A survey of known and new cubature formulas for the unit disk

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Abstract

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1 Introduction

We are concerned with the approximation of integrals over the unit disk with constant weight function

$$I[f] = \int_\Omega f(x,y)dxdy \quad , \quad \Omega = \{(x,y) : x^2 + y^2 \leq 1\} \tag{1.1}$$

by weighted sums of function values

$$Q[f] = \sum_{j=1}^{N} w_j f(x_j, y_j). \tag{1.2}$$

We are especially interested in cubature formulas of algebraic degree, i.e., approximations that are exact for all polynomials of total degree at most d:

$$Q[f] = I[f], \forall f \in \mathcal{P}_d^2$$

where

$$\mathcal{P}_d^2 = \text{span}\{x^i y^j : 0 \leq i + j \leq d, i, j \in \mathbb{N} \cup \{0\}\}.$$

It is known (see, e.g., [1]) that the number of points N in a cubature formula of degree d for the integral (1.1) satisfies

$$\begin{array}{lll} N \geq \frac{k(k+1)}{2} & \text{if} & d = 2k-2,\ k \in \mathbb{N} \\ N \geq \frac{k(k+1)}{2} + \left\lfloor \frac{k}{2} \right\rfloor & \text{if} & d = 2k-1,\ k = 1,2,3,4,6,8,\ldots \\ N \geq \frac{k(k+1)}{2} + \left\lfloor \frac{k}{2} \right\rfloor + 1 & \text{if} & d = 2k-1,\ k = 5,7,\ldots \end{array} \tag{1.3}$$

The motivation for this paper comes from [17]. In that paper a method to construct cubature formulas (1.2) of algebraic degree of precision to approximate the integral (1.1) is presented. Furthermore, in addition to some new cubature formulas, a survey of existing cubature formulas is presented. We will give some additional new cubature formulas constructed using this method and mention some errors in [17]. We also present a more complete survey of known cubature formulas for this region.

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2 Survey of known cubature formulas

2.1 Spherical product rules

A classical way to deal with the integral (1.1) is to transform it to a square. Define

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Using this transformation the integral (1.1) becomes

$$\int_{-1}^{1}\int_{-\pi/2}^{\pi/2} f(r\cos\theta, r\sin\theta)|r|d\theta dr$$

and therefore

$$I[x^i y^j] = \int_\Omega x^i y^j dxdy = \int_{-1}^{1}\int_{-\pi/2}^{\pi/2} |r|r^{i+j}\cos^i(\theta)\sin^j(\theta)d\theta dr.$$

This integral can be rewritten as the product of two 1-dimensional integrals:

$$\int_{-1}^{1} |r|r^{i+j}dr \tag{2.4}$$

$$\int_{-\pi/2}^{\pi/2} \cos^i(\theta)\sin^j(\theta)d\theta. \tag{2.5}$$

Note that we only have to worry about the cases where both i and j are even. For odd i or j, the moment $I[x^i y^j] = 0$. Define $t = \sin\theta$ and use this transformation for the integral (2.5). This gives

$$\int_{-1}^{1} (1-t^2)^{-1/2}(1-t^2)^{i/2}t^j dt. \tag{2.6}$$

Gauss quadrature formulas of degree $2k-1$ with k points that take into account the weight functions $|r|$ and $(1-t^2)^{-1/2}$ respectively, exist for the integrals (2.4) and (2.6). Combining these gives a cubature formula of degree $2k-1$ with

$$\begin{array}{ll} N = k^2 & \text{if } k \text{ is even, or} \\ N = k^2 - k + 1 & \text{if } k \text{ is odd} \end{array} \tag{2.7}$$

points inside the unit disk and all weights positive. These points are located on $\lfloor\frac{k}{2}\rfloor$ regular polygons with $2k$ vertices. If k is odd, then the center is also a point of the cubature formula.

This type of formulas is easy to construct but requires more points than indicated by the lower bound (1.3). We refer to [25] for more details.

2.2 An automatic procedure to construct efficient rules

Using a very special structure for the points and weights of the cubature formulas, in [6] an algorithm is presented that allows straightforward computation (i.e., without human intervention or starting points) of formulas of arbitrary odd degree for (amongst others) the unit disk. The points of these formulas are the vertices of regular polygons, but there may be different types and orientations.

A special case of the special structure uses the vertices of k congruent regular polygons with k vertices. Such a formula always exists and has all points inside the unit disk and all weights positive. Furthermore, it has the same number of points as the spherical product formula (2.7). If k is odd, one of these polygons has zero radius, i.e., the points collapse in the center.

More clever choices of the polygons result in cubature formulas with less points. There is then no theoretical guarantee that the cubature formula exists and has real points inside the unit disk. In practice it turns out that in most cases all points are inside the unit disk and all weights are positive. The cubature formulas with the lowest number of points that are obtained with this algorithm up to degree 31 correspond to those tabulated in [14], see Table 1 in the following subsection.

2.3 Tables of published cubature formulas

The survey of known cubature formulas of the form (1.2) for (1.1) given in [17] is far from complete. A survey of all known cubature formulas up to 1971 for standard regions was given in [25]. An update of this overview was presented in [10]. Some corrections and additions are presented in [2]. The unit disk is only one of the regions treated in these surveys. In Table 1 below we give the union of all this information for the unit disk.

In the table, we give the following information: in addition to the degree d and the corresponding number of points N, we give an indication of the quality and the references. An asterisk is appended to the number of points if this is known to be the theoretical minimim number of points, see (1.3).

We briefly repeat the notion of quality of a cubature formula. The first symbol gives information about the weights:

P: all the weights are positive;
E: all the weights are equal (and thus positive);
N: some weights are negative;
?: the weights are not given explicitely.

The second symbol gives information about the location of the points:

I: all points are inside the unit disk;
B: some points lie on the unit circle, the others inside the disk;
O: some points are outside the unit disk.

As a general rule we do not include published cubature formulas of degree $2k-1$ which use more than k^2 points since the spherical product Gaussian rule of degree $2k-1$ has k^2 points inside the unit disk and all weights positive. Exceptions to this rule are cubature formulas with properties that some users may desire, such as equal weights. Between brackets we sometimes give the number of cubature formulas with the same d and N and quality.

As a principal reference, we give the first appearance of the cubature formula in a journal or book, even though it has previously appeared in an internal report or thesis. Additional references to journals and books are included to help the user of this table to look up the formula. Internal reports and theses or normaly only mentioned if the cubature formula did not appear in a book or journal. For more information we refer to [10, 2].

In [17] three cubature formulas are given that are wrong. The error cannot easily be corrected: there seems *not* to exist cubature formulas with the given symmetry and number of points. In Table 1 these have an additional "Error" after the reference.

Table 1: Overview of published cubature formulas for the unit disk.

Degree	N	Quality	References
2	3*	PI	[25]
3	4*	PI(3)	[25]
		PB	[25]
4	6*	PI	[25]
		?O	[24]
	10	EI	[12]
5	7*	PI(2)	[25]
	8	PO	[3]
		PO	[25]
	9	PI(2)	[25]
		PB	[25]
	12	EI	[25]
		EI	[12]
		EI	[11]
6	10*	PO	[23][26]
		PO	[18]
	11	PO	[26][20]
	14	EI	[12]
7	12*	PI	[25]
	16	EI	[12]
		PI	[25]
	18	EI	[12]
	20	EI	[12]
8	16	PI	[20][26]
9	18*	PO	[22]
	19	PI	[25]
		PI	[19]
		PI	[16]
		PI	[20]
	20	PI(3)	[3]
		PO	[25]
		PO	[3]
		PO	[3][17]
	21	PI	[3]
		PI	[25]
		PO	[25]

Degree	N	Quality	References
11	25	PO	[15]
		NO	[17] Error
	26	PI	[21]
		PI	[15]
	28	PI(3)	[25]
	32	PI	[25]
13	34	PO	[9]
	35	PB	[9]
	36	PI	[7] [17]
	37	PI	[25]
	41	PI	[25]
		PI	[14]
15	44	PI	[25]
		PI	[17]
	48	PI	[25]
17	56	PO	[17] Error
	57	PI	See §3, Table 2
	60	PO	[17]
	61	PI	[25]
19	68	NO	[17] Error
	69	PO	See §3, Table 3
	71	PO	[14]
	72	PI	[17]
	76	PI	[14] [13]
21	88	PO	See §3, Table 4
	90	PI	[14]
	99	PI	[14] [4]
23	97	PO	[14]
	108	PI	[14] [13]
25	127	PI	[14]
27	140	PI	[14] [13]
31	172	PI	[14] [13]

Embedded cubature formulas			
Degrees	N	Quality	References
5-7	8-16	PNI(2)	[5]
17-27	71-183	PNI	[8]

3 New cubature formulas

In this section we present new cubature formulas of degree 17, 19 and 21 with a number of points lower than any known cubature formula. They are constructed using the algorithm described in [17].

In the Tables 2–4 we give one point for each orbit, a so-called generator, and its weight. The other points of an orbit are obtained by applying the symmetry operations of the square: whenever (x_i, y_i) is a point, so are $(-x_i, y_i)$, $(x_i, -y_i)$, $(-x_i, -y_i)$, (y_i, x_i), $(-y_i, x_i)$, $(y_i, -x_i)$ and $(-y_i, -x_i)$ and all have the same weight. In general orbits thus have 8 points, but for special choices of x_i and y_i, this can be reduced to 4 or even 1. One has to consider 4 types of generators. In the tables we mention the type, but we refer to [17] for more details.

Table 2: New formula of degree 17 with 57 points

weight	x_i	y_i	type
0.11498334179998566	0.0	0.0	I
0.042666281539386779	0.88696766316393713	0.0	II
0.087938325357145539	0.42463390374323367	0.0	II
0.076206570461793249	0.69446902308083445	0.0	II
0.019156522218855521	0.68785354082699271	0.68785354082699271	III
0.062085722273139239	0.59664767781455707	0.59664767781455707	III
0.095664962820418119	0.23562252091530831	0.23562252091530831	III
0.085162533604288747	0.31294754888343992	0.54894025523701459	IV
0.020201237989565462	0.96121228504617867	0.17385745088683603	IV
0.056834571713156972	0.30538732225214729	0.79035487531148609	IV
0.024268628331345539	0.84937290409632805	0.46270056598293749	IV

Table 3: New formula of degree 19 with 69 points

weight	x_i	y_i	type
0.092826182741729107	0.0	0.0	I
0.061210649714171629	0.79836610957931832	0.0	II
0.084701305074092631	0.30042292336770777	0.0	II
0.0064918529741922661	0.99615333438631671	0.0	II
0.014104347317113714	0.69583191637449732	0.69583191637449732	III
0.077590938349196292	0.28670391168237706	0.28670391168237706	III
0.046841922848831439	0.61667878163587778	0.61667878163587778	III
0.071523458803010660	0.47214212637530880	0.47214212637530880	III
0.077267777905420501	0.56686082157858617	0.15587783378762243	IV
0.034350583657896668	0.92555127310854838	0.17347695471172525	IV
0.028368949384516057	0.83822144355055265	0.46025996563724600	IV
0.0018669074320751099	0.96670876580048578	0.33737925901740466	IV
0.058009352935795362	0.73773390012389973	0.32685240870845720	IV

Table 4: New formula of degree 21 with 88 points

weight	x_i	y_i	type
0.064712088156233115	0.18089963670914432	0.0	II
0.012203257346077736	0.98391537714755427	0.0	II
4.6214466764876138 (10^{-5})	0.77364389614737803	0.77364389614737803	III
0.020941123674084990	0.67975937134823609	0.67975937134823609	III
0.044321766954122515	0.59230560995561076	0.59230560995561076	III
0.052715691149269700	0.41774444740353380	0.41774444740353380	III
0.028690025849304919	0.89789289495208579	0.13375169526590821	IV
0.013819709344648730	0.93927352486965313	0.27985732696474981	IV
0.037034888202855274	0.79099671385625722	0.39385576573951899	IV
0.047520136607474831	0.56605481445792489	0.10885005806786229	IV
0.014296165120136605	0.83180776464845906	0.51224072541565606	IV
0.042923763669888949	0.76010583069266321	0.12830595415488073	IV
0.047444099811669938	0.62209339812108792	0.34691040719842391	IV
0.063500222219468438	0.36200059276768541	0.16676191877222966	IV

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