see, again, [3]). The results are depicted in figure 1, right. The dots represent the PS–points, the shaded triangles are optimal PS–triangles. It can be shown that these are calculated from the Bézier ordinates by

$$\begin{aligned} Q_{i,1} &= V_i \\ Q_{i,2} &= \frac{2}{3}V_{i,i_3+1} + \frac{1}{3}V_i \\ Q_{i,3} &= \frac{2}{3}V_{(i+1)_3+1,i} + \frac{1}{3}V_i \end{aligned} \tag{7}$$

where $k_3 = k \bmod 3$ and $V_{i,j} = \frac{1}{2}(V_i + V_j)$. The B–spline control points can be found from these equations by replacing the Bézier ordinates with the corresponding control points.

§4. Quadrics as NURPS

Representing quadrics as rational Bézier patches has been investigated in [1], [2], [4], [5], [6] and [9]. Here we present a constructive approach to the problem of representing quadrics as NURPS. The purpose is to derive closed formulae for the control triangles of (patches on) a cylinder, cone and sphere of given dimension, which is usefull in CAGD systems. The calculations have been made using modern computer techniques, such as the symbolic mathematical manipulation package Maple. We restrict ourselves to investigate the case of the cylinder, and mention the results for a cone and a sphere only briefly.

4.1. The Cylinder

Using symmetry operations, we brake up the surface into isometrical triangular parts. Then we calculate the rational Bézier representation

$$\boldsymbol{b}(t) = \frac{1}{q(t)} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \frac{\sum_{i=0}^{5} w_i \boldsymbol{P_i} B_i^2(t)}{\sum_{i=0}^{5} w_i B_i^2(t)}, \quad t = (t_1, t_2, t_3) \in \tau \tag{8}$$

of one such triangle. Note that we use a different indexing here then in former equations, in order to simplify notation. Application of (7) immediately yields the NURPS representation.

Figure 4 shows the octant of the cylinder with radius r and height equal to h:

$$\left(\frac{x(t)}{q(t)}\right)^2 + \left(\frac{y(t)}{q(t)}\right)^2 = r^2. \tag{9}$$

The Bézier control points follow from the fact that the control net is tangent to the surface at the corners of the patch:

$\boldsymbol{P_0} = (r, 0, 0)$ $\quad$ $\boldsymbol{P_1} = (r, r, 0)$ $\quad$ $\boldsymbol{P_2} = (0, r, 0)$
$\boldsymbol{P_3} = (r, r, \frac{h}{2})$ $\quad$ $\boldsymbol{P_4} = (r, 0, h)$ $\quad$ $\boldsymbol{P_5} = (r, 0, \frac{h}{2})$

Setting $w_0 = w_2 = w_4 = w_5 = 1$ since the corresponding control points are on the surface, we can find the weights w_1 and w_3 by imposing that the edge curves satisfy (9), e.g.,

$$\begin{aligned} R_1 &\leftrightarrow x^2(t_1, 1-t_1, 0) + y^2(t_1, 1-t_1, 0) - q^2(t_1, 1-t_1, 0) = 0 \\ &\leftrightarrow t_1^4(4w_1^2 - 2) + t_1^3(4 - 8w_1^2) + t_1^2(4w_1^2 - 2) = 0 \\ &\leftrightarrow w_1 = \frac{\sqrt{2}}{2} \end{aligned}$$

Analogeously we find $w_3 = \frac{\sqrt{2}}{2}$. Substituting these weights in (8), one can verify that (9) is satisfied for all $t \in \tau$. The corresponding B–spline control points for a cylinder of radius r and heigth h can be found from (7), referring to figure 1, right:

$\boldsymbol{i}$	$\boldsymbol{c_{i,1}}$	$w_{i,1}$	$\boldsymbol{c_{i,2}}$	$w_{i,2}$	$\boldsymbol{c_{i,3}}$	$w_{i,3}$
1	$(r, 0, 0)$	1	$(r, \frac{2}{3}r, 0)$	$\frac{\sqrt{2}+1}{3}$	$(r, 0, \frac{1}{3}h)$	1
2	$(0, r, 0)$	1	$(\frac{2}{3}r, r, \frac{1}{3}h)$	$\frac{\sqrt{2}+1}{3}$	$(\frac{2}{3}r, r, 0)$	$\frac{\sqrt{2}+1}{3}$
3	$(r, 0, h)$	1	$(r, 0, \frac{2}{3}h)$	1	$(r, \frac{2}{3}r, \frac{2}{3}h)$	$\frac{\sqrt{2}+1}{3}$

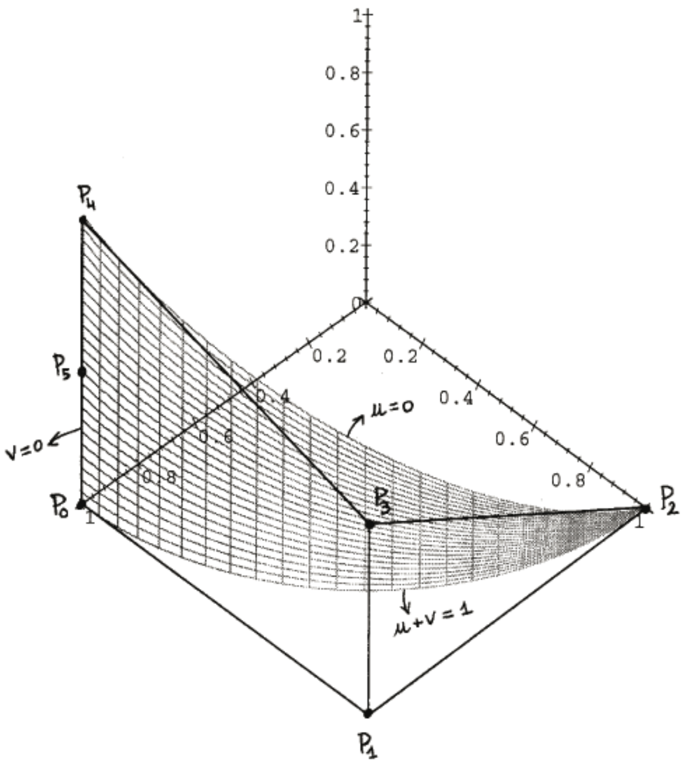

Fig. 4. Quadric segment of a cylinder.

By combining 8 isometrical patches, we have a complete NURPS cylinder (see figure 5, left).

4.2. The cone

The NURPS control points for the 1/6 part of a cone with radius r and height h are:

i	$c_{i,1}$	$w_{i,1}$	$c_{i,2}$	$w_{i,2}$	$c_{i,3}$	$w_{i,3}$
1	$(r,0,0)$	1	$(r,\frac{2\sqrt{3}}{9}r,0)$	$\frac{\sqrt{3}+1}{3}$	$(\frac{1}{3}r,0,\frac{2}{3}h)$	1
2	$(r,\frac{\sqrt{3}}{2}r,0)$	1	$(\frac{1}{6}r,\frac{\sqrt{3}}{6}r,\frac{2}{3}h)$	1	$(\frac{5}{6}r,\frac{7\sqrt{3}}{18}r,0)$	$\frac{\sqrt{3}+1}{3}$
3	$(0,0,h)$	1	$(0,0,h)$	1	$(0,0,h)$	1

The complete cone is shown in figure 5, middle.

4.3. The Sphere

It is not possible to represent a complete sphere by collecting non-degenerate quadratic rational Bézier triangles (see [1], [2], [9]). The NURPS patch given

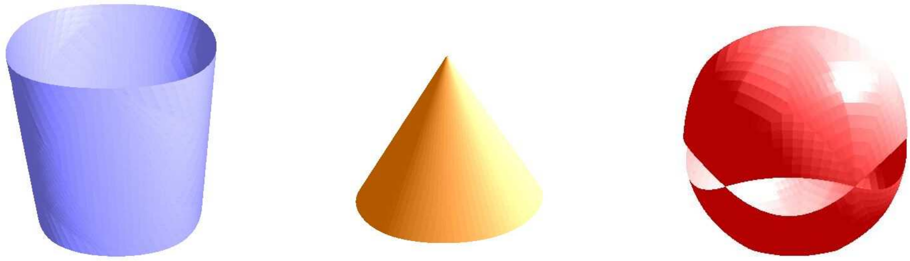

Fig. 5. Quadric NURPS.

in the table below however allows to approximate the sphere, leaving small gaps around the equator plane (see figure 5, right).

i	$c_{i,1}$	$w_{i,1}$	$c_{i,2}$	$w_{i,2}$	$c_{i,3}$	$w_{i,3}$
1	$(\frac{\sqrt{3}}{2}r, -\frac{r}{2}, 0)$	1	$(\frac{11\sqrt{3}}{18}r, -\frac{1}{6}r, \frac{2}{9}r)$	$\frac{5}{6}$	$(\frac{\sqrt{3}}{2}r, -\frac{1}{2}r, \frac{2}{3}r)$	$\frac{\sqrt{2}+1}{3}$
2	$(\frac{\sqrt{3}}{2}r, \frac{1}{2}r, 0)$	1	$(\frac{\sqrt{3}}{2}r, \frac{1}{2}r, \frac{2}{3}r)$	$\frac{\sqrt{2}+1}{3}$	$(\frac{11\sqrt{3}}{18}r, \frac{1}{6}r, \frac{2}{9}r)$	$\frac{5}{6}$
3	$(0, 0, r)$	1	$(\frac{\sqrt{3}}{3}r, -\frac{1}{3}r, r)$	$\frac{\sqrt{2}+1}{3}$	$(\frac{\sqrt{3}}{3}r, \frac{1}{3}r, r)$	$\frac{\sqrt{2}+1}{3}$

4.4. Parametric Continuity

In spite of the visual smoothness of the surfaces obtained by combining isometric NURPS patches, , the $q(t)$ component from (8) in the global representation as a piecewise rational Bézier surface, or as a piecewise 4D Bézier surface in homogeneous space, is not C^1–continuous. Hence, there does not exist a global NURPS representation with respect to the given parameter space.

References

1. Boehm, W., and Hansford, D., Bézier patches on quadrics, in *NURBS for Curve and Surface Design*, G. Farin (ed.), SIAM, Philadelphia, PA, 1991, 1–14.
2. Boehm, W., Some remarks on quadrics, Comput. Aided Geom. Design **10** (1993), 231–236.
3. Dierckx, P., On calculating normalized Powell–Sabin B–splines, Comput. Aided Geom. Design **15** (1997), 61–78.
4. Dietz, R., Hoschek, J., and Jüttler, B., An algebraic approach to curves and surfaces on the sphere and on other quadrics, Comput. Aided Geom. Design **10** (1993), 211–229.
5. Dietz, R., Hoschek, J., and Jüttler, B., Rational patches on quadric surfaces, Computer-Aided Design **27** (1995), 27–40.
6. Farin, G., Piper, B., and Worsey, A. J., The octant of a sphere as a nondegenerate triangular Bézier patch, Comput. Aided Geom. Design **4** (1987), 329–332.

7. Farin, G., *Curves And Surfaces For Computer Aided Geometric Design: A Practical Guide*, 4th edition, Academic Press, Boston, 1997.

8. Powell, M. J. D. and Sabin, M. A., Piecewise quadratic approximations on triangles, ACM Trans. Math. Software **3** (1977), 316–325.

9. Sederberg, T. W., and Anderson, D. C., Steiner surface patches, IEEE Comp. Graphics Appl. **5** (1985), 23–36.

10. Windmolders, J., and Dierckx, P., From PS–splines to NURPS, *Curve and Surface Fitting: Saint-Malo 1999*, Albert Cohen, Christophe Rabut, and Larry L. Schumaker (eds.), Vanderbilt University Press, Nashville, 2000.

Joris Windmolders and Paul Dierckx
Department of Computer Sciences
Celestijnenlaan 200A
B-3001 Herverlee
Belgium
joris.windmolders@cs.kuleuven.ac.be
paul.dierckx@cs.kuleuven.ac.be