Bernstein equiconvergence and Fejér type theorems for general rational Fourier series

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Abstract

Let $w(\theta)$ be a positive weight function on the interval $[-\pi, \pi)$ and associate the positive definite inner product on the unit circle of the complex plane by $\langle f, g\rangle_w = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}w(\theta)\mathrm{d}\theta$. For a sequence of points $\{\alpha_k\}_{k=1}^{\infty}$ included in a compact subset of the open unit disk, we consider the orthogonal rational functions (**ORF**) $\{\phi_k\}_{k=0}^{\infty}$ that are obtained by orthogonalization of the sequence $\{1, z/\pi_1, z^2/\pi_2, \ldots\}$ where $\pi_k(z) = \prod_{j=1}^{k}(1-\overline{\alpha}_j z)$, with respect to this inner product.

In this paper we prove that $\mathbf{s_n(z)} - \mathbf{S_n(z)}$ tends to zero in $|z| \leq 1$ if n tends to ∞, where $\mathbf{s_n}$ is the nth partial sum of the expansion of a bounded analytic function F in terms of the **ORF** $\{\phi_k\}_{k=0}^{\infty}$ and $\mathbf{S_n}$ is the nth partial sum of the ordinary power series expansion of F. The main condition on the weight is that it satisfies a Lipschitz-Dini condition and that it is bounded away from zero. This generalizes a theorem given by Szegő in the polynomial case, that is when all $\alpha_k = 0$.

As an important consequence we find that under the above conditions on the weight w and the points $\{\alpha_k\}_{k=1}^{\infty}$, the Cesàro means of the series $\mathbf{s_n}$ converge uniformly to the function F in $|z| \leq 1$ if the boundary function $f(\theta) := F(e^{i\theta})$ is continous on $[0, 2\pi]$. This can be seen as a generalization of Fejér's Theorem.

Keywords : equiconvergence, Fejér theorem, orthogonal rational functions
AMS Subject Classification : Primary : 42A20, Secondary : 26A15.

1 Introduction

First we define the spaces of rational functions that play a central role in this paper. For a given sequence of points $\{\alpha_k\}_{k=1}^{\infty}$, we define the factors

$$\pi_0 = 1, \quad \pi_n = \prod_{i=1}^{n}(1 - \overline{\alpha}_i z), \quad n = 1, 2, \ldots$$

If Π_n denotes the space of all polynomials of degree at most n, then we set

$$\mathcal{L}_n = \{p_n(z)/\pi_n(z) : p_n \in \Pi_n\}.$$

There are several ways to give bases for these spaces. We shall use here the Blaschke products defined as

$$B_0 = 1, \quad B_n(z) = \zeta_1(z) \cdots \zeta_n(z), \quad n = 1, 2, \ldots$$

with Blaschke factors, defined for $k = 1, 2, \ldots$ as

$$\zeta_k(z) = z_k \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \quad z_k = \begin{cases} \overline{\alpha}_k / |\alpha_k|, & \text{if } \alpha_k \neq 0 \\ 1, & \text{otherwise.} \end{cases} \tag{1.1}$$

Then it is clear that $\mathcal{L}_n = \operatorname{span}\{B_0, \ldots, B_n\}$.

We shall now consider some weight function w on $[-\pi, \pi)$ and the corresponding inner product

$$\langle f, g \rangle_w = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} w(\theta) \mathrm{d}\theta$$

and orthogonalize the basis $B_0, B_1, \ldots$ with respect to this inner product, to get the system of orthonormal rational functions (**ORF**) $\phi_0, \phi_1, \ldots$. Thus

$$\langle \phi_k, \phi_l \rangle_w = \delta_{kl},$$

where δ_{kl} is the Kronecker-delta.

We suppose that the weight function w we consider here is uniformly positive and uniformly bounded, i.e., there are positive numbers m and M such that

$$0 < m \leq w(\theta) \leq M < \infty, \quad \forall \theta \in [-\pi, \pi).$$

Moreover, we assume that it satisfies a Lipschitz-Dini condition

$$|w(\theta + \delta) - w(\theta)| < L |\log \delta|^{-1-\lambda}, \tag{1.2}$$

where L is a fixed positive constant and $\lambda > 0$. This form of the Lipschitz-Dini condition is stronger than the one defined in [5, p 227] in the sense that (1.2) implies (1.3)

$$\int_0^\pi \frac{\omega(w;\delta)}{\delta} \mathrm{d}\delta < \infty, \tag{1.3}$$

where $\omega(w;\delta)$ denotes the modulus of continuity

$$\omega(w;\delta) = \sup\{|w(x) - w(y)| : |x-y| < \delta\}.$$

A third form of the Lipschitz-Dini condition can be found in [4]

$$\lim_{\delta\to 0} \omega(w;\delta)\log(\delta) = 0. \tag{1.4}$$

This one is weaker than (1.3) in the sense that (1.3) implies (1.4). So we have the following implications

$$(1.2) \Rightarrow (1.3) \Rightarrow (1.4).$$

Unless stated otherwise, when we refer to the Lipschitz-Dini condition we mean in this paper the condition (1.2). It is the most informative since it describes the convergence precisely by prescribing the parameter λ.

We denote the unit circle as $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the open unit disk as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The closure is denoted as $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$.

Because $\log w \in L^1(\mathbb{T})$, we can define the spectral factor

$$\sigma(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{z+\zeta}{z-\zeta} \log w(\theta)\mathrm{d}\theta\right\}, \quad \zeta = e^{i\theta}.$$

This σ is an outer function in $H^2(\mathbb{D})$ and it has a boundary value on $\mathbb{T}$ and $|\sigma(e^{i\theta})|^2 = w(\theta)$ a.e.

Without loss of generality, we shall also assume that $\int_{-\pi}^{\pi} w(\theta)\mathrm{d}\theta = 2\pi$, so that we have $\phi_0 = 1$. For the sequence of points $\{\alpha_k\}_{k=1}^\infty$ we make the following two assumptions:

(1) The $\{\alpha_k\}_{k=1}^\infty$ are compactly included in the open unit disk, thus $|\alpha_k| \le 1 - d$ with $d > 0$, independent of k;
(2) The counting measures ν_n^α, that is the discrete measure $\nu_n^\alpha := \frac{1}{n}\sum_{i=1}^n \delta_{\alpha_i}$ that assigns a mass $1/n$ to the points α_k for $k = 1, \ldots, n$, has a weak star limit ν^α, that is $\lim_{n\to\infty} \int f(z)\mathrm{d}\nu_n^\alpha(z) = \int f(z)\mathrm{d}\nu^\alpha(z)$ for all continuous functions f. We shall denote this as $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. This is a mild condition on the distribution of the $\{\alpha_k\}_{k=1}^\infty$.

Under these conditions on the weight w and on the points $\{\alpha_k\}_{k=1}^{\infty}$, we gave in a previous paper [3], the asymptotics for the **ORF** $\{\phi_k\}_{k=0}^{\infty}$ on the unit circle. We even got the rate of convergence, namely $O(\log n)^{-\lambda}$. The main result was the following.

Theorem 1.1 *Suppose that the weight is uniformly bounded and satisfies a Lipschitz-Dini condition (1.2). Suppose furtheron that the above conditions on the* $\{\alpha_k\}_{k=1}^{\infty}$ *are satisfied. Then*

$$\rho_n \frac{(1-\overline{\alpha}_n)\phi_n^*(z)}{\sqrt{1-|\alpha_n|^2}} = \frac{1}{\sigma(z)} + O(\log n)^{-\lambda},$$

where $\phi_n^*(z) = B_n(z)\overline{\phi_n(1/\overline{z})}$ *denotes the generalized reciprocal of* ϕ_n.

We shall in this paper consider the uniform convergence of a general Fourier expansion with respect to the **ORF** $\{\phi_k\}_{k=0}^{\infty}$. Consider a function $F(z)$ analytic in $|z|<1$ and that has a (bounded) boundary value for $|z|\to 1$, which we also denote as $F(z)$. Obviously, since we have an orthonormal system, we get for such a function the formal expansion $\sum_{k=0}^{\infty} \langle F, \phi_k\rangle_w \phi_k$. Let us denote the partial sums as

$$\mathbf{s}_n(z) = \sum_{k=0}^{n} \langle F, \phi_k\rangle_w \phi_k(z).$$

We are interested in finding conditions under which $\mathbf{s}_n$ converges uniformly to F in $\overline{\mathbb{D}}$. It is clear that in the simplest possible case, that is when we consider the weight $w\equiv 1$ and set all $\alpha_k = 0$, then the **ORF** are just the powers $f_k(z) = z^k$, and even then the uniform convergence of $\mathbf{s}_n$ to $F(z)$ is not guaranteed on $\mathbb{T}$. Note that in this case we obtain the Maclaurin series expansion of F. Thus if F is analytic in the open unit disk, then we have uniform convergence there, but it is not guaranteed that there is uniform convergence on the circle itself. Let us introduce a notation for this special case. The inner product with $w\equiv 1$, thus using the Lebesgue measure, is denoted as $\langle\cdot,\cdot\rangle$, rather than $\langle\cdot,\cdot\rangle_1$. The partial sum in the expansion with respect to the orthogonal functions $f_k(z)\equiv z^k$ is denoted as

$$\mathbf{S}_n(z) = \sum_{k=0}^{n} \langle F, f_k\rangle f_k(z).$$

Rather than proving that $\mathbf{s}_n$ converges to F, we shall first prove that $\mathbf{s}_n - \mathbf{S}_n$ converges uniformly to zero under a somewhat stronger Lipschitz-Dini condition on the weight w: it shall be assumed that $\lambda > 1$. For the function F it will only be required that it is regular and bounded in $\mathbb{D}$. This means that $F \in H^\infty(\mathbb{D})$.

We are now ready to formulate our main theorem, but first we shall introduce some notation that is useful in the rest of the paper.

The $\{\alpha_k\}_{k=1}^{\infty}$ and the **ORF** $\{\phi_k\}_{k=0}^{\infty}$ will always satisfy the properties given above. Furthermore, we define for any complex function f the para-hermitian conjugate as $f_*(z) = \overline{f(1/\overline{z})}$ and we define $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\}$. Finally we define for a function $f_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the reciprocal function as $f_n^*(z) = B_n(z) f_{n*}(z)$.

In this paper we first prove the following main theorem.

Theorem 1.2 *Let w be a weight function on $[-\pi, \pi)$ that satisfies the following Lipschitz-Dini condition ($\delta > 0$)*

$$|w(\theta + \delta) - w(\theta)| < L|\log \delta|^{-1-\lambda}, \tag{1.5}$$

where $L > 0$ and $\lambda > 1$ are fixed numbers. Suppose furthermore that w is uniformly bounded, i.e.

$$\exists\ m, M \in \mathbb{R}: \quad 0 < m \leq w(\theta) \leq M < \infty, \quad \forall \theta \in [-\pi, \pi).$$

For the $\{\alpha_k\}_{k=1}^{\infty}$ we assume that they are all in a compact subset of $\mathbb{D}$ and that the associated counting measure ν_n^{α} converges in the weak star topology to the measure ν^{α}.

*Let $F \in H^{\infty}(\mathbb{D})$. If $\mathbf{s}_n$ denotes the nth partial sum of the expansion of the boundary values $F(z)$, $z \in \mathbb{T}$, in terms of the **ORF** $\{\phi_k\}_{k=0}^{\infty}$ associated with w, and if $\mathbf{S}_n$ is the nth partial sum of the ordinary power series expansion of F, then we have*

$$\lim_{n \to \infty} \{\mathbf{s}_n(z) - \mathbf{S}_n(z)\} = 0,$$

uniformly in the whole closed unit disc $\overline{\mathbb{D}}$.

In the second section we deduce the lemmas we need to prove this theorem. The proof is then given in the third section. In the last section a number of corollaries of our main theorem are given. The most important result here is a generalization of Fejér's Theorem [4] that states that under the conditions of Theorem 1.2 on the weight w and the points $\{\alpha_k\}_{k=1}^{\infty}$, the Cesàro means of the partial sums $\mathbf{s}_n$ converge uniformly on $\mathbb{D} \cup \mathbb{T}$ to the function F if this function F is bounded and analytic in $\mathbb{D}$ and if its boundary function $f(\theta) := F(e^{i\theta})$ is continous on $[0, 2\pi]$. This means that F is in the disc algebra $\mathcal{A}(\mathbb{D})$.

2 Some preliminaries

Notice that it is sufficient to discuss the statement for $z \in \mathbb{T}$, because we are dealing with analytic functions. The following integral expressions for the partial sums are easy to obtain. Setting $\zeta = e^{i\theta}$ we get

$$\begin{aligned} \mathbf{s}_n(z) &= \sum_{i=0}^{n} \langle F, \phi_i \rangle_w \, \phi_i(z) \\ &= \sum_{i=0}^{n} \phi_i(z) \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\zeta)\overline{\phi_i(\zeta)} w(\theta) \mathrm{d}\theta \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\zeta) k_n(w; z, \zeta) w(\theta) \mathrm{d}\theta.$$

We have used the notation $k_n(w; z, \zeta)$ to denote the kernel $k_n(w; z, \zeta) = \sum_{i=0}^{n} \phi_i(x) \overline{\phi_i(\zeta)}$. Note that k_n is the reproducing kernel for $\mathcal{L}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle_w$. For the power series expansion we have (recall $f_k(z) = z^k$)

$$\begin{aligned} \mathbf{S}_n(z) &= \sum_{i=0}^{n} \langle F, f_i \rangle z^i \\ &= \sum_{i=0}^{n} z^i \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\zeta) \overline{\zeta}^i \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\zeta) \frac{1 - (\overline{\zeta} z)^{n+1}}{1 - \overline{\zeta} z} \mathrm{d}\theta. \end{aligned}$$

Here the reproducing kernel is

$$K_n(z, \zeta) = \sum_{i=0}^{n} f_i(z) \overline{f_i(\zeta)} = \sum_{i=0}^{n} z^i \overline{\zeta}^i = \frac{1 - (\overline{\zeta} z)^{n+1}}{1 - \overline{\zeta} z}.$$

We denote with $\Delta_n(z, \zeta)$ the difference of the kernels times the weight functions

$$\Delta_n(z, \zeta) = k_n(w; z, \zeta) w(\theta) - \frac{1 - (\overline{\zeta} z)^{n+1}}{1 - \overline{\zeta} z}.$$

Thus our main theorem will be proved if we show that (recall $\zeta = e^{i\theta}$)

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} F(\zeta) \Delta_n(z, \zeta) \mathrm{d}\theta = 0, \quad \forall z \in \mathbb{T}.$$

We now give a sequence of lemma's that are necessary to prove the main theorem. We recall that the conditions on the weight w and on the points $\{\alpha_k\}_{k=1}^{\infty}$ are assumed to hold throughout this paper.

From [2] we get the Christoffel-Darboux relations for orthonormal rational functions.

Lemma 2.1 *The following relation holds between the reproducing kernel $k_n(w; z, \zeta)$ and the **ORF** $\{\phi_k\}_{k=0}^{\infty}$:*

$$k_n(w; z, \zeta) = \frac{\phi_{n+1}^*(z) \overline{\phi_{n+1}^*(\zeta)} - \phi_{n+1}(z) \overline{\phi_{n+1}(\zeta)}}{1 - \zeta_{n+1}(z) \overline{\zeta_{n+1}(\zeta)}}.$$

Remember that the ζ_n are the Blaschke factors (1.1).

We recall the simple fact that if w satisfies a Lipschitz-Dini condition then also $1/w$ satisfies a Lipschitz-Dini condition (see [3,7]).

We now prove the following lemma, which says that also the spectral factor σ satisfies a Lipschitz-Dini condition.

Lemma 2.2 *If the weight w satisfies the Lipschitz-Dini condition (1.5), then the spectral factor σ satisfies the following Lipschitz-Dini condition*

$$|s(\theta+\delta)-s(\theta)| < L'|\log\delta|^{-\lambda},$$

where $s(\theta) := \sigma(e^{i\theta})$ and where L' is a positive constant and $\lambda > 1$ is the same as in (1.5).

PROOF. This lemma was proved in [7]. It relies on a Jackson [6] Theorem for trigonometric polynomials, usually referred to as the Jackson III Theorem [4, p 144]. We give a brief sketch of the proof.

Let n be an arbitrary positive integer. Applying the Jackson Theorem to the function $1/w$, we find a trigonometric polynomial g_n of order n, so that

$$|w(\theta) - 1/g_n(\theta)| < P(\log n)^{-1-\lambda},$$

where we used the Lipschitz-Dini condition of the function $1/w$. We know that there exists a polynomial H_n of degree n, so that $g_n(\theta) = |H_n(e^{i\theta})|^2$. We can show that [7]

$$|\sigma(z) - 1/H_n(z)| < Q(\log n)^{-\lambda}, \qquad (2.1)$$

uniformly for $|z| \leq 1$. The constant Q only depends on the minimum and maximum of the weight w as well as on L and λ, the parameters of the Lipschitz-Dini condition (1.5). Making use of (2.1) we obtain

$$|\sigma(e^{i(\theta+\delta)}) - \sigma(e^{i\theta})| < 2Q(\log n)^{-\lambda} + |H_n(e^{i(\theta+\delta)})^{-1} - H_n(e^{i\theta})^{-1}|.$$

By the Theorem of Bernstein [7, Theorem 1.22.1] the second term on the right-hand side is equal to $\delta O(n)$. So we found the bound $O(\log n)^{-\lambda} + \delta O(n)$. When we put $n = O(\delta^{-1}|\log\delta|^{-\lambda})$ the statement of the lemma follows. □

The previous proof can be given using rational functions as well. Indeed, a Jackson III type Theorem was derived in [1, Lemma 4.6] and except for technicalities, the proof can be given along the same line. This is however an unnecessary complication.

We remark here that by (2.1) and using the same argumentation as above, we can obtain the following more general inequality

$$|\sigma(z_1) - \sigma(z_2)| < L'\left|\log|z_1 - z_2|\right|^{-\lambda}, \quad \forall z_1, z_2 \in \mathbb{D} \cup \mathbb{T}. \qquad (2.2)$$

We now derive an approximation of the **ORF** $\{\phi_k\}_{k=0}^{\infty}$ in terms of the spectral factor σ.

Lemma 2.3 *With the notations of Theorem 1.1, we find for $z \in \mathbb{T}$ and for $n \to \infty$*

$$\begin{aligned}\phi_n^*(z) &= \overline{\rho}_n \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z} \frac{1}{\sigma(z)} + O(\log n)^{-\lambda};\\ \phi_n(z) &= \rho_n z_n \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z} \frac{z B_{n-1}(z)}{\overline{\sigma(z)}} + O(\log n)^{-\lambda},\end{aligned}$$

where $\rho_n \in \mathbb{T}$ is for normalization and $z_n \in \mathbb{T}$ as defined in (1.1).

PROOF. The proof is simply a rewriting of the results from Theorem 1.1. The first equation is obtained by writing

$$\rho_n \frac{(1-\overline{\alpha}_n z)\phi_n^*(z)}{\sqrt{1-|\alpha_n|^2}} = \frac{1}{\sigma(z)} + O(\log n)^{-\lambda}$$

and bringing all the factors to the right-hand side. The second equation can be obtained by the definition of the reciprocal function (note that $1/\overline{z} = z$ on $\mathbb{T}$).

$$\begin{aligned}\phi_n(z) &= B_n(z)\overline{\phi_n^*(z)}\\ &= B_n(z)\rho_n \frac{\sqrt{1-|\alpha_n|^2}}{1-\alpha_n \overline{z}} \frac{1}{\overline{\sigma(z)}} + O(\log n)^{-\lambda}\\ &= \rho_n z_n \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z} \frac{z B_{n-1}(z)}{\overline{\sigma(z)}} + O(\log n)^{-\lambda}.\end{aligned}$$

This proves the lemma. $\square$

The next lemma gives us an explicit form of $\Delta_n(z,\zeta)$ for $z,\ \zeta \in \mathbb{T}$.

Lemma 2.4 *For $z \in \mathbb{T}$ the following equality holds uniformly for ζ in compact subsets of $\mathbb{T}\backslash\{z\}$*

$$\Delta_n(z,\zeta) = \frac{\sigma(\zeta)/\sigma(z) - 1}{1-\overline{\zeta}z} - (\overline{\zeta}z)^{n+1} \frac{\frac{\overline{\sigma(\zeta)}}{\overline{\sigma(z)}} \frac{B_n(z)}{z^n} \frac{\zeta^n}{B_n(\zeta)} - 1}{1-\overline{\zeta}z} + \frac{O(\log n)^{-\lambda}}{1-\overline{\zeta}z}, \quad n \to \infty.$$

PROOF. Suppose $z,\ \zeta \in \mathbb{T}$, but $z \neq \zeta = e^{i\theta}$. From the Christoffel-Darboux relation (Lemma 2.1) we find

$$\Delta_n(z,\zeta) = \frac{\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(\zeta)} - \phi_{n+1}(z)\overline{\phi_{n+1}(\zeta)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(\zeta)}}|\sigma(\zeta)|^2 - \frac{1-(\overline{\zeta}z)^{n+1}}{1-\overline{\zeta}z}. \tag{2.3}$$

Recall that the **ORF** are denoted by ϕ_n and the ζ_n are the Blaschke factors given in (1.1). First we consider the denominator

$$1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(\zeta)} = \frac{(\zeta - z)(1-|\alpha_{n+1}|^2)}{(1-\overline{\alpha}_{n+1}z)(\zeta-\alpha_{n+1})}. \tag{2.4}$$

From Lemma 2.3, we get for $n \to \infty$

$$\phi_n^*(z) = \overline{\rho}_n \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z}\frac{1}{\sigma(z)} + O(\log n)^{-\lambda}; \tag{2.5}$$

$$\phi_n(z) = \rho_n z_n \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z} z B_{n-1}(z) \frac{1}{\overline{\sigma(z)}} + O(\log n)^{-\lambda}. \tag{2.6}$$

If we combine (2.3)-(2.6), we find

$$\begin{aligned}\Delta_n(z,\zeta) &= \frac{\sigma(\zeta)/\sigma(z) - \frac{\overline{\sigma(\zeta)}}{\overline{\sigma(z)}}\frac{zB_n(z)}{\zeta B_n(\zeta)}}{1-\overline{\zeta}z} - \frac{1-(\overline{\zeta}z)^{n+1}}{1-\overline{\zeta}z} + \frac{O(\log n)^{-\lambda}}{1-\overline{\zeta}z}\\ &= \frac{\sigma(\zeta)/\sigma(z)-1}{1-\overline{\zeta}z} - (\overline{\zeta}z)^{n+1}\frac{\frac{\overline{\sigma(\zeta)}}{\overline{\sigma(z)}}\frac{B_n(z)}{z^n}\frac{\zeta^n}{B_n(\zeta)}-1}{1-\overline{\zeta}z} + \frac{O(\log n)^{-\lambda}}{1-\overline{\zeta}z}.\end{aligned}$$

This proves the lemma. □

The last lemma of this section gives an upper bound for the kernel $k_n(w;z,\zeta)$.

Lemma 2.5 *If $k_n(w;z,\zeta)$ denotes the kernel for the **ORF** $\{\phi_k\}_{k=0}^{\infty}$, then for $n \to \infty$*

$$|k_n(w;z,\zeta)| \le O(n), \quad z,\zeta \in \mathbb{D}\cup\mathbb{T}.$$

PROOF. From [3, Lemma 4.8] we find (for $k \to \infty$)

$$\max_{t\in\mathbb{T}} |\phi_k(t)| = O(1).$$

Because ϕ_k is analytic in $\mathbb{D}$, we find for $z,\zeta \in \mathbb{D}\cup\mathbb{T}$

$$\begin{aligned}|k_n(w;z,\zeta)| &\le \sum_{k=0}^{n} |\phi_k(z)|\left|\overline{\phi_k(\zeta)}\right|\\ &\le \sum_{k=0}^{n} O(1)\\ &= O(n).\end{aligned}$$

□

3 Proof of the main theorem

Now we are able to prove the main theorem

Proof of Theorem 1.2 We have to prove that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} F(\zeta)\Delta_n(z,\zeta)\mathrm{d}\theta = 0, \quad \zeta = e^{i\theta},$$

uniformly for $z \in \mathbb{T}$. We split the integral into two parts

$$\int_E F(\zeta)\Delta_n(z,\zeta)\mathrm{d}\theta + \int_{E'} F(\zeta)\Delta_n(z,\zeta)\mathrm{d}\theta,$$

where $E = E(n,\varepsilon,z)$ is the set $\{\zeta \in \mathbb{T} : \ |z-\zeta| \geq \varepsilon n^{-1}\}$ and $E' = E'(n,\varepsilon,z)$ is the complementary set $\mathbb{T} \setminus E$. Here ε is an arbitrary small positive number.

The second integral is easy to bound. By Lemma 2.5 we find that $\Delta_n(z,\zeta) = O(n)$. So we have

$$\int_{E'} F(\zeta)\Delta_n(z,\zeta)\mathrm{d}\theta = O(n)\varepsilon n^{-1} = \varepsilon O(1).$$

This is arbitrarily small as $\varepsilon \to 0$.

Before looking at the first integral, we take a look at the integrals

$$\int_{-\pi}^{\pi} F(\zeta)\frac{\sigma(\zeta)/\sigma(z) - 1}{1-\overline{\zeta}z}\mathrm{d}\theta \quad \text{and} \quad \int_{-\pi}^{\pi} F(\zeta)(\overline{\zeta}z)^{n+1}\frac{\overline{\frac{\sigma(\zeta)}{\sigma(z)}}\frac{B_n(z)}{z^n}\frac{\zeta^n}{B_n(\zeta)} - 1}{1-\overline{\zeta}z}\mathrm{d}\theta.$$

Because $F \in H^\infty(\mathbb{D})$ and because also the rest of the integrand is analytic inside the unit disk (the pole $\zeta = z$ is canceled in the numerator) we can apply Cauchy's Theorem to find the following result

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\zeta)\frac{\sigma(\zeta)/\sigma(z) - 1}{1-\overline{\zeta}z}\mathrm{d}\theta = \frac{1}{2\pi i}\int_{|\zeta|=1} F(\zeta)\frac{\sigma(\zeta)/\sigma(z) - 1}{1-\overline{\zeta}z}\frac{\mathrm{d}\zeta}{\zeta} = 0. \tag{3.1}$$

Next we take a look at the function F_n, defined as ($z \in \mathbb{T}$ is a parameter)

$$F_n(\zeta) := \frac{\overline{\frac{\sigma(\zeta)}{\sigma(z)}}\frac{B_n(z)}{z^n}\frac{\zeta^n}{B_n(\zeta)} - 1}{1-\overline{\zeta}z}.$$

We see that the numerator of F_n has a simple zero at $\zeta = z$. This cancels the simple zero of the denominator and thus F_n is a rational function which has all his poles strictly inside the unit disc (this is a consequence of the assumption that the $\{\alpha_k\}_{k=1}^{\infty}$ are all in a compact subset of $\mathbb{D}$ and because σ is uniformly bounded). Therefore we can use Cauchy's Theorem to find ($\zeta = e^{i\theta}$)

$$\int_{-\pi}^{\pi} F_n(\zeta)\mathrm{d}\theta = 0.$$

Thus $F_n \in L^1(\mathbb{T})$. Because F is bounded, we see that FF_n is also in $L^1(\mathbb{T})$. According to the Riemann-Lebesgue Lemma (see [8, p. 45]), we now find

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} F(\zeta)(\overline{\zeta}z)^{n+1} F_n(\zeta)\mathrm{d}\theta = 0. \tag{3.2}$$

When we combine Lemma 2.4, (3.1) and (3.2), we find

$$\int_{-\pi}^{\pi} F(\zeta)\left(\Delta_n(z,\zeta) - \frac{O(\log n)^{-\lambda}}{1-\overline{\zeta}z}\right)\mathrm{d}\theta = o(1).$$

We use this equation to bound the first integral as follows

$$\begin{aligned}
\int_E F(\zeta)\Delta_n(z,\zeta)\mathrm{d}\theta = {} & O(\log n)^{-\lambda}\int_E \frac{\mathrm{d}\theta}{|1-\overline{\zeta}z|} \\
& - \int_{E'} F(\zeta)\left\{\frac{\sigma(\zeta)/\sigma(z)-1}{1-\overline{\zeta}z} - (\overline{\zeta}z)^{n+1}\frac{\overline{\frac{\sigma(\zeta)}{\sigma(z)}}\frac{B_n(z)}{z^n}\frac{\zeta^n}{B_n(\zeta)} - 1}{1-\overline{\zeta}z}\right\}\mathrm{d}\theta + o(1) \\
= {} & O(\log n)^{1-\lambda} - O(1)\int_{E'}\frac{|\log|z-\zeta||^{-\lambda}}{1-\overline{\zeta}z}\mathrm{d}\theta + \int_{E'} F(\zeta)(\overline{\zeta}z)^{n+1}F_n(\zeta)\mathrm{d}\theta + o(1) \\
= {} & O(\log n)^{1-\lambda} + o(1) = o(1), \quad n\to\infty.
\end{aligned}$$

The second equality follows from (2.2) and the third from (3.2). This proves our statement. □

4 Some important consequences

In this last section we give some consequences of Theorem 1.2. We have shown that under some conditions on the weight w and the points $\{\alpha_k\}_{k=1}^{\infty}$ and under some mild conditions for the function F, the general **ORF**-Fourier series and the ordinary Fourier series for F behave in the same way. Thus, if we impose some extra conditions on the function F that guarantee that $\mathbf{S}_n$ converges uniformly to F in $\overline{\mathbb{D}} = \mathbb{D}\cup\mathbb{T}$, then also $\mathbf{s}_n$ shall convergerge uniformly to F in $\overline{\mathbb{D}}$.

Two examples are given below: either the boundary function $f(\theta) := F(e^{i\theta})$ is 2π-periodic and satisfies a Lipschitz-Dini condition of the form (1.4) or it is continuous and of bounded variation.

As we mentioned before, since F is analytic in $\mathbb{D}$, it is sufficient to consider convergence on $\mathbb{T}$, since this immediately implies convergence in $\overline{\mathbb{D}}$. Indeed, $F - \mathbf{S}_n$ is analytic in $\mathbb{D}$ and by the maximum modulus Theorem, the maximum is reached on $\mathbb{T}$.

Since we are interested in uniform convergence of the generalized Fourier series $\mathbf{s}_n$ on the unit circle $\mathbb{T}$, we need to impose some constraints on the boundary function $f(\theta) := F(e^{i\theta})$.

If we assume Lipschitz-Dini conditions for the boundary function f we get the following generalization of the Lipschitz-Dini Theorem [4, p 146].

Theorem 4.1 (Lipschitz-Dini) *Suppose that the weight w and the points $\{\alpha_k\}_{k=1}^{\infty}$ satisfy the conditions of theorem 1.2. If $F \in \mathcal{A}(\mathbb{D})$ and its boundary function $f(\theta) := F(e^{i\theta})$ satisfies a Lipschitz-Dini condition of the form (1.4)*

$$\omega(f;\delta)\log(\delta) \to 0, \quad \delta \to 0,$$

then the series $\mathbf{s}_n$ converges uniformly to F in $\overline{\mathbb{D}}$.

PROOF. This can be proved by combining Theorem 1.2 and the Lipschitz-Dini Theorem for ordinary Fourier series (see e.g. [4, p 146] or [8, p 63]). This theorem states that under the same conditions the series $\mathbf{S}_n$ converges uniformly to F on $\mathbb{T}$. □

When the boundary function f is of bounded variation we can find the following.

Theorem 4.2 *Suppose that the weight w and the points $\{\alpha_k\}_{k=0}^{\infty}$ satisfy the conditions of Theorem 1.2. If $F \in \mathcal{A}(\mathbb{D})$ and its boundary function $f(\theta) := F(e^{i\theta})$ is of bounded variation over $[0, 2\pi]$, then the series $\mathbf{s}_n$ converges uniformly to F in $\overline{\mathbb{D}}$.*

PROOF. This can also be proved by combining two theorems, namely Theorem 1.2 and the Dirichlet-Jordan-test [8, Theorem 8.14] that states that under the same conditions the series $\mathbf{S}_n$ converges uniformly to F on $\mathbb{T}$. □

Finally we generalize the Fejér Theorem which says that the Cesàro means of the Fourier series $\mathbf{S}_n$ converges uniformly for a continuous function. We now prove that under our conditions on the weight w and the points $\{\alpha_k\}_{k=1}^{\infty}$, this also holds in the general case of an **ORF** Fourier series.

First we prove the next lemma.

Lemma 4.3 *If a series $r_n(z)$ converges uniformly on $\mathbb{T}$, then also the Cesàro means converge uniformly on $\mathbb{T}$, to the same limit.*

PROOF. Without loss of generality we may assume that $\lim_{n\to\infty} r_n(z) = 0$. Take $\varepsilon > 0$. So we know that there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 : |r_n(z)| < \varepsilon$, where n_0 does not depend on z. Thus

$$\begin{aligned}\left|\frac{1}{N+1}\sum_{n=0}^{N} r_n(z)\right| &\leq \frac{1}{N+1}\sum_{n=0}^{n_0-1}|r_n(z)| + \frac{1}{N+1}\sum_{n=n_0}^{N}|r_n(z)| \\ &< \frac{n_0 R}{N+1} + \frac{N-n_0}{N+1}\varepsilon \;<\; \frac{n_0 R}{N+1} + \varepsilon,\end{aligned}$$

where $R = \max\{R_0, \ldots, R_{n_0}\}$ with $R_k = \max_{z\in\mathbb{T}}\{|r_k(z)|\},\ k = 0, \ldots, n_0$. Letting $N \to \infty$ we find that $\rho_N(z) \to 0$ where $\rho_N(z) = \frac{1}{N+1}\sum_{k=0}^{N} r_k(z)$. This proves the lemma. $\square$

We can now prove a generalization of Fejér's Theorem (see e.g. [8, p 89] or [4, p 123]).

Theorem 4.4 (Fejér) *Suppose that the weight w and the points $\{\alpha_k\}_{k=1}^{\infty}$ satisfy the conditions of Theorem 1.2. If $F \in \mathcal{A}(\mathbb{D})$, then we find ($\sigma_N$ denotes the Nth Cesàro mean of $\mathbf{s}_n$)*

$$\lim_{N\to\infty}\max_{z\in\mathbb{T}}|F(z) - \sigma_N(z)| = 0.$$

Thus the Cesàro means of $\mathbf{s}_n$ converge uniformly to the function F in $\overline{\mathbb{D}}$.

PROOF. This is a combination of Theorem 1.2, Lemma 4.3 and the well-known Fejér Theorem, that states that if f is continous, then the Cesàro means of $\mathbf{S}_n$ converge uniformly to F in $\overline{\mathbb{D}}$. $\square$

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