

# Quantum Complexity of Integration

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## Abstract

We study the computation of the integral of functions from the classical Hölder classes  $F_d^{k,\alpha}$  on  $[0, 1]^d$  and define  $\gamma$  by  $\gamma = (k + \alpha)/d$ . The known optimal orders for the complexity of deterministic and (general) randomized methods are

$$\text{comp}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-1/\gamma}$$

and

$$\text{comp}^{\text{random}}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-2/(1+2\gamma)}.$$

For a quantum computer we prove

$$\text{comp}_{\text{query}}^{\text{quant}}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-1/(1+\gamma)}$$

and

$$\text{comp}^{\text{quant}}(F_d^{k,\alpha}, \varepsilon) \leq C \varepsilon^{-1/(1+\gamma)} (\log \varepsilon^{-1})^{2/(1+\gamma)}.$$

For restricted Monte Carlo (only coin tossing instead of general random numbers) we prove

$$\text{comp}^{\text{coin}}(F_d^{k,\alpha}, \varepsilon) \leq C \varepsilon^{-2/(1+2\gamma)} (\log \varepsilon^{-1})^{1/(1+2\gamma)}.$$

**Keywords :** complexity of numerical integration, quantum computing, Monte Carlo methods

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# QUANTUM COMPLEXITY OF INTEGRATION

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To summarize the results one can say that

- there is a (roughly) quadratic speed-up of quantum algorithms over randomized classical methods, if  $\gamma$  is small;
- there is an exponential speed-up of quantum algorithms over deterministic (classical) algorithms, if  $\gamma$  is small.

## 1. INTRODUCTION AND RESULTS

1.1. **Computation of the mean.** Consider the following problem: Compute the mean

$$S_n(x) = \frac{1}{n} \sum_{i=1}^n x_i$$

of  $n$  numbers  $x_i$ , where  $|x_i| \leq 1$ , up to some error  $0 < \varepsilon < 1/2$ . The complexity of this problem depends on  $n$  and  $\varepsilon$  and in the real number model we obtain

$$(1) \quad \text{comp}(n, \varepsilon) \approx n \cdot (1 - \varepsilon).$$

Here we consider the worst case setting, with the worst case cost and the worst case error. With randomized methods we can do much better, at least if  $n$  is large

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compared to  $\varepsilon^{-2}$ . The cost is of the order

$$(2) \quad \text{comp}^{\text{random}}(n, \varepsilon) \asymp \min(n, \varepsilon^{-2}).$$

Now the error of a method is a random variable and the requirement is that its expectation is bounded by  $\varepsilon$ . The statements (1) and (2) follow easily from well known upper and lower bounds. See, for example, Novak (1988).

If we allow only random bits (restricted Monte Carlo methods, coin tossing) instead of arbitrary randomized methods then one gets the upper bound

$$(3) \quad \text{comp}^{\text{coin}}(n, \varepsilon) \leq C \cdot \min(n, \varepsilon^{-2} \log n)$$

which follows easily from (2).

A further improvement is possible by a quantum computer. The upper bound  $\varepsilon^{-1}$  for the query complexity, defined by the number of times the real valued oracle is accessed to solve the problem, is proved in Brassard, Høyer, Mosca, Tapp (2000). See Theorem 12 of this paper. Using also the lower bounds from Nayak, Wu (1998) one can see that the exact order of this query complexity is

$$(4) \quad \text{comp}_{\text{query}}^{\text{quant}}(n, \varepsilon) \asymp \min(n, \varepsilon^{-1}).$$

The bound (4) is very important for the present paper. Grover (1998) states the upper bound  $\varepsilon^{-1}$  and says that it is “up to polylogarithmic factors”.

If we consider, for the quantum computer, the bit number model then we need a slightly larger cost. Here we need  $\varepsilon^{-1} \log \varepsilon^{-1}$  oracle calls. An additional factor  $\log n$  comes in by the definition of the cost and one gets

$$(5) \quad \text{comp}^{\text{quant}}(n, \varepsilon) \leq C \cdot \min(n \log \varepsilon^{-1}, \varepsilon^{-1} \log n \log \varepsilon^{-1}),$$

see Section 2 for details on the model of computation. The output of a quantum algorithm is a random variable  $A(x, \varepsilon)$ , we always request that

$$(6) \quad |A(x, \varepsilon) - S_n(x)| \leq \varepsilon \quad \text{with probability at least } 3/4.$$

Of course we can run the algorithm several times to increase the probability of success.

**1.2. Computation of integrals.** How can we apply these results to the problem of numerical integration? Let us first consider the computation of the integral

$$I(f) = \int_{[0,1]^d} f(x) dx$$

for functions from the Hölder classes

$$F_d^\alpha = \{f : [0, 1]^d \rightarrow \mathbb{R} \mid \|f\|_\infty \leq 1, |f(x) - f(y)| \leq \|x - y\|^\alpha\},$$

$0 < \alpha \leq 1$ . Consider, for  $d = 1$ , the midpoint rule

$$Q_\ell^1(f) = \frac{1}{\ell} \sum_{i=1}^{\ell} f\left(\frac{2i-1}{2\ell}\right)$$

and for  $d > 1$  the respective tensor product  $Q_n^d$  that uses  $n = \ell^d$  function values. By well known estimates for  $d = 1$  together with the technique of Haber (1970, p. 489) we get the estimate

$$(7) \quad e(Q_n^d, F_d^\alpha) \leq C \cdot d \cdot n^{-\alpha/d}$$

for the worst case error of the product rule. To obtain  $e(Q_n^d, F_d^\alpha) \approx \varepsilon$ , we have to take

$$(8) \quad n(F_d^\alpha, \varepsilon) \approx \left( \frac{C d}{\varepsilon} \right)^{d/\alpha}.$$

We can now use the results from above to obtain upper bounds for the complexity of numerical integration. Using the (trivial) result (1) we get a bound for the (worst case) complexity of integration,

$$(9) \quad \text{comp}(F_d^\alpha, \varepsilon) \leq \left( \frac{C d}{\varepsilon} \right)^{d/\alpha}.$$

With (2) we obtain

$$(10) \quad \text{comp}^{\text{random}}(F_d^\alpha, \varepsilon) \leq C \cdot \varepsilon^{-2}$$

and if we only allow random bits then we obtain, using (3) and (8),

$$(11) \quad \text{comp}^{\text{coin}}(F_d^\alpha, \varepsilon) \leq C \frac{d}{\alpha} \varepsilon^{-2} (\log d + \log \varepsilon^{-1}).$$

In the same way we obtain, for the quantum computer, the upper bounds

$$(12) \quad \text{comp}_{\text{query}}^{\text{quant}}(F_d^\alpha, \varepsilon) \leq C \varepsilon^{-1}$$

and

$$(13) \quad \text{comp}^{\text{quant}}(F_d^\alpha, \varepsilon) \leq C \frac{d}{\alpha} \varepsilon^{-1} (\log d + \log \varepsilon^{-1}) \log \varepsilon^{-1}.$$

Observe that all these bounds (9)–(13) are just upper bounds which we get by a particular proof technique. Actually it is known that the order in (9) is optimal,

$$(14) \quad \text{comp}(F_d^\alpha, \varepsilon) \approx C_{d,\alpha} \varepsilon^{-d/\alpha},$$

while the upper bounds for Monte Carlo methods are not optimal, we have

$$(15) \quad \text{comp}^{\text{random}}(F_d^\alpha, \varepsilon) \approx C_{d,\alpha} \varepsilon^{-2d/(2\alpha+d)}$$

and

$$(16) \quad \text{comp}^{\text{coin}}(F_d^\alpha, \varepsilon) \leq C_{d,\alpha} \varepsilon^{-2d/(2\alpha+d)} \log \varepsilon^{-1}.$$

For the proof of (15) see Heinrich (1993), Novak (1988), or Traub, Wasilkowski, Woźniakowski (1988). It is not difficult to show that (16) follows from (15). Actually we will improve the exponent in the log-term slightly and prove such an upper bound with the factor  $(\log \varepsilon^{-1})^{1/(1+2\alpha/d)}$ , see (21).

**1.3. The problem and the results.** Can the upper bounds (12) and (13) be improved, similarly as the upper bounds (10) and (11)? What is the optimal rate of convergence (or the rate of the complexity) for numerical integration with a quantum computer?

In this paper we answer this question for classes such as the  $F_d^\alpha$ . We consider the more general Hölder classes

$$F_d^{k,\alpha} = \{f : [0, 1]^d \rightarrow \mathbb{R} \mid \|f\|_\infty \leq 1, f \in C^k, |D^i f(x) - D^i f(y)| \leq \|x - y\|^\alpha, \forall D^i\},$$

where  $D^i$  runs through the set of all partial derivatives of order  $k$  and  $k \in \mathbb{N}_0$ ,  $0 < \alpha \leq 1$ . For  $k = 0$  we obtain  $F_d^{0,\alpha} = F_d^\alpha$ . It is convenient to use the notation

$$\gamma = \frac{k + \alpha}{d},$$

because this number is a good measure for the smoothness and appears in all the estimates. First of all, the optimal orders for deterministic and (general) randomized methods are known, see, e.g., Novak (1988). We have

$$(17) \quad \text{comp}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-1/\gamma}$$

and

$$(18) \quad \text{comp}^{\text{random}}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-2/(1+2\gamma)}.$$

Therefore we have to study only the quantities  $\text{comp}^{\text{quant}}$ ,  $\text{comp}_{\text{query}}^{\text{quant}}$ , and  $\text{comp}^{\text{coin}}$ . For the upper bounds we use a technique called “variance reduction” in the literature on Monte Carlo methods. For the lower bound we use a decomposition technique of Bakhvalov, together with the lower bound of Nayak and Wu, see (4). We obtain the following optimal rates of convergence.

**Theorem 1.** *Define  $\gamma = (k + \alpha)/d$ , as above.*

$$(19) \quad \text{comp}_{\text{query}}^{\text{quant}}(F_d^{k,\alpha}, \varepsilon) \asymp \varepsilon^{-1/(1+\gamma)},$$

$$(20) \quad \text{comp}^{\text{quant}}(F_d^{k,\alpha}, \varepsilon) \leq C_{d,k,\alpha} \varepsilon^{-1/(1+\gamma)} (\log \varepsilon^{-1})^{2/(1+\gamma)},$$

$$(21) \quad \text{comp}^{\text{coin}}(F_d^{k,\alpha}, \varepsilon) \leq C_{d,k,\alpha} \varepsilon^{-2/(1+2\gamma)} (\log \varepsilon^{-1})^{1/(1+2\gamma)}.$$

To summarize the results one can say that

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1.4. **Some comments.** So far, most papers on quantum computing deal with discrete problems, such as factoring numbers or searching a database. Quantum computing also helps for the *continuous problems of numerical analysis* or *information-based complexity*. Grover (1998) studies, among other things, the computation of the mean of finitely many real numbers. Related problems and algorithms are investigated also in the papers Boyer, Brassard, Høyer, Tapp (1998), Brassard, Høyer, Tapp (1998), Grover (1996), and Mosca (1998). The paper Nayak, Wu (1998) contains new lower bounds, while the recent paper Brassard, Høyer, Mosca, Tapp (2000) contains new upper bounds.

Excellent surveys on quantum computing are Shor (1998) and Cleve, Ekert, Henderson, Macchiavello, Mosca (1999). Also the paper Abrams, Williams (1999) discusses the computation of sums and integrals, “as long as the function is not pathological”.

In numerical analysis and information-based complexity we usually assume the real number model with an oracle that gives function values, see Traub, Wasilkowski, Woźniakowski (1988) and, more formally, Novak (1995). Concerning the “allowed randomness” of the algorithms we may distinguish between three different cases. If a random number generator is available that can produce random  $\omega \in [0, 1]$  according to the Lebesgue measure then we obtain the well known result (18). Also the other extreme case, where no randomness is available, is well studied and we obtain the result (17). Hence we only have to consider the case of *restricted Monte Carlo methods* where coin tossing is allowed (and has unit cost), but not general random number generators. This case somehow corresponds to quantum computation where such a randomness can be easily realized.

In Section 2 we give a little tutorial on quantum computation and present the search algorithm of Grover. We explain the model of computation and the cost of a quantum computation. Our proofs are contained in Section 3. We add a section where we discuss the computation of arbitrary bounded random variables by a quantum computer with a random number generator. Here we use a rather unrealistic model of computation because we assume that there is a random number generator without cost.

## 2. THE MODEL OF COMPUTATION AND THE SEARCH ALGORITHM OF GROVER

In this section we describe the model of computation and the search algorithm of Grover (1996), see also Boyer et al. (1998). This section does not contain new results.

Let  $H_1$  be a 2-dimensional Hilbert space over  $\mathbb{C}$  and let  $e_0$  and  $e_1$  be two orthonormal vectors in  $H_1$ . The space  $H_1$  represents a quantum bit, in the Dirac notation we have

$$e_0 = |0\rangle \quad \text{and} \quad e_1 = |1\rangle.$$

For  $m \in \mathbb{N}$  quantum bits we use the  $2^m$ -dimensional tensor product space

$$H_m = H_1 \otimes \cdots \otimes H_1$$

with  $m$  factors. An orthonormal basis is given by the  $2^m$  vectors

$$b_\ell = e_{i_1} \otimes \cdots \otimes e_{i_m},$$

where  $i_j \in \{0, 1\}$  and

$$(22) \quad \ell = \sum_{j=1}^m i_j 2^{m-j}, \quad \ell = 0, \dots, 2^m - 1.$$

There are  $2^m$  different  $b_\ell$  and this corresponds to the  $2^m$  different possibilities of an information that is given by  $m$  classical bits. The Dirac notation for  $b_\ell$  is just  $|\ell\rangle$ , instead of  $e_{i_1} \otimes e_{i_2}$  one finds  $|i_1, i_2\rangle$  or also  $|i_1\rangle |i_2\rangle$ . The formally different objects  $(i_1, \dots, i_m)$  and  $\ell$  or  $b_\ell$  are often identified and called “classical state”.

One more piece of Dirac-notation is often used:  $|x\rangle \langle y|$  is a mapping, defined by

$$(|x\rangle \langle y|)(|z\rangle) := (y, z) \cdot |x\rangle.$$

Here we write  $(y, z) = \langle y | z \rangle$  for the scalar product. Therefore the projection  $P_x$  on a normed vector  $x$  is written as  $|x\rangle \langle x|$ . It is defined by  $y \mapsto (x, y) x$ .

The Fourier series of  $x \in H_m$  is given by

$$(23) \quad x = \sum_{i_j \in \{0,1\}} \alpha_{(i_1, \dots, i_m)} e_{i_1} \otimes \cdots \otimes e_{i_m} = \sum_{\ell=0}^{2^m-1} \beta_\ell b_\ell.$$

We are only interested in normed vectors,  $\|x\| = 1$ . All such vectors are called “quantum states”. For each quantum state there is a probability distribution on the classical states: the probability of  $\ell$  is  $|\beta_\ell|^2$ .

A quantum algorithm starts with a classical state  $k \in \{0, \dots, 2^m - 1\}$  which we identify with  $b_k \in H_m$ . Then a number of unitary transformations  $U_1, \dots, U_r$  are applied, the result is the quantum state

$$x_k = U_r \dots U_1(b_k)$$

and can be written in the form (23). Allowed are only those unitary transformations that are “efficient” in the sense that at most two quantum bits are changed. This means that, for example,  $U_i$  changes the first two bits and is of the form

$$U_i(v_1 \otimes \cdots \otimes v_m) = \tilde{U}_i(v_1 \otimes v_2) \otimes v_3 \cdots \otimes v_m,$$

for some unitary  $\tilde{U}_i : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ . In the quantum bit number model, which we use for the numbers  $\text{comp}^{\text{quant}}$ , one such unitary operation has cost one. The output of the algorithm, given by a final measurement, is a classical state  $\ell \in \{0, \dots, 2^m - 1\}$ , or certain bits of  $\ell$ . The probability of  $\ell$  is  $|\beta_\ell|^2$ , where  $\beta_\ell$  is the respective Fourier coefficient of  $x_k = U_r \dots U_1(b_k)$ . We say that a quantum algorithm computes a given function if the probability of a correct output is at least  $3/4$ .

Now we describe the search problem and *quantum computations with an oracle*. Let  $m \in \mathbb{N}$  and  $X_m = \{0, 1, \dots, 2^m - 1\}$ . Assume that  $f : X_m \rightarrow \{0, 1\}$  is an arbitrary mapping which, of course, can be identified with a subset of  $X_m$ . We define a corresponding unitary mapping  $S_f$  on  $H_m$  by

$$S_f(b_\ell) = -b_\ell \quad \text{if } f(\ell) = 1 \quad \text{and} \quad S_f(b_\ell) = b_\ell \quad \text{if } f(\ell) = 0.$$

We also put  $S_0 = S_f$  for  $f(\ell) = \delta_{0,\ell}$ . A *black box* or *oracle*  $Q_f$  for  $f$  is defined on  $H_{m+1}$  by

$$Q_f(b_\ell \otimes e_i) = b_\ell \otimes e_{i+f(\ell)}.$$

Here the plus sign in  $e_{i+f(\ell)}$  means addition modulo 2, also called exclusive or. Then one can easily show that

$$S_f(b_\ell) \otimes e_0 = Q_f P Q_f (b_\ell \otimes e_0),$$

and therefore the oracle  $Q_f$  can be used to compute function values of  $S_f$ . Here  $P$  is defined by

$$P(b_\ell \otimes e_i) = (-1)^i b_\ell \otimes e_i.$$

One can even simulate  $S_f$  with  $Q_f$  using only one application of  $Q_f$ : simply apply  $Q_f$  on  $(b_\ell \otimes (e_0 - e_1))$ .

A *search problem* is defined as follows. Let  $F_m$  be the set of all  $f_\ell : X_m \rightarrow \{0, 1\}$  with  $f_\ell(j) = 1$  iff  $j = \ell$ . Of course we may identify the sets  $F_m$  and  $X_m$ , and to each  $\ell \in X_m$  or  $f_\ell \in F_m$  there is exactly one  $b_\ell$ . The problem is to find  $\ell$  if  $f = f_\ell$  is given by the oracle  $Q_f$ .

The *algorithm of Grover* works as follows. First we define the Walsh-Hadamard transform  $W_1 : H_1 \rightarrow H_1$  by

$$W_1(e_i) = \frac{1}{\sqrt{2}}(e_0 + (-1)^i e_1)$$

and  $W_m = W_1 \otimes \dots \otimes W_1$ . Now the algorithm is defined by

$$(24) \quad (-W_m S_0 W_m S_f)^k (W_m(b_0)),$$

where  $k$  has the order  $2^{m/2}$ . It is shown in Boyer, Brassard, Høyer, Tapp (1998) that  $k \approx \pi 2^{m/2-2}$  is a very good choice that leads to a high probability of success.

Since every iteration in (24) costs about  $m$ , the total cost (to find the element  $\ell$  with high probability) is about  $m \cdot 2^{m/2}$  or, with  $N = 2^m$ , about  $\sqrt{N} \log N$ . Formally the algorithm is slightly different because  $S_f$  is only given by the oracle  $Q_f$ , hence we work with  $m + 1$  instead of  $m$  quantum bits.

For the problem “compute the mean of  $x_1, \dots, x_n$ ” we assume, when we consider the quantum bit number model and the numbers  $\text{comp}^{\text{quant}}$ , that there is a Boolean oracle that can answer questions of the form “is  $x_i \leq c$ ?”. For the quantum query complexity we allow real valued oracles and only count the number of oracle calls.

Also for all classical algorithms we simply allow real-valued oracles for the  $x_i$ .

## 3. PROOF OF THEOREM 1

We fix a space  $F_d^{k,\alpha}$ . First we prove the upper bounds for quantum computers. We use an algorithm of the form

$$(25) \quad A(f) = I(P_n f) + \tilde{Q}_N^d(f - P_n f).$$

Here  $I(P_n f)$  is the integral of a function  $P_n f$  and  $Q_N^d$  is the  $d$ -dimensional midpoint rule, as in Section 1.2, which we apply to  $(f - P_n f)$ . We now explain the operators  $P_n$  and  $\tilde{Q}_N^d$ . By  $P_n$  we mean a projection operator by interpolation, one can use piecewise polynomials, which uses  $n$  function values and gives an order

$$\|f - P_n f\|_\infty \asymp n^{-\gamma},$$

for  $f \in F_d^{k,\alpha}$ . It is well known that this is the optimal order of convergence, see Novak (1988). By  $\tilde{Q}_N^d$  we mean that we do not really apply the midpoint rule  $Q_N^d$ . Instead we evaluate this midpoint rule by a quantum computer up to some error  $\varepsilon_1 \cdot n^{-\gamma}$  with the cost

$$\text{cost}_{\text{query}}^{\text{quant}}(N, \varepsilon_1) \leq C \cdot \varepsilon_1^{-1}$$

or

$$\text{cost}_{\text{query}}^{\text{quant}}(N, \varepsilon_1) \leq C \cdot \varepsilon_1^{-1} \log N \log \varepsilon_1^{-1},$$

respectively. The error of this method is bounded by

$$(26) \quad e(f) \leq n^{-\gamma} \cdot (N^{-\beta} + \varepsilon_1), \quad f \in F_d^{k,\alpha},$$

which is the sum of the error by discretization (the integral being replaced by  $Q_N^d$ ) and the error made by the approximate evaluation of  $Q_N^d$ . This error bound is valid for all  $\beta \leq \alpha$ , if  $k = 0$ , and  $\beta = 1$ , if  $k > 0$ . In addition we have to assume that  $N$  is at least of the order  $n$ . To simplify the presentation we use a  $\beta$  which is always smaller than 1. The complete cost of the method is bounded by

$$(27) \quad \text{cost}_{\text{query}}^{\text{quant}}(N, \varepsilon_1) \leq C \cdot (n + \varepsilon_1^{-1})$$

or

$$(28) \quad \text{cost}_{\text{query}}^{\text{quant}}(N, \varepsilon_1) \leq C \cdot n + C \cdot \varepsilon_1^{-1} \log N \log \varepsilon_1^{-1},$$

respectively. In the query-complexity case we simply choose

$$N^\beta \approx \varepsilon_1^{-1} \approx n$$

and observe that we may apply (26), because of  $\beta < 1$ . We obtain a cost of the order  $n$  and an error of the order  $n^{-\gamma-1}$  and so obtain the upper bound in (19).

In the quantum bit number model we take

$$n \approx \varepsilon_1^{-1} (\log \varepsilon_1^{-1})^2,$$

again with

$$N^\beta \approx \varepsilon_1^{-1}.$$

Then we get the cost bound

$$C \varepsilon_1^{-1} (\log \varepsilon_1^{-1})^2$$

and the error bound

$$e(f) \leq \varepsilon_1^{\gamma+1} (\log \varepsilon_1^{-1})^{-2\gamma}.$$

We obtain

$$e(f)^{-1/(\gamma+1)} = \varepsilon_1^{-1} (\log \varepsilon_1^{-1})^{2\gamma/(\gamma+1)}$$

and therefore

$$\text{comp}^{\text{quant}}(F_d^{k,\alpha}, \varepsilon) \leq C \varepsilon^{-1/(1+\gamma)} (\log \varepsilon^{-1})^{2-2\gamma/(\gamma+1)}.$$

Now we prove the lower bound in (19). The space  $F_d^{k,\alpha}$  contains  $n \asymp \varepsilon_1^{-1/\gamma}$  functions  $f_1, f_2, \dots, f_n$  with disjoint supports such that

- $\int_{[0,1]^d} f_i dx = \varepsilon_1^{1-1/\gamma}$  and
- $\sum_{i=1}^n \lambda_i f_i \in F_d^{k,\alpha}$  if  $|\lambda_i| \leq 1$ ,

see Novak (1988, p. 35). Consider now the following problem. Compute the mean value of the integrals  $\int_{[0,1]^d} \lambda_i f_i dx$ , where  $|\lambda_i| \leq 1$ , up to some error  $\varepsilon_2$ . We can apply the lower bound of Nayak, Wu (1999), see (4), to obtain the lower bound

$$\text{cost} \geq C \min(\varepsilon_1^{-1/\gamma}, \varepsilon_1^{1-1/\gamma} \varepsilon_2^{-1}).$$

Of course we simply put  $\varepsilon_2 = \varepsilon_1$  and obtain

$$\text{cost} \geq C \varepsilon_1^{-1/\gamma}.$$

What we estimated was the cost to compute the mean value. The cost to compute  $\int_{[0,1]^d} \sum_{i=1}^n \lambda_i f_i dx$  is actually the sum and hence the error is to be multiplied by  $n$ , hence  $\varepsilon = \varepsilon_1 n \approx \varepsilon_1^{1-1/\gamma}$ . Since the cost to obtain this error is at least of the order  $\varepsilon_1^{-1/\gamma}$  we obtain

$$\text{comp}_{\text{query}}^{\text{quant}}(F_d^{k,\alpha}, \varepsilon) \geq C \varepsilon^{-1/(1+\gamma)}.$$

We finally prove the upper bound for restricted Monte Carlo. Instead of the arbitrary random numbers of a (general) Monte Carlo method we can only use random bits or coin tossing. We use a discretized version of a well known variance reduction technique and write the method in the form

$$(29) \quad A(f) = I(P_n f) + \tilde{Q}_N^d(f - P_n f),$$

and only the meaning of  $\tilde{Q}_N^d$  is different from (25). By  $\tilde{Q}_N^d$  we mean that we do not really apply the midpoint rule  $Q_N^d$ . Instead we evaluate this midpoint rule by the classical Monte Carlo method, again up to some error  $\varepsilon_1 \cdot n^{-\gamma}$  with the cost

$$\text{cost}^{\text{coin}}(N, \varepsilon_1) \leq C \cdot \varepsilon_1^{-2} \log N + n.$$

The bound  $\varepsilon_1^{-2}$  is the classical Monte Carlo bound, the factor  $\log N$  comes in because we need (about)  $\log N$  random bits to select one node from the possible  $N$  nodes. The error of this method is bounded by

$$e(f) \leq n^{-\gamma} \cdot (N^{-\beta} + \varepsilon_1), \quad f \in F_d^{k,\alpha},$$

see (26). Again we use a  $\beta$  which is always smaller than 1. We put  $N^{-\beta} \approx \varepsilon_1$  and obtain

$$\text{cost}^{\text{coin}}(N, \varepsilon_1) \leq C \cdot \varepsilon_1^{-2} \log \varepsilon_1^{-1} + n$$

and

$$e(f) \leq n^{-\gamma} \varepsilon_1.$$

With the choice  $n \approx \varepsilon_1^{-2} \log \varepsilon_1^{-1}$  we obtain

$$e(f)^{-2/(1+2\gamma)} \leq \varepsilon_1^{-2} \cdot (\log \varepsilon_1^{-1})^{2\gamma/(1+2\gamma)}$$

and therefore (21).

#### 4. A REMARK ON RANDOMIZED QUANTUM ALGORITHMS

We present an algorithm to compute the expectation of arbitrary bounded random variables up to some error  $\varepsilon > 0$ . A classical randomized method needs time  $\varepsilon^{-2}$ . The proposed algorithm uses the algorithm of Grover and a random generator. We assume that the random generator is for free and this is certainly not a realistic assumption. Let  $(X, B, m)$  be a probability space and let

$$F = \{f : X \rightarrow \mathbb{R} \mid f \text{ is measurable and } \|f\|_\infty \leq 1\}.$$

We want to compute the integral of a function (or the expectation of a random variable)

$$I(f) = \int_X f(x) dm(x),$$

for  $f \in F$ . With (5) one obtains the complexity bound

$$(30) \quad \text{comp}_{\text{quant}}^{\text{random}}(F, \varepsilon) \leq C \varepsilon^{-1} (\log \varepsilon^{-1})^2,$$

with a constant  $C$  that does not depend on the particular space  $(X, B, m)$ . If we use the quantum query model, together with a free random generator, then we get in the same way the upper bound  $C \varepsilon^{-1}$ . We prove (30).

To compute an approximation  $A(f, \varepsilon)$  of  $I(f)$  for  $f \in F$  and  $0 < \varepsilon \leq 1/2$  (with the understanding that (6) should be true) we proceed as follows:

- First we randomly select  $x_1, \dots, x_n \in X$  using the random generator, where

$$(31) \quad n = \lceil 72 \varepsilon^{-2} \rceil.$$

If we put<sup>1</sup>

$$Q_n(f) = \frac{1}{n} \sum_{i=1}^n f(x_i)$$

then it follows from Chebyshev's inequality that

$$(32) \quad |I(f) - Q_n(f)| \leq \frac{\varepsilon}{3} \quad \text{with probability at least } 7/8.$$

- We assume that an oracle is available for the computation of  $f(x_i)$ , up to a maximal error of  $\varepsilon/3$ . On input  $i$  the oracle gives  $\tilde{f}(x_i)$  such that

$$|f(x_i) - \tilde{f}(x_i)| \leq \frac{\varepsilon}{3}.$$

- Quantum computation. Let

$$\tilde{Q}_n(f) = \frac{1}{n} \sum_{i=1}^n \tilde{f}(x_i).$$

Then we know  $|Q_n(f) - \tilde{Q}_n(f)| \leq \varepsilon/3$  and therefore

$$|I(f) - \tilde{Q}_n(f)| \leq \frac{2}{3}\varepsilon \quad \text{with probability at least } 7/8.$$

Hence we compute an approximation  $A(f, \varepsilon)$  of  $\tilde{Q}_n(f)$  such that

$$(33) \quad |A(f, \varepsilon) - \tilde{Q}_n(f)| \leq \frac{\varepsilon}{3} \quad \text{with probability at least } 7/8$$

and get (6). With (5) obtain the bound (30) for the complexity of the problem.

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<sup>1</sup>Of course  $Q_n$  is simply the classical Monte Carlo algorithm for the computation of the integral. Here we only *define* the points  $x_1, \dots, x_n$ , we do not compute  $Q_n(f)$ . This expression is only used for the analysis and the intuition.

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