A relation between orthogonal rational functions on the unit circle and the interval $[-1,1]$

P. Van gucht and A. Bultheel

Report TW 299, January 2000

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

A relation between orthogonal rational functions on the unit circle and the interval $[-1,1]$

P. Van gucht and A. Bultheel[*]

Report TW299, January 2000

Department of Computer Science, K.U.Leuven

Abstract

In his famous book on orthogonal polynomials (OP), Szegő gave a relation between OP on the unit circle and OP on the interval $[-1,1]$ by use of a simple projection. The aim of this note is to show that this can be extended to orthogonal rational functions (ORF). The poles of the ORF on the unit circle and the poles of the ORF on the interval are real and outside the interval $[-1,1]$.

Keywords : orthogonal rational functions
AMS(MOS) Classification : Primary : 42C05

---

[*] This work is partially supported by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

A relation between orthogonal rational functions on the unit circle and the interval $[-1,1]$

P. Van gucht and A. Bultheel *

Department of Computer Science, K.U.Leuven, Belgium

E-mail: {Patrick.Vangucht—Adhemar.Bultheel}@cs.kuleuven.ac.be

In his famous book on orthogonal polynomials (OP), Szegő gave a relation between OP on the unit circle and OP on the interval $[-1,1]$ by use of a simple projection. The aim of this note is to show that this can be extended to orthogonal rational functions (ORF). The poles of the ORF on the unit circle and the poles of the ORF on the interval are real and outside the interval $[-1,1]$.

Keywords: orthogonal rational functions

AMS Subject classification: 42C05

1. Introduction

Orthogonal polynomials (OP) are studied in many books by many different authors with many different interests. The topic seems a never drying source for research. Even nowadays there are many questions and (until now) unsolved problems about special cases, generalizations and applications of the *old* subject.

In a recent monograph [2], a rational generalization of the OP is treated in detail. Herein the polynomials are replaced by rational functions with fixed poles and many of the properties of OP can be *translated* for these orthogonal rational functions (ORF). These properties are e.g. recurrence relations, Favard theorems, quadrature formulas, interpolation properties and moment problems.

In this note we will give some further generalization of OP to ORF that are not covered in [2]. The relation that Szegő gave in his book [3] to connect OP on the unit circle and OP on the interval $[-1,1]$ can be generalized to ORF when we choose the poles of the system in an apropriate way. The transformation used by Szegő to transfrom the variable z on the unit circle into a variable x on $[-1,1]$ is the Joukowski transform $x=(z+1/z)/2$. To obtain a direct generalization of the polynomial case, we shall see that the transformation that we need for the poles is the same. The poles in the circular case are outside the closed unit disk, and so will the poles in the case of the interval be outside the closed interval $[-1,1]$, but by the previous transformation it can be arranged such that they are real. To obtain this, we use real poles in the circular case real but we also have to double the multiplicity of every pole. Note that in the polynomial case the poles are at infinity, both for the interval and for the circle.

This article is built up as follows. In the next section notations are introduced and the ORF are properly defined. The third section deals with the placement of the poles. A fourth section contains

---

* This work is partially supported by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

the main result. The last section contains an example of these relations.

2. Preliminaries

In this article we take a look at a generalization of a relation between OP on the unit circle and OP on the real interval $[-1,1]$. Szegő proved the following [3, Theorem 11.5, p. 294].

Theorem 2.1. *Let $u(x)$ be a weight function on the interval $[-1,1]$ and let*

$$w(\theta) = u(\cos\theta)|\sin\theta|.$$

Further let $\{p_n(x)\}$ and $\{q_n(x)\}$ be the sets of polynomials which are orthonormal on $[-1,1]$ with respect to $u(x)$ and $(1-x^2)u(x)$ respectively, and let $\{\phi_n(z)\}$ be the orthonormal set associated with $w(\theta)$ on $z=e^{i\theta}$. Then, by writing $x=\frac{1}{2}(z+z^{-1})$, we have for $n\geq 1$

$$\begin{aligned} p_n(x) &= (2\pi)^{-1/2}\left\{1+\frac{\phi_{2n}(0)}{\kappa_{2n}}\right\}^{-\frac{1}{2}}\{z^{-n}\phi_{2n}(z)+z^n\phi_{2n}(z^{-1})\} \\ &= (2\pi)^{-1/2}\left\{1-\frac{\phi_{2n}(0)}{\kappa_{2n}}\right\}^{-\frac{1}{2}}\{z^{-n+1}\phi_{2n-1}(z)+z^{n-1}\phi_{2n-1}(z^{-1})\}; \\ q_n(x) &= (2/\pi)^{1/2}\left\{1-\frac{\phi_{2n+2}(0)}{\kappa_{2n+2}}\right\}^{-\frac{1}{2}}\frac{z^{-n-1}\phi_{2n+2}(z)-z^{n+1}\phi_{2n+2}(z^{-1})}{z-z^{-1}} \\ &= (2/\pi)^{1/2}\left\{1+\frac{\phi_{2n+2}(0)}{\kappa_{2n+2}}\right\}^{-\frac{1}{2}}\frac{z^{-n}\phi_{2n+1}(z)-z^{n}\phi_{2n+1}(z^{-1})}{z-z^{-1}}. \end{aligned}$$

Here κ_n denotes the coefficient of z^n in $\phi_n(z)$.

Before we state the generalization of this theorem to deal with orthogonal rational functions we introduce some notation where we try to stay close to the notation in [2].

With $\mathbb{R}$, $\hat{\mathbb{R}}$ and $\mathbb{C}$ we denote the real line, the extended real line and the complex plane, respectively. We also use the following sets

$$\begin{aligned} \mathbb{D} &= \{z\in\mathbb{C} : |z|<1\}; \\ \mathbb{T} &= \{z\in\mathbb{C} : |z|=1\}; \\ \mathbb{E} &= \{z\in\mathbb{C} : |z|>1\}; \\ I &= [-1,1]; \\ I^c &= \hat{\mathbb{R}}\setminus I. \end{aligned}$$

The bar-notation denotes a complex conjugate.

We will have to make the distinction between *complex* and *real* functions. We will use the variable x as a real variable and z as a complex variable.

Let u be a weight function on I with inner product

$$\langle f,g\rangle_u = \int_{-1}^{1} f(x)\overline{g(x)}u(x)dx.$$

With this u we can define a weight function w on $[-\pi,\pi]$ as $w(\theta) := u(\cos\theta)|\sin\theta|$, with inner product

$$\langle f,g\rangle_w = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}w(\theta)d\theta.$$

Suppose that $\alpha = \{\alpha_0, \alpha_1, \alpha_2, \ldots\} \subset \mathbb{D}$, with $\alpha_0 = 0$, and $\beta = \{\beta_0, \beta_1, \beta_2, \ldots\} \subset I^c$, with $\beta_0 = \infty$, are given. We can now define the Blaschke factors.

real:

$$Z_0(x) = 1;\ Z_i(x) = \frac{x}{1 - x/\beta_i}, \quad i > 0;$$

complex:

$$\zeta_0(x) = 1;\ \zeta_i(z) = \frac{z - \alpha_i}{1 - \overline{\alpha}_i z}, \quad i > 0.$$

With these Blaschke factors we can define Blaschke products ($n \geq 0$).

real:

$$b_n(x) = Z_0(x) Z_1(x) \cdots Z_n(x);$$

complex:

$$B_n(z) = \zeta_0(z) \zeta_1(z) \cdots \zeta_n(z),$$

and spaces of rational functions

real:

$$\mathcal{L}_n^r = \operatorname{span}\{b_0(x), b_1(x), \ldots, b_n(x)\};$$

complex:

$$\mathcal{L}_n^c = \operatorname{span}\{B_0(z), B_1(z), \ldots, B_n(z)\}.$$

We define the para-hermitian conjugate of a complex function g as

$$g_*(z) = \overline{g(1/\overline{z})}.$$

Note that for z on $\mathbb{T}$ this coincides with the complex conjugate: $g_*(z) = \overline{g(z)}$, $z \in \mathbb{T}$. Now we can define the following (complex) spaces

$$\mathcal{L}_{n*}^c = \operatorname{span}\{B_{0*}(z), B_{1*}(z), \ldots, B_{n*}(z)\}.$$

This allows us to define spaces of rational functions that generalize the notion of Laurent polynomials. For m and n nonnegative integers, denote

$$\mathcal{R}_{m,n} = \mathcal{L}_{m*}^c + \mathcal{L}_n^c.$$

Note that $B_{n*} = 1/B_n$, which we denote as B_{-n} and hence $\mathcal{R}_{m,n} = \operatorname{span}\{B_{-m}, \ldots, B_{-1}, 1, B_1, \ldots, B_n\}$ and $\mathcal{R}_{0,n} = \mathcal{L}_n^c$. When all $\alpha_k = 0$, then $\mathcal{R}_{m,n} = \Lambda_{m,n}$, a space of Laurent polynomials and $\mathcal{L}_n^c = \Pi_n$, a space of ordinary polynomials.

With the sequence α, we associate the sequence

$$\tilde{\alpha} = \{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \ldots\} = \{\alpha_0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots\},$$

i.e., $\tilde{\alpha}_0 = \alpha_0$ and $\tilde{\alpha}_{2k} = \tilde{\alpha}_{2k-1} = \alpha_k$, $k = 1, 2, \ldots$. The associated Blaschke factors and products are denoted with a tilde. Thus $\tilde{B}_{2n} = (B_n)^2$ and $\tilde{B}_{2n+1} = B_n B_{n+1}$. Furthermore, the corresponding spaces spanned by $\{\tilde{B}_0, \tilde{B}_1, \ldots \tilde{B}_n\}$ are denoted as $\tilde{\mathcal{L}}_n^c$.

We now introduce the orthogonal rational functions. With the weight function $u(x)$ on I, we obtain by orthonormalization of the basis $\{b_n(x)\}$ a system of orthonormal rational functions $\{p_n(x)\}$.

With the weight function w, we obtain by orthogonalization of the basis $\{\tilde{B}_n(z)\}$ a sequence of orthonormal rational functions on the unit circle that we denote as $\{\tilde{\phi}_n(z)\}$.

The natural rational generalization of the orthogonal Laurent polynomials is to orthogonalize (with respect to the weight function w) the basis

$$B_0, B_1, B_{-1}, B_2, B_{-2}, \ldots \quad \text{or} \quad B_0, B_{-1}, B_1, B_{-2}, B_2, \ldots$$

which generate the nested sequences $\mathcal{R}_{0,0} \subset \mathcal{R}_{0,1} \subset \mathcal{R}_{1,1} \subset \mathcal{R}_{1,2} \subset \mathcal{R}_{2,2} \subset \cdots$ and $\mathcal{R}_{0,0} \subset \mathcal{R}_{1,0} \subset \mathcal{R}_{1,1} \subset \mathcal{R}_{2,1} \subset \mathcal{R}_{2,2} \subset \cdots$ respectively. In the first case, this gives rise to an orthogonal system $\{\sigma_n\}$ and in the second case we find an orthogonal system $\{\tau_n\}$. These orthogonal functions were studied in [1]. It is shown there that

$$\sigma_{2m} = B_m \tilde{\phi}_{2m*} \quad \text{and} \quad \sigma_{2m+1} = B_{m*} \tilde{\phi}_{2m+1}$$

while for the τ_n we have

$$\tau_{2m} = B_{m*} \tilde{\phi}_{2m} \quad \text{and} \quad \tau_{2m+1} = B_m \tilde{\phi}_{2m+1*}.$$

Note that $\tau_{2m} = \sigma_{2m*}$ and $\tau_{2m+1} = \sigma_{2m+1*}$. Thus we can continue using only σ_m, disregarding τ_m. For further properties and recurrence relations of these ORFs we refer to [1].

Finally, with u, we can also associate an array of varying weight functions v_k on I defined by

$$v_k(x) := \frac{[(\beta_k^2 - 1)(\beta_{k+1}^2 - 1)]^{1/2}(1 - x^2)}{|(\beta_k - x)(\beta_{k+1} - x)|} u(x).$$

We also allow values $\beta_k = \infty$. To see what happens in that case, divide by β_k in numerator and denominator and let β_k tend to infinity. Note that if all $\beta_k = \infty$, i.e., in the polynomial case, then $v_k(x) = (1 - x^2)u(x)$ does not depend on k. In the general rational case however, we shall have to deal with a varying measure. The orthonormal rational functions with respect to v_n are denoted as $\{q_n(x)\}$. By orthonormality with respect to the varying measure v_k it is meant that for all $n = 0, 1, \ldots$

$$\int_{-1}^{1} q_n(x) x^k v_n(x) \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n - 1,$$

and

$$\int_{-1}^{1} |q_n(x)|^2 v_n(x) \mathrm{d}x = 1.$$

We also assume that the orthonormal functions $\tilde{\phi}_n$ are normalized such that their leading coefficients with respect to the basis $\{\tilde{B}_k\}$ are positive.

These settings are defined for general α and β. In the next section we will discuss the relation between the two, which is necessary for a relation between $\{p_n(x)\}$, $\{q_n(x)\}$, $\{\tilde{\phi}_n(z)\}$, $\{\sigma_n(z)\}$, and $\{\tau_n(z)\}$.

3. Placement of the poles

We want to find the orthonormal rational functions $\{p_n(x)\}$ and $\{q_n(x)\}$ in terms of the orthonormal rational functions $\{\phi_n(z)\}$. Let us only look at the p_n. The q_n will give rise to an analogue condition as mentioned later on. Thus we want the following orthogonality conditions to be satisfied.

$$\int_{-1}^{1} p_n(x) \overline{b_k(x)} u(x) \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n - 1. \tag{3.1}$$

We now use the Joukowski transformation $x = \frac{1}{2}(z + z^{-1})$, where $z = e^{i\theta}$, and $\xi = -\theta$ to find $(k = 0, 1, \dots, n-1)$

$$-\int_{-\pi}^{0} p_n(\cos\theta)\overline{b_k\left(\frac{z+1/z}{2}\right)}u(\cos\theta)\sin\theta \mathrm{d}\theta = 0;$$
$$\int_{0}^{\pi} p_n(\cos\xi)\overline{b_k\left(\frac{1/z+z}{2}\right)}u(\cos\xi)\sin\xi \mathrm{d}\xi = 0.$$

When we combine these we find the following sufficient condition to satisfy (3.1).

$$\int_{-\pi}^{\pi} p_n(\cos\theta)\overline{b_k\left(\frac{z+1/z}{2}\right)}w(\theta)\mathrm{d}\theta = 0, \quad k = 0, \dots, n-1. \tag{3.2}$$

What we will need is a link between the real Blaschke products $b_k(x)$ and the complex Blaschke products $B_j(z)$, which results in a condition on the placement on the poles as shown in Lemma 3.1.

A first remark: if we do want the integral $\int_{-1}^{1} p_n(x)u(x)\mathrm{d}x$ to exist (it has to be zero for $k > 0$) without special conditions on the weight function, we should avoid singularities of the integrand, i.e., we have to place the points β outside the interval I. If we want furthermore the functions p_k to be real on the real interval I, it is a most natural choice to also make the β real. Thus we should have $\beta \subset I^c$.

We first look at the relation between real and complex Blaschke factors, which will give a condition concerning the placement of the poles. The following lemma says that the Blaschke factor Z_k can be written as a linear combination of ζ_k, ζ_0 and ζ_k^{-1}. Note also that if we want $\beta \subset I^c$ we need that $\alpha \subset I$.

Lemma 3.1. *With the above notations consider the Blaschke factors $Z_k(x)$ and $\zeta_k(z)$, where $\beta \subset I^c$ and $\tilde{\alpha}$ some arbitrary sequence in $\mathbb{D}$. Then there exist constants A and B only depending on $\tilde{\alpha}$ and β such that*

$$Z_k\left(\frac{z+1/z}{2}\right) = A\tilde{\zeta}_{2k}(z) + B + A\tilde{\zeta}_{2k-1}^{-1}(z),$$

iff the points $\tilde{\alpha}_k$ and β_k satisfy ($k > 0$)

$$\tilde{\alpha}_{2k} = \tilde{\alpha}_{2k-1} \quad \textit{and} \quad \beta_k = \frac{\tilde{\alpha}_{2k-1} + 1/\tilde{\alpha}_{2k}}{2}. \tag{3.3}$$

Setting $\alpha_k = \tilde{\alpha}_{2k} = \tilde{\alpha}_{2k-1}$, then the above relation reads

$$Z_k\left(\frac{z+1/z}{2}\right) = A\zeta_k(z) + B + A\zeta_k^{-1}(z),$$

with $\beta_k = (\alpha_k + 1/\alpha_k)/2$, and the explicit forms of the constants A and B are

$$A = \frac{(1+\alpha_k^2)^2}{2(1-\alpha_k^2)^2} \quad \textit{and} \quad B = \frac{2\alpha_k(1+\alpha_k^2)}{(1-\alpha_k^2)^2}.$$

Proof. For the real Blaschke factor Z_k we find

$$Z_k\left(\frac{z+1/z}{2}\right) = \frac{z+1/z}{2 - \frac{z+1/z}{\beta_k}} = \frac{-\beta_k(z^2+1)}{z^2 - 2\beta_k z + 1}.$$

The denominator of this expression has the following factorization

$$z^2 - 2\beta_k z + 1 = \left(z - (\beta_k - \sqrt{\beta_k^2 - 1})\right)\left(z - (\beta_k + \sqrt{\beta_k^2 - 1})\right).$$

If $\beta_k > 1$, we choose $\tilde{\alpha}_{2k-1} = \beta_k - \sqrt{\beta_k^2 - 1} \in I$ and $1/\tilde{\alpha}_{2k} = \beta_k + \sqrt{\beta_k^2 - 1} \in I^c$, otherwise, if $\beta_k < -1$, we choose $\tilde{\alpha}_{2k-1} = \beta_k + \sqrt{\beta_k^2 - 1} \in I$ and $1/\tilde{\alpha}_{2k} = \beta_k - \sqrt{\beta_k^2 - 1} \in I^c$. This yields in both cases the relation (3.3). The linear combination is now easily verified by looking at the coefficients of the powers of z in the numerator. □

It is now easy to see that if the condition on the placement of the poles (3.3) is satisfied, then the real Blaschke product $b_k(x)$ is a linear combination of

$$B_{-k}, B_{-(k-1)}, \ldots, B_{-1}, 1, B_1, \ldots, B_{k-1}, B_k.$$

Thus $b_k((z + z^{-1})/2) \in \mathcal{R}_{k,k}$. To satisfy (3.1) it is sufficient to satisfy

$$\int_{-\pi}^{\pi} p_n(\cos\theta) B_k(e^{i\theta}) w(\theta) \mathrm{d}\theta = 0, \quad k = -(n-1), \ldots, n-1. \tag{3.4}$$

By the fact that $\beta \subset I^c$, we also know that p_n and q_n have real coefficients with respect to the basis $\{b_k\}$ [2, Lemma 11.1.1]. Thus it is easily seen (p_n is real and $B_{k*} = \overline{B}_k$ on $\mathbb{T}$) that the conditions for positive and negative k in (3.4) are the same.

Because w is even and $\alpha \subset I$ it follows that the ORF $\tilde{\phi}_n$ also has real coefficients with respect to the basis $\tilde{B}_k$ as stated in the next lemma.

Lemma 3.2. *Suppose that w is even. If $\alpha \subset I$, then $\tilde{\phi}_n$ has real coefficients with respect to the basis $\{\tilde{B}_0, \tilde{B}_1, \ldots\}$. Thus*

$$\tilde{\phi}_n(z) = \sum_{i=0}^{n} \tilde{a}_i \tilde{B}_i(z), \quad \text{with } \tilde{a}_i \in \mathbb{R}.$$

Proof. First we notice that for the Blaschke products $\tilde{B}_k(z)$ the following holds

$$\tilde{B}_k(z) = \overline{\tilde{B}_k(\overline{z})}. \tag{3.5}$$

We prove this lemma by induction on n.
For $n = 0$ we find $\tilde{\phi}_0(z) = \tilde{\kappa}_0 \tilde{B}_0(z)$, with $\tilde{\kappa}_0 > 0$.
Suppose that it is valid for $i < n$. Then we find from the Gram-Schmidt procedure

$$\tilde{\phi}_n(z) = \frac{\tilde{\varphi}_n(z)}{||\tilde{\varphi}_n||}, \quad \text{with } \tilde{\varphi}_n(z) = \tilde{B}_n(z) - \sum_{i=0}^{n-1} \left\langle \tilde{B}_n, \tilde{\phi}_i \right\rangle_w \tilde{\phi}_i(z).$$

It is sufficient to prove that $\left\langle \tilde{B}_n, \tilde{\phi}_i \right\rangle_w \in \mathbb{R}$. From the fact that w is even and (3.5), we find

$$\begin{aligned}
\left\langle \tilde{B}_n, \tilde{\phi}_i \right\rangle_w &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{B}_n(z) \overline{\tilde{\phi}_i(z)} w(\theta) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \overline{\int_{-\pi}^{\pi} \overline{\tilde{B}_n(z)} \tilde{\phi}_i(z) w(\theta) \mathrm{d}\theta} \\
&= \frac{1}{2\pi} \overline{\int_{-\pi}^{\pi} \overline{\tilde{B}_n(\overline{z})} \tilde{\phi}_i(\overline{z}) w(\theta) \mathrm{d}\theta} \\
&= \frac{1}{2\pi} \overline{\int_{-\pi}^{\pi} \tilde{B}_n(z) \overline{\tilde{\phi}_i(z)} w(\theta) \mathrm{d}\theta} = \overline{\left\langle \tilde{B}_n, \tilde{\phi}_i \right\rangle_w}.
\end{aligned}$$

This concludes the proof. □

For the q_n it is sufficient to see that with $z = e^{i\theta}$

$$
\begin{aligned}
v_n(x) &= \frac{[(\beta_n^2-1)(\beta_{n+1}^2-1)]^{1/2}(1-x^2)}{|(\beta_n - x)(\beta_{n+1}-x)|}u(x) \\
&= \frac{-1}{4}\Big(\zeta_n(z) - \zeta_{n*}(z)\Big)\Big(\zeta_{n+1}(z) - \zeta_{n+1*}(z)\Big)u(x).
\end{aligned}
$$

Thus the following relation is the analogue of (3.4): q_n is orthogonal to $\mathcal{L}_{n-1}^r$ with respect to the weight v_n iff it holds for $k = -(n-1), \dots, n-1$ that

$$
\int_{-\pi}^{\pi} q_n(\cos\theta)B_k(e^{i\theta})\Big(\zeta_n(e^{i\theta}) - \zeta_{n*}(e^{i\theta})\Big)\Big(\zeta_{n+1}(e^{i\theta}) - \zeta_{n+1*}(e^{i\theta})\Big)w(\theta)\mathrm{d}\theta = 0. \qquad (3.6)
$$

Again we have to check this only for nonpostitive k. The case of nonnegative k is the same.

4. Main result

Now we are able to state our main result.

Theorem 4.1. *Suppose the weight functions u, v_k and w, with orthonormal rational functions p_n, q_n and $\tilde{\phi}_n$ respectively, are defined as above. Also define the orthogonal rational functions σ_n as before. If further the points $\alpha \subset I$ and $\beta \subset I^c$ satisfy (3.3), then the following relations hold*

$$
\begin{aligned}
p_n(x) &= (2\pi)^{-1/2}\left\{1 + \frac{\tilde{\phi}_{2n}(\alpha_n)}{\tilde{\kappa}_{2n}}\right\}^{-\frac{1}{2}}\{B_{n*}(z)\tilde{\phi}_{2n}(z) + B_n(z)\tilde{\phi}_{2n*}(z)\} \\
&= (2\pi)^{-1/2}\left\{1 + \frac{\sigma_{2n*}(\alpha_n)}{\sigma_{2n}(\alpha_n)}\right\}^{-\frac{1}{2}}\{\sigma_{2n*}(z) + \sigma_{2n}(z)\} \\
&= (2\pi)^{-1/2}\left\{1 - \frac{\tilde{\phi}_{2n}(\alpha_n)}{\tilde{\kappa}_{2n}}\right\}^{-\frac{1}{2}}\{B_{n-1*}(z)\tilde{\phi}_{2n-1}(z) + B_{n-1}(z)\tilde{\phi}_{2n-1*}(z)\} \\
&= (2\pi)^{-1/2}\left\{1 - \frac{\sigma_{2n*}(\alpha_n)}{\sigma_{2n}(\alpha_n)}\right\}^{-\frac{1}{2}}\{\sigma_{2n-1}(z) + \sigma_{2n-1*}(z)\}; \\
q_n(x) &= C_n\frac{B_{n+1*}(z)\tilde{\phi}_{2n+2}(z) - B_{n+1}(z)\tilde{\phi}_{2n+2*}(z)}{\zeta_{n+1}(z) - \zeta_{n+1*}(z)} \\
&= C_n\frac{\sigma_{2n+2*}(z) - \sigma_{2n+2}(z)}{\zeta_{n+1}(z) - \zeta_{n+1*}(z)} \\
&= D_n\frac{B_{n*}(z)\tilde{\phi}_{2n+1}(z) - B_n(z)\tilde{\phi}_{2n+1*}(z)}{\zeta_{n+1}(z) - \zeta_{n+1*}(z)} \\
&= D_n\frac{\sigma_{2n+1*}(z) - \sigma_{2n+1}(z)}{\zeta_{n+1}(z) - \zeta_{n+1*}(z)}.
\end{aligned}
$$

Here $\tilde{\kappa}_n = \tilde{B}_n(z)\tilde{\phi}_{n*}(z)\Big|_{z=\tilde{\alpha}_n}$ denotes the leading coefficient, i.e., the coefficient of $\tilde{B}_n(z)$ in the expansion of $\tilde{\phi}_n(z)$ with respect to the basis $\{\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_n\}$. The coefficients C_n and D_n have the following explicit form

$$
C_n = (2/\pi)^{1/2}\Big(\frac{1-\alpha_{n+1}^2}{1-\alpha_n^2}\Big)^{1/2} \times
$$

$$
\left\{\left\|\frac{z-\alpha_{n+1}}{z-\alpha_n}\tilde{\phi}_{2n+2}(z)\right\|_w^2 - \frac{\mathrm{Re}\left\langle\frac{z-\alpha_{n+1}}{z-\alpha_n}\tilde{\phi}_{2n+2}(z), \frac{z-\alpha_{n+1}}{z-\alpha_n}\tilde{k}_{2n+2}(z,\alpha_{n+1})\right\rangle_w}{\tilde{\kappa}_{2n+2}}\right\}^{-1/2}
$$

$$D_n = C_n \Big(\frac{\tilde{\kappa}_{2n+2} - \tilde{\phi}_{2n+2}(\alpha_{n+1})}{\tilde{\kappa}_{2n+2} + \tilde{\phi}_{2n+2}(\alpha_{n+1})} \Big)^{1/2},$$

where $\tilde{k}_n(z,t)$ *denotes the reproducing kernel of* $\mathcal{L}_n^c$*. If all the* α_i *are the same or more general if* $\alpha_n = \alpha_{n+1}$*, then the coefficients* C_n *and* D_n *simplify considerable.*

$$C_n = (2/\pi)^{1/2} \left\{ 1 - \frac{\tilde{\phi}_{2n+2}(\alpha_{n+1})}{\tilde{\kappa}_{2n+2}} \right\}^{-\frac{1}{2}}$$

$$D_n = (2/\pi)^{1/2} \left\{ 1 + \frac{\tilde{\phi}_{2n+2}(\alpha_{n+1})}{\tilde{\kappa}_{2n+2}} \right\}^{-\frac{1}{2}}.$$

Proof. The even lines in these equalities are equal to the lines just preceeding it by introducing the definition of the σ_k. So we have to prove only the odd lines.

First we look at p_n. For the orthogonality we have to prove that (3.1) holds and for the normalization we have to prove that

$$\int_{-1}^{1} p_n^2(x) u(x) \mathrm{d}x = 1. \tag{4.1}$$

To prove the orthogonality (3.1) , we need to verify (3.4). We find with $z = e^{i\theta}$ and $x = \frac{z+1/z}{2}$, that for $k = 0, \ldots, n-1$, there is some constant C such that

$$\begin{aligned}
&\int_{-\pi}^{\pi} p_n(\cos\theta) B_{k*}(z) w(\theta) \mathrm{d}\theta \\
&= C \int_{-\pi}^{\pi} \left[B_{n*}(z) B_{k*}(z) \tilde{\phi}_{2n}(z) + B_n(z) B_{k*}(z) \tilde{\phi}_{2n*}(z) \right] w(\theta) \mathrm{d}\theta \\
&= C \int_{-\pi}^{\pi} \left[B_{k*}(z) \sigma_{2n*}(z) + B_{k*}(z) \sigma_{2n}(z) \right] w(\theta) \mathrm{d}\theta \\
&= 2\pi C \left[\langle 1/B_k, \sigma_{2n} \rangle_w + \langle \sigma_{2n}, B_k \rangle_w \right] = 0.
\end{aligned}$$

This last equality follows because $k \leq n-1$ and hence $1/B_k$ and B_k are in $\mathcal{R}_{n-1,n-1}$ while σ_{2n} is orthogonal to that space.

It remains to check the normalization (4.1). We have

$$\begin{aligned}
&\int_{0}^{\pi} |B_{n*}(z) \tilde{\phi}_{2n}(z) + B_n(z) \tilde{\phi}_{2n*}(z)|^2 w(\theta) \mathrm{d}\theta \\
&= 2\pi + \mathrm{Re} \left\{ \int_{-\pi}^{\pi} B_{n*}(z) \tilde{\phi}_{2n}(z) \overline{B_n(z) \tilde{\phi}_{2n*}(z)} w(\theta) \mathrm{d}\theta \right\} \\
&= 2\pi + \mathrm{Re} \left\{ \int_{-\pi}^{\pi} \tilde{\phi}_{2n}(z) \overline{\tilde{B}_{2n}(z) \tilde{\phi}_{2n*}(z)} w(\theta) \mathrm{d}\theta \right\} \\
&= 2\pi + 2\pi \left\langle \tilde{\phi}_{2n}, \tilde{\phi}_{2n}^* \right\rangle_w \\
&= 2\pi + 2\pi \frac{\tilde{\phi}_{2n}(\alpha_n)}{\tilde{\kappa}_{2n}}.
\end{aligned}$$

The last line follows from [2, Theorem 2.2.3] which states that $\tilde{\kappa}_{2n} \tilde{\phi}_{2n}^*(z) = \tilde{k}_{2n}(z, \alpha_{2n})$, where $\tilde{k}_{2n}(z,t)$ denotes the reproducing kernel of $\mathcal{L}_{2n}^c$.

The second relation for p_n follows easily from

$$\tilde{\phi}_{2n}(z) = \frac{\tilde{\kappa}_{2n}}{\tilde{\kappa}_{2n-1}} \left[\frac{\overline{\tilde{\phi}_{2n}^*(\alpha_n)}}{\tilde{\kappa}_{2n}} \frac{z - \alpha_n}{1 - \alpha_n z} \tilde{\phi}_{2n-1}(z) + \frac{\tilde{\phi}_{2n}(\alpha_n)}{\tilde{\kappa}_{2n}} \tilde{\phi}_{2n-1}^*(z) \right]$$

which is derived from [2, p. 77, formula (4.8)]. This formula allows to write $\tilde{\phi}_{2n}$ in terms of $\tilde{\phi}_{2n-1}$ and $\tilde{\phi}_{2n-1*}$. When this is introduced in the previous formula for p_n, the second formula follows.

For the q_n we have to check the orthogonality by verifying (3.6). We have with $z = e^{i\theta}$ and $x = \cos\theta$, that there is a constant C such that for $k = 0, \ldots, n-1$,

$$\begin{aligned} q_n(x)B_{k*}(z)&\frac{(\zeta_n(z) - \zeta_{n*}(z))(\zeta_{n+1}(z) - \zeta_{n+1*}(z))}{4} \\ &= C(\zeta_n(z) - \zeta_{n*}(z))B_{k*}(z)\left[\sigma_{2n+2*}(z) - \sigma_{2n+2}(z)\right]. \end{aligned}$$

Because $k \le n-1$,

$$(\zeta_n(z) - \zeta_{n*}(z))B_{k*}(z) \in \mathcal{R}_{n,n}.$$

Thus, because σ_{2n+2} is orthogonal to $\mathcal{R}_{n,n}$ and so is $\tau_{2n+2} = \sigma_{2n+2*}$, we find that the condition of (3.6) holds. To check the normalization of the q_n, we proceed as in the case of the p_n. Since exactly the same arguments as in the case of p_n are used, we leave this to the reader. Also the fourth odd relation is deduced from [2, p. 77, formula (4.8)] as in the case of p_n. □

Remark 4.2. Because $\sigma_{2n*}(z) + \tau_n\sigma_{2n}(z)$ for $\tau_n \in \mathbb{T}$ is a para-orthogonal rational function, we know that it has $2n$ simple zeros on the unit circle [1]. Because the coeffcients are real these zeros appear in complex conjugate pairs or are real. But the zeros of $\tilde{\phi}_{2n}(z)$ are strictly inside the open unit disk, so $\tilde{\phi}_{2n}(1) \neq 0$, and therefore also $\sigma_{2n}(1) = \sigma_{2n*}(1) \neq 0$. Hence $\sigma_{2n*}(1) + \sigma_{2n}(1) \neq 0$, and thus also $\sigma_{2n*}(-1) + \sigma_{2n}(-1) \neq 0$. In other words the zeros appear in pairs of complex conjugate numbers on the unit circle. Suppose they are $e^{\pm i\theta_k}$, $k = 1, \ldots, n$ then $x_k = \cos\theta_k$, $k = 1, \ldots, n$ are the zeros of $p_n(x)$. This means that the zeros of $p_n(x)$ are all strictly inside the open interval $(-1, 1)$.

A similar argument can be used for $q_n(x)$, but now $z = \pm 1$ do belong to the zeroset of the para-orthogonal rational function $\sigma_{2n+2}(z) - \sigma_{2n+2*}(z)$. But the denominator $\zeta_n(z) - \zeta_{n*}(z)$ also has the zeros $z = \pm 1$, (in fact these are the only ones) and thus, ± 1 will cancel out as zeros of $q_n(x)$, thus again the zeros of $q_n(x)$ are strictly inside $(-1, 1)$.

These remarks are obvious in the light of the fact that the numerators of the orthogonal rational functions can be considered as the polynomials orthogonal to positive varying measures.

Remark 4.3. It is remarkable that q_n belongs to $\mathcal{L}_n^r$, and this does only depend on $\beta_1, \ldots, \beta_n$ and not on β_{n+1}, thus not on α_{n+1}. However, the varying measure for which there is orthogonality does depend on β_{n+1} and also the expressions for q_n in our main theorem depend on ζ_{n+1}, hence on α_{n+1}. In the latter this does cancel out in numerator and denominator. So if we are interested in q_n (up to a constant) for a fixed n, then we could as well choose $\alpha_{n+1} = 0$ and thus $\beta_{n+1} = \infty$. In that case $v_n(x)$ becomes

$$v_n(x) = \frac{(1 - x^2)\sqrt{\beta_n^2 - 1}}{|\beta_n - x|}u(x).$$

The inverse relations can also easily be achieved as shown in the following corollary.

Corollary 4.4. *Suppose the weight functions u, v_k and w, with orthonormal rational functions p_n, q_n and $\tilde{\phi}_n$ respectively, are defined as above. If further the points $\alpha \subset I$ and $\beta \subset I^c$ satisfy (3.3), then the following relations hold*

$$2B_{n*}(z)\tilde{\phi}_{2n}(z) = Ap_n\left(\frac{1}{2}(z + 1/z)\right) + C(\zeta_n(z) - \zeta_{n*}(z)q_{n-1}\left(\frac{1}{2}(z + 1/z)\right);$$

$$\begin{aligned}
2B_n(z)\tilde{\phi}_{2n*}(z) &= Ap_n\left(\frac{1}{2}(z+1/z)\right) - C(\zeta_n(z) - \zeta_{n*}(z)q_{n-1}\left(\frac{1}{2}(z+1/z)\right);\\
2B_{n-1*}(z)\tilde{\phi}_{2n-1}(z) &= Bp_n\left(\frac{1}{2}(z+1/z)\right) + D(\zeta_n(z) - \zeta_{n*}(z)q_{n-1}\left(\frac{1}{2}(z+1/z)\right);\\
2B_{n-1}(z)\tilde{\phi}_{2n-1*}(z) &= Bp_n\left(\frac{1}{2}(z+1/z)\right) - D(\zeta_n(z) - \zeta_{n*}(z)q_{n-1}\left(\frac{1}{2}(z+1/z)\right).
\end{aligned}$$

The constants A, B, C and D are real and their explicit form is as follows

$$\begin{aligned}
A &= \sqrt{2\pi}\Big\{1 + \frac{\tilde{\phi}_{2n}(\alpha_n)}{\tilde{\kappa}_{2n}}\Big\}^{1/2};\\
B &= \sqrt{2\pi}\Big\{1 - \frac{\tilde{\phi}_{2n}(\alpha_n)}{\tilde{\kappa}_{2n}}\Big\}^{1/2};\\
C &= C_{n-1}^{-1};\\
D &= D_{n-1}^{-1}.
\end{aligned}$$

Proof. The result is easily deduced by combining the formulas for p_n and q_{n-1} of Theorem 4.1. □

5. An example

The simplest possible example is to take $w(\theta) = 1$ and hence $u(x) = 1/\sqrt{1-x^2}$. The orthogonal rational functions on the circle with respect to the Lebesgue measure are given by

$$\tilde{\phi}_n(z) = \tilde{\kappa}_n \frac{z\tilde{B}_n(z)}{z - \tilde{\alpha}_n} = \begin{cases} \tilde{\kappa}_{2k}\dfrac{zB_{k-1}(z)B_k(z)}{1-\alpha_k z}, & n = 2k\\ \tilde{\kappa}_{2k-1}\dfrac{zB_{k-1}^2(z)}{1-\alpha_k z}, & n = 2k-1. \end{cases}$$

According to our main theorem, with $z = e^{i\theta}$ and $x = \cos\theta$, we have

$$\begin{aligned}
p_n(x) &= A_n\left[\frac{zB_{n-1}(z)}{1-\alpha_n z} + \frac{1}{(z-\alpha_n)B_{n-1}(z)}\right]\\
&= A_n\left[\frac{z(z-\alpha_n)\prod_{j=1}^{n-1}(z-\alpha_j)^2 + (1-\alpha_n z)\prod_{j=1}^{n-1}(1-\alpha_j z)^2}{\prod_{j=1}^{n}(z-\alpha_j)\prod_{j=1}^{n}(1-\alpha_j z)}\right]\\
&= A_n\left[\frac{z^{-(n-1)}(z-\alpha_n)\prod_{j=1}^{n-1}(z-\alpha_j)^2 + z^{n-1}(z^{-1}-\alpha_n)\prod_{j=1}^{n-1}(z^{-1}-\alpha_j)^2}{\prod_{j=1}^{n}(z-\alpha_j)\prod_{j=1}^{n}(z^{-1}-\alpha_j)}\right]\\
&= A_n\left[\frac{\lambda_n(z) + \lambda_{n*}(z)}{\prod_{j=1}^{n}(z-\alpha_j)\prod_{j=1}^{n}(z^{-1}-\alpha_j)}\right]
\end{aligned}$$

where

$$A_n = (2\pi)^{-1/2}\left\{1 + \frac{\alpha_n B_{n-1}^2(\alpha_n)}{1-\alpha_n^2}\right\}^{-1/2}\tilde{\kappa}_{2n}$$

is a normalizing constant and

$$\lambda_n(z) = z^{-(n-1)}(z-\alpha_n)\prod_{j=1}^{n-1}(z-\alpha_j)^2.$$

It is easily checked that if $\alpha_k \neq 0$ and with $\beta_k = (\alpha_k + \alpha_k^{-1})/2$,

$$(z - \alpha_k)(z^{-1} - \alpha_k) = 2\alpha_k(x - \beta_k).$$

This expression is just 1 when $\alpha_k = 0$. Thus, writing $K_n = C_n / \prod_l (2\alpha_l)$ where α_l runs over all nonzero α_k's, we have

$$p_n(x) = K_n \frac{\lambda_n(z) + \lambda_{n*}(z)}{\prod_{j=1}^n (x - \beta_j)}.$$

If we write

$$\lambda_n(z) = \sum_{k=-(n-1)}^{n} c_k z^k, \quad c_n = 1,$$

then the numerator equals

$$\begin{aligned}
\lambda_n(z) + \lambda_{n*}(z) &= \sum_{k=-(n-1)}^{n} c_k (z^k + z^{-k}) \\
&= 2 \sum_{k=-(n-1)}^{n} c_k \cos k\theta = 2 \sum_{k=1}^{n-1} (c_k + c_{-k}) \cos k\theta + 2c_0 + 2\cos n\theta \\
&= 2 \sum_{k=1}^{n-1} (c_k + c_{-k}) T_k(x) + 2(T_n(x) + c_0),
\end{aligned}$$

where $T_k(x) = \cos k\theta$ are the Chebyshev polynomials of the first kind.

As an example, take $\alpha_k = 0$ for $k \geq 2$ and $\alpha_1 = \alpha$, then

$$\lambda_n(z) + \lambda_{n*}(z) = z^n - 2\alpha z^{n-1} + \alpha^2 z^{n-2} + \alpha^2 z^{-(n-2)} - 2\alpha z^{-(n-1)} + z^{-n}.$$

Thus the numerator of $p_n(x)$ is $2\Big(T_n(x) - 2\alpha T_{n-1}(x) + \alpha^2 T_{n-2}(x)\Big)$.

If we have two nonzero α's, namely $\alpha_1 = -\alpha_2 = \alpha \neq 0$, then

$$\lambda_n(z) + \lambda_{n*}(z) = z^n - 2\alpha^2 z^{n-2} + \alpha^4 z^{n-4} + \alpha^4 z^{-(n-4)} - 2\alpha^2 z^{-(n-2)} + z^{-n}.$$

Hence the numerator of $p_n(x)$ is $2\Big(T_n(x) - 2\alpha^2 T_{n-2}(x) + \alpha^4 T_{n-4}(x)\Big)$.

Similar arguments lead to the result that

$$q_n(x) = C_n' \left[\frac{\lambda_{n+1}(z) - \lambda_{n+1*}(z)}{\prod_{j=1}^n (z - \alpha_j) \prod_{j=1}^n (z^{-1} - \alpha_j)(z - z^{-1})}. \right]$$

for an appropriate constant C_n' and with $\lambda_n(z)$ as before. This implies that for some constant K_n'

$$q_n(x) = K_n' \frac{[\lambda_{n+1}(z) - \lambda_{n+1*}(z)]/[z - z^{-1}]}{\prod_{j=1}^n (x - \beta_j)}.$$

If we expand $\lambda_{n+1}(z)$ as before, then

$$\begin{aligned}
\frac{\lambda_{n+1}(z) - \lambda_{n+1*}(z)}{z - z^{-1}} &= \sum_{k=-n}^{n+1} c_k \frac{z^k - z^{-k}}{z - z^{-1}} \\
&= \sum_{k=1}^{n} (c_k - c_{-k}) \frac{\sin k\theta}{\sin\theta} + \frac{\sin(n+1)\theta}{\sin\theta} \\
&= \sum_{k=0}^{n-1} (c_{k+1} - c_{-(k+1)}) U_k(x) + U_n(x),
\end{aligned}$$

where $U_{k-1}(x) = \sin k\theta / \sin\theta$ is a Chebyshev polynomial of the second kind.

For example, if $\alpha_1 = \alpha \neq 0$ and $\alpha_k = 0$ for $k \geq 2$, then the numerator of $q_n(x)$ is $U_n(x) - 2\alpha U_{n-1}(x) + \alpha^2 U_{n-2}(x)$. If $\alpha_1 = -\alpha_2 = \alpha \neq 0$ and all other α_k's are zero, then the numerator of $q_n(x)$ is $U_n(x) - 2\alpha^2 U_{n-2}(x) + \alpha^4 U_{n-4}(x)$. Note also that if all the α_k are zero, then $q_n = U_n$. If $\alpha_n = 0$, then orthogonality of the q_n holds with respect to the weight function $v_n(x) = \sqrt{1-x^2}$, which is independent of n.

References

[1] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions and interpolatory product rules on the unit circle. I. Recurrence and interpolation. *Analysis*, 18:167–183, 1998.

[2] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[3] G. Szegő. *Orthogonal polynomials*, volume 33 of *Amer. Math. Soc. Colloq. Publ.* Amer. Math. Soc., Providence, Rhode Island, 4rd edition, 1975. First edition 1939.