Boundary asymptotics for orthogonal rational functions on the unit circle

A. Bultheel and P. Van gucht

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Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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Abstract

Let $w(\theta)$ be a positive weight function on the unit circle of the complex plane. For a sequence of points $\{\alpha_k\}_{k=1}^{\infty}$ included in a compact subset of the unit disk, we consider the orthogonal rational functions ϕ_n that are obtained by orthogonalization of the sequence $\{1, z/\pi_1, z^2/\pi_2, \ldots\}$ where $\pi_k(z) = \prod_{j=1}^{k}(1 - \overline{\alpha}_j z)$, with respect to the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}w(\theta)\mathrm{d}\theta$. We discuss in this paper the behaviour of $\phi_n(t)$ for $|t| = 1$ and $n \to \infty$ under certain conditions. The main condition on the weight is that it satisfies a Lipschitz-Dini condition and that it is bounded away from zero. This generalizes a theorem given by Szegő in the polynomial case, that is when all $\alpha_k = 0$.

Keywords : orthogonal rational functions, asymptotics
AMS(MOS) Classification : Primary : 42C05 42A10

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A. Bultheel and P. Van gucht *

Department of Computer Science, K.U.Leuven, Belgium

E-mail: {Adhemar.Bultheel—Patrick.Vangucht}@cs.kuleuven.ac.be

Let $w(\theta)$ be a positive weight function on the unit circle of the complex plane. For a sequence of points $\{\alpha_k\}_{k=1}^{\infty}$ included in a compact subset of the unit disk, we consider the orthogonal rational functions ϕ_n that are obtained by orthogonalization of the sequence $\{1, z/\pi_1, z^2/\pi_2, \ldots\}$ where $\pi_k(z) = \prod_{j=1}^{k}(1 - \overline{\alpha}_j z)$, with respect to the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}w(\theta)\mathrm{d}\theta$. We discuss in this paper the behaviour of $\phi_n(t)$ for $|t| = 1$ and $n \to \infty$ under certain conditions. The main condition on the weight is that it satisfies a Lipschitz-Dini condition and that it is bounded away from zero. This generalizes a theorem given by Szegő in the polynomial case, that is when all $\alpha_k = 0$.

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1. Introduction

The asymptotics of orthogonal polynomials on the unit circle have been discussed in many papers and monographs. Consider the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mathrm{d}\mu(\theta),$$

where μ represents a positive measure on the interval $[-\pi, \pi]$. By Gram-Schmidt orthogonalization, we can orthogonalize the sequence $\{1, z, z^2, \ldots\}$ and obtain orthogonal polynomials $\{\phi_k\}_{k=0}^{\infty}$.

An important problem in the theory of orthogonal polynomials is to describe the limiting behaviour of $\phi_n(z)$ as $n \to \infty$. These results can be obtained under various conditions on the measure μ. Typically, there is a distinction between the cases where $z \in \mathbb{D}$, $z \in \mathbb{T}$ or $z \in \mathbb{E}$ where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}.$$

In $\mathbb{D}$, the polynomials converge locally uniformly to zero. This holds under rather weak conditions like $\mu' > 0$ a.e. in $[-\pi, \pi]$.

The behaviour in $\mathbb{E}$ is more interesting. Therefore we need the definition of the the Szegő function or spectral factor of the measure.

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Assuming that $\log \mu' \in L^1$ (Szegő's condition), then the spectral factor is defined by

$$S(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\mu'(\theta)\mathrm{d}\theta\right\}.$$

This is an outer function in $H^2(\mathbb{D})$, which implies that S and $1/S$ are analytic functions in $\mathbb{D}$ and that the radial limit of $S(re^{i\theta})$ for $r \to 1$ exists. If we denote the boundary function again as $S(e^{i\theta})$, or $s(\theta)$, thus $s(\theta) = \lim_{r\to 1^-} S(re^{i\theta}) = S(e^{i\theta})$, then it holds that $\mu'(\theta) = |s(\theta)|^2$ almost everywhere on $[-\pi, \pi]$.

A typical behaviour is that locally uniformly in $\mathbb{E}$, we have $\phi_n(z)/z^n$ converges to the function $1/S_*(z)$, where for any function f, we define its para-hermitian conjugate f_* as $f_*(z) = \overline{f(1/\overline{z})}$. This can also be expressed as $z^n\phi_{n*}(z) \to 1/S(z)$ uniformly in compact subsets of the unit disk $\mathbb{D}$.

These results are standard and so are the conditions on the measure under which these conditions hold.

For the asymptotics of $\phi_n(t)$ when $t \in \mathbb{T}$, the literature is more subtle on the conditions that should be imposed for the measure. Under various conditions, various asymptotics were derived. See for example [4–7].

Many results for orthogonal polynomials were generalized to orthogonal rational functions. These generalizations have shown usefull for numerical quadrature and in several signal processing and system theoretic applications. A survey of currently available results and some applications are given in the recent monograph [2].

The polynomial situation is generalized as follows. Given a sequence of complex points $\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{D}$, one constructs the Blaschke factors $\zeta_k(z)$ and the finite Blaschke products $B_n(z)$ as follows:

$$\zeta_k(z) = z_k\frac{z-\alpha_k}{1-\overline{\alpha}_k z}, \quad k = 1,2,\ldots \quad \text{with} \quad z_k = \begin{cases}\overline{\alpha}_k/|\alpha_k| & \text{if } \alpha_k \neq 0\\ 1 & \text{otherwise}\end{cases}$$

$$B_0 = 1, \quad B_k(z) = \zeta_1(z)\cdots\zeta_k(z), \quad k = 1,2,\ldots$$

The space Π_n of the polynomials of degree at most n is replaced by the space $\mathcal{L}_n$ of rational functions:

$$\mathcal{L}_n = \mathrm{span}\{B_0, \ldots, B_n\}.$$

When we introduce the notation (it is consistently used throughout the paper)

$$\pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z) \quad \text{and} \quad \pi_n^*(z) = \prod_{k=1}^{n}(z-\alpha_k), \quad k = 1,2,\ldots$$

then we can also express $\mathcal{L}_n$ as $\mathcal{L}_n = \{p_n/\pi_n : p_n \in \Pi_n\}$. By orthogonalization of the sequence $\{B_0, B_1, \ldots\}$, one obtains the orthogonal rational functions $\{\phi_0, \phi_1, \ldots\}$. Note that if all the $\alpha_k = 0$, then the rational situation reduces to the polynomial case. In analogy with the polynomial situation, we shall call the coefficient a_n in

$$f(z) = a_n B_n(z) + a_{n-1}B_{n-1}(z) + \cdots + a_0 \in \mathcal{L}_n$$

the leading coefficient of f. We also denote for a function $f_n \in \mathcal{L}_n$ the reciprocal function

$$f_n^*(z) = B_n(z)f_{n*}(z) = \overline{a}_n + \overline{a}_{n-1}B_n(z)/B_{n-1}(z) + \cdots + \overline{a}_0 B_n.$$

The leading coefficient of ϕ_n will be denoted as κ_n: $\phi_n = \kappa_n B_n + \cdots$. Note that $\overline{\kappa}_n = \phi_n^*(\alpha_n)$.

The previous theorems about the asymptotics for the orthogonal polynomials were generalized to the rational case (see e.g. [2]). Inside $\mathbb{D}$ the orthogonal rational functions converge locally uniformly to zero, while the behaviour of ϕ_n in $\mathbb{E}$ is expressed by the behaviour of ϕ_n^* in $\mathbb{D}$: It holds that for appropriate $\rho_n \in \mathbb{T}$

$$\rho_n \frac{(1-\overline{\alpha}_n z)\phi_n^*(z)}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{S(z)} \quad \text{locally uniformly in } \mathbb{D}. \tag{1.1}$$

However, in the monograph [2], there is no theorem giving the asymptotics of ϕ_n on $\mathbb{T}$. In this paper we shall give the rational form of a theorem that can be found in Szegő's book [7, p. 297] for the polynomial case. If we suppose that $\mathrm{d}\mu(\theta) = w(\theta)\mathrm{d}\theta$ is absolutely continuous and the weight w satisfies $0 < m \leq w(\theta) \leq M < \infty$ uniformly in $[-\pi, \pi]$, and if it satisfies the Lipschitz-Dini condition then with some additional constraints on the asymptotics of the prescribed points $\{\alpha_i\}$ we shall prove that (1.1) also holds uniformly on $\mathbb{T}$ and we shall give the rate of convergence.

For the asymptotics of the points $\alpha = \{\alpha_k : k \in \mathbb{N}\}$, we shall assume some limiting distribution that is contained in a compact subset of $\mathbb{D}$. Thus $1 - |\alpha_k| \geq d > 0$ for some $d > 0$ that does not depend on k. Moreover, we assume that the counting measure, that is the discrete measure $\nu_n^\alpha := \frac{1}{n}\sum_{i=1}^n \delta_{\alpha_i}$ that assigns a mass $1/n$ to the points α_k for $k = 1, \ldots, n$, has a weak star limit ν^α, that is $\lim_{n\to\infty} \int f(z)\mathrm{d}\nu_n^\alpha(z) = \int f(z)\mathrm{d}\nu^\alpha(z)$ for all continuous functions f. We shall denote this as $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$.

Thus, unless stated otherwise, we shall assume the following conditions

(AC) The measure μ is *absolutely continuous*: $\mathrm{d}\mu(\theta) = w(\theta)\mathrm{d}\theta$, with

$$0 < m \leq w(\theta) \leq M < \infty \text{ uniformly in } \theta \in [-\pi, \pi].$$

(LD) The 2π-periodic function $w(\theta)$ satisfies a *Lipschitz-Dini* condition:

$$\forall \delta > 0 : \exists L > 0 : \quad |w(\theta+\delta) - w(\theta)| < L|\log\delta|^{-1-\lambda}, \quad \lambda > 0,$$

where L does not depend on θ or δ.

(CI) The point set $\alpha = \{\alpha_k : k \in \mathbb{N}\}$ is *compactly included* in $\mathbb{D}$, i.e. $\alpha \subset C$ with C a compact subset of $\mathbb{D}$, and the associated counting measures ν_n^α converge to some ν^α in weak star sense: $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$.

Note that (AC) implies that the Szegő condition $\log w(\theta) \in L^1_{2\pi}$ is satisfied. From (CI) it also follows that the Blaschke condition $\sum(1-|\alpha_k|) = \infty$ is satisfied, which means that the infinite Blaschke product $B(z) = \prod_{k=1}^\infty \zeta_k(z)$ diverges. The requirement $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ allows us to write the root asymptotics for the polynomials π_n^* in terms of logarithmic potentials of the ν^α. Recall that for a measure ν, its logarithmic potential is defined by $V_\nu(z) = -\int \log|z-\xi|\mathrm{d}\nu(\xi)$. It can be shown [1] that $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ implies the root asymptotics

$$\lim_{n\to\infty} |\pi_n^*(z)|^{1/n} = \exp\{-V_{\nu^\alpha}(z)\}, \quad z \in \mathbb{C} - \mathrm{supp}(\nu^\alpha),$$

and

$$\limsup_{n\to\infty} |\pi_n^*(z)|^{1/n} \leq \exp\{-V_{\nu^\alpha}(z)\}, \quad z \in \mathbb{C}$$

uniformly in each compact subset of the indicated regions.

The following is then a generalization of Theorem 12.1.3 of [7, p. 297].

Theorem 1.1. Suppose that the conditions (AC), (LD), and (CI) are satisfied.

Then there exist a sequence of unimodular constants $\rho_n \in \mathbb{T}$ such that the orthogonal rational functions ϕ_n satisfy

$$\rho_n \frac{(1-\overline{\alpha}_n z)\phi_n^*(z)}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{S(z)} \quad \text{uniformly for } z \in \mathbb{D} \cup \mathbb{T}.$$

The rate of convergence is $O(\log n)^{-\lambda}$. Thus also, with the same rate of convergence

$$\frac{P(t,\alpha_n)}{|\phi_n(t)|^2} \to |S(t)|^2 = w(\theta), \quad t = e^{i\theta}, \qquad \text{where} \qquad P(t,w) = \frac{1-|w|^2}{|t-w|^2}, \quad t \in \mathbb{T}$$

is the Poisson kernel.

The proof of this theorem will be given in the next sections. The idea is as follows. First we generalize the notion of trigonometric or Laurent polynomials as being elements from the space $\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_{n*}$ where $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\}$. Thus if $f \in \mathcal{R}_n$, then $f(z) = q(z)/[\pi_n(z)\pi_n^*(z)]$ where $q \in \Pi_{2n}$ is a polynomial of degree at most $2n$. If all α_k are zero, then $\mathcal{R}_n$ is the space of Laurent polynomials of degree at most n.

Then the inverse of the weight function: $1/w(\theta)$ is approximated arbitrary close by a positive trigonometric rational function, say $g_n(\theta)$. Thus $w(\theta) \approx w_n(\theta) = 1/g_n(\theta) = 1/|h_n(\theta)|^2$. The approximant $g_n(\theta) = G_n(e^{i\theta})$ with $G_n \in \mathcal{R}_{n-1}$ and $h_n(\theta) = H_n(e^{i\theta})$ where $H_n \in \mathcal{L}_{n-1}$ is the outer spectral factor of g_n. The nth orthogonal rational function ψ_n for the weight w_n can be explicitly written in terms of h_n and it can be shown that H_n converges to the inverse $1/S$ of the spectral factor of w, not only in $\mathbb{D}$ but also on $\mathbb{T}$. This gives the asymptotics for ψ_n. It then remains to show that ψ_n and ϕ_n have the same asymptotics.

2. Approximation of the weight

First we prove that if (AC) and (LD) hold then also the inverse $1/w$ satisfies a Lipschitz-Dini condition of the same order.

Lemma 2.1. Assume that the 2π-periodic function satisfies $0 < m \leq w(\theta) \leq M < \infty$ uniformly in $[-\pi, \pi]$, and if for all $\delta > 0$ there is some constant $L > 0$ such that

$$|w(\theta+\delta) - w(\theta)| < L|\log\delta|^{-1-\lambda}, \quad \lambda > 0,$$

then there is some constant K such that for all $\delta > 0$

$$\left|\frac{1}{w(\theta+\delta)} - \frac{1}{w(\theta)}\right| < K|\log\delta|^{-1-\lambda}$$

holds.

Proof. This is obvious since

$$\left|\frac{1}{w(\theta+\delta)} - \frac{1}{w(\theta)}\right| = \left|\frac{w(\theta+\delta)-w(\theta)}{w(\theta)w(\theta+\delta)}\right| < Lm^{-2}|\log\delta|^{-1-\lambda}.$$

Thus the lemma holds with $K = Lm^{-2}$. □

We now want to find an approximation for the orthogonal rational functions (**ORF**) with respect to the given weight function w. We know that the space $\bigcup_{n=0}^{\infty} \mathcal{R}_n$ is dense in $C(\mathbb{T})$ with respect to the supremum norm if and only if the Blaschke product diverges [2, Theorem 7.1.2]. Thus, it should be possible to find some approximant in $\mathcal{R}_n$ for w that is as close to w as we want.

The next theorem is a Jackson III type of theorem [3, p. 144]. It says how good such an approximation is as a function of n. It depends on the smoothness of w. In the polynomial case it states the following. Let $f \in C_{2\pi}$ be a continuous 2π-periodic function with modulus of continuity $\omega(f;\delta)$, that is $\omega(f;\delta) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta\}$. Then the best approximation in the set trigonometric polynomials gives an error that is at most $\omega(f; \frac{2\pi}{n})$ in $[-\pi, \pi]$. A similar theorem was obtained in [1, lemma 4.6] for the rational case. It requires some extra assumption on the distribution of the α_k so that asymptotics $|\pi_n^*(z)|^{1/n}$ can be estimated. This is where the condition $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ comes in.

We include it here without proof.

Theorem 2.2. Suppose the point set α satisfies the condition (CI). Then every real 2π-periodic continuous function $f \in C_{2\pi}$ can be approximated by a trigonometric rational function $r_n(\theta) = R_n(e^{i\theta})$ with $R_n \in \mathcal{R}_{n-1}$ such that for n large enough there is some constant K_1 such that

$$\sup_{[-\pi,\pi]} |f(\theta) - r_n(\theta)| \leq K_1 \omega(f; \frac{\pi}{n})$$

where $\omega(f;\delta)$ denotes the modulus of continuity for f.

Obviously the function $r_n(\theta)$ in this theorem is of the form $r_n = T_{n-1}(\theta)/|\pi_{n-1}(e^{i\theta})|^2$ with T_{n-1} a trigonometric polynomial of degree at most $n-1$.

According to this theorem we can find a function $G_n \in \mathcal{R}_{n-1}$ such that $g_n(\theta) := G_n(e^{i\theta})$ satisfies

$$\left| \frac{1}{w(\theta)} - g_n(\theta) \right| \leq K_2 \omega\Big(\frac{1}{w}; \frac{\pi}{n}\Big). \tag{2.1}$$

This function $g_n(\theta)$ is uniformly bounded and positive if w is uniformly bounded and positive. This is proved in the following lemma.

Lemma 2.3. Assume that the conditions (AC), (LD), and (CI) hold and that g_n is a function as characterized by the Jackson III theorem 2.2. Then there exist positive constants g, G, and n_0 such that for $n \geq n_0$

$$0 < g \leq g_n(\theta) \leq G < \infty.$$

The constants g and G do not depend on n or θ.

Proof. First we prove that $g_n(\theta)$ is uniformly bounded from above. From (2.1) we get

$$\left| \frac{1}{w(\theta)} - g_n(\theta) \right| \leq K_2 \sup_{\delta < \frac{\pi}{n}} \left| \frac{1}{w(\theta+\delta)} - \frac{1}{w(\theta)} \right| \leq 2K_2 m^{-2}.$$

This gives an upper bound for g_n since

$$|g_n(\theta)| \leq 2K_2 m^{-2} + m^{-1} = G.$$

Next we prove that there is a constant g such that $g_n(\theta) \geq g > 0$ if n is large enough. Suppose this is not true, then for every ϵ_1, such that $0 < \epsilon_1 < 1/M$, there exists a θ_1 so that $g_n(\theta_1) < \epsilon_1$. From (2.1) we find for $\theta = \theta_1$

$$\frac{1}{w(\theta_1)} < \epsilon_1 + K_2 \sup_{|\delta|<\pi/n} \left| \frac{1}{w(\theta+\delta)} - \frac{1}{w(\theta)} \right| \leq \epsilon_1 + \frac{K_2 L}{m^2} |\log(\pi/n)|^{-1-\lambda}.$$

The left-hand side is larger than $\frac{1}{M}$. So we get for $n > \pi$

$$\frac{m^2(1-\epsilon)}{MK_2L} < \left(\log \frac{n}{\pi}\right)^{-1-\lambda}, \quad \epsilon = M\epsilon_1.$$

This yields

$$\log \frac{n}{\pi} < \left(\frac{MK_2L}{m^2(1-\epsilon)}\right)^{\frac{1}{1+\lambda}}.$$

Herefrom we get

$$n < \pi e^{\left(\frac{MK_2L}{m^2(1-\epsilon)}\right)^{\frac{1}{1+\lambda}}}.$$

This gives a contradiction since the right-hand side is bounded and we can take n larger than that. So there must be some g such that $g_n(\theta) \geq g > 0$. □

Next we show how good the approximation of $w(\theta)$ by $1/g_n(\theta)$ is.

Lemma 2.4. Suppose the conditions (AC), (LD) and (IC) hold. Then there is a constant K_3 depending only on m, M, λ and L so that

$$|w(\theta) - g_n^{-1}(\theta)| < K_3(\log n)^{-1-\lambda}.$$

Proof. First note that

$$\left|\frac{1}{w(\theta)} - g_n(\theta)\right| = \left|\frac{w(\theta) - g_n^{-1}(\theta)}{w(\theta)g_n^{-1}(\theta)}\right| \geq \frac{|w(\theta) - g_n^{-1}(\theta)|}{\max_\theta |w(\theta)| \max_\theta |g_n^{-1}(\theta)|}.$$

Because the denumerator on the right-hand side is bounded, this inequality and (2.1) imply that there exists a positive constant K_4 such that

$$\left|w(\theta) - \frac{1}{g_n(\theta)}\right| \leq K_4\, \omega\Big(\frac{1}{w}; \frac{\pi}{n}\Big) = K_4 \sup_{|\theta_1-\theta_2|<\frac{\pi}{n}} \left|\frac{1}{w(\theta_2)} - \frac{1}{w(\theta_1)}\right|.$$

From Lemma 2.1 we get

$$|w(\theta) - g_n^{-1}(\theta)| \leq K_4 K \sup_{|\theta_1-\theta_2|<\frac{\pi}{n}} |\log|\theta_1 - \theta_2||^{-1-\lambda}.$$

If n is large and if we put $K_5 = K_4 K$, then

$$|w(\theta) - g_n^{-1}(\theta)| \leq K_5 |\log(\pi/n)|^{-1-\lambda} = K_5\, (\log(n/\pi))^{-1-\lambda} \leq K_3(\log n)^{-1-\lambda}.$$

Where $K_3 = 2^{1+\lambda}K_5$. The bound $\log(n/\pi) \geq (\log n)/2$ holds for $n > \pi^2$. Thus the proof is complete. □

Let us now introduce the spectral factors

$$S(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log w(\theta) \mathrm{d}\theta\right\} \quad \text{and} \quad H_n(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log g_n(\theta) \mathrm{d}\theta\right\}.$$

Their boundary functions are denoted as $s(\theta) = S(e^{i\theta})$ and $h_n(\theta) = H_n(e^{i\theta})$. So $w(\theta) = |s(\theta)|^2$ and $g_n(\theta) = |h_n(\theta)|^2$.

We now prove that $1/H_n(z)$ converges to $S(z)$ uniformly in $|z| \leq 1$. More precisely, we prove

Lemma 2.5. Assume that (AC), (LD), and (CI) hold. With the notation just introduced we have

$$|S(z) - 1/H_n(z)| < Q(\log n)^{-\lambda}$$

uniformly in $|z| \leq 1$. The constant Q depends on L, λ, as well as on m and M, but not on n or z.

Proof. It is sufficient to give the proof for $z \in \mathbb{T}$. So we switch to the notation $h_n(\theta)$ and $s(\theta)$.

We first prove

$$\Big||s(\theta)| - |h_n^{-1}(\theta)|\Big| = O(\log n)^{-1-\lambda}, \quad n \to \infty. \tag{2.2}$$

Noting that $w = |s|^2$ and $g_n = |h_n|^2$, we have

$$\Big||s(\theta)| - |h_n^{-1}(\theta)|\Big| = \Big|w(\theta) - g_n^{-1}(\theta)\Big| \Big/ \Big||s(\theta)| + |h_n^{-1}(\theta)|\Big|.$$

The numerator is $O(\log n)^{-1-\lambda}$ by Lemma 2.4 and the denominator is $O(1)$ because by (LD) and Lemma 2.3, there are constants m_2 and M_2 such that $0 < m_2 \leq ||s(\theta)| + |h_n^{-1}(\theta)|| \leq M_2 < \infty$. This proves the convergence of the moduli as in (2.2).

It remains to show the convergence of the arguments. Therefore we need a bound for

$$\operatorname{sgn} h_n(\theta)\{\operatorname{sgn} s(\theta) - \operatorname{sgn}[h_n(\theta)]^{-1}\} = \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \log[w(\omega)g_n(\omega)] \cot\frac{\theta - \omega}{2} \mathrm{d}\omega\right\} - 1.$$

In other words, we should bound

$$\int_{-\pi}^{\pi} \log[w(\omega)g_n(\omega)] \cot\frac{\theta - \omega}{2} \mathrm{d}\omega.$$

We split this into two terms

$$\int_{E_n} \log\frac{w(\omega)g_n(\omega)}{w(\theta)g_n(\theta)} \cot\frac{\theta - \omega}{2} \mathrm{d}\omega + \int_{E_n'} \log[w(\omega)g_n(\omega)] \cot\frac{\theta - \omega}{2} \mathrm{d}\omega.$$

Here E_n is the subset of $[-\pi, \pi]$ where $|\theta - \omega| < n^{-1}$ and $E_n' = [-\pi, \pi] \setminus E_n$. Because $|w(\theta) - [g_n(\theta)]^{-1}| = O(\log n)^{-1-\lambda}$ and the fact that both w and g_n are bounded away from 0 and ∞, we get $|\log[w(\theta)g_n(\theta)]| = O(\log n)^{-1-\lambda}$. Hence, we have for the second integral

$$\left|\int_{E_n'} \log[w(\omega)g_n(\omega)] \cot\frac{\theta - \omega}{2} \mathrm{d}\omega\right| \leq O(\log n)^{-1-\lambda} \int_{E_n'} \left|\cot\frac{\theta - \omega}{2}\right| \mathrm{d}\omega = O(\log n)^{-\lambda}.$$

The first integral over E_n contains a singularity at $\omega = \theta$. Here we use that w satisfies the (LD) condition $|w(\omega) - w(\theta)| \leq L|\log|\omega - \theta||^{-1-\lambda}$, and therefore, for $|\omega - \theta|$ small, also $|\log(w(\omega)/w(\theta))|$ has a bound of the order $O(|\log|\omega - \theta||^{-1-\lambda})$ as well. Because of Lemma 2.1, the same argument can be used for bounding $|\log(g_n(\omega)/g_n(\theta))|$ by the same bound. Thus the first integral is bounded by

$$O(1)\int_{E_n} |\log|\theta - \omega||^{-1-\lambda} |\theta - \omega|^{-1} \mathrm{d}\omega = O(\log n)^{-\lambda}.$$

This concludes the proof. □

3. Two systems of orthogonal rational functions

We now have an approximation $w_n = 1/g_n$ for w. We consider the orthogonal rational functions (**ORF**) ϕ_k for w and the **ORF** ψ_{nk} for w_n. We then derive the asymptotics for ϕ_n from the asymptotics for $\psi_n = \psi_{nn}$.

So, we suppose that we have an approximant $w_n = 1/g_n$ as described in the previous section. Assume that the spectral factors of w and g_n are denoted as before by $S(z)$ and $H_n(z)$, and that we set $s(\theta) := S(e^{i\theta})$ and $h_n(\theta) := H_n(e^{i\theta})$. Thus

$$w_n(\theta) = \frac{1}{H_n(e^{i\theta})H_{n*}(e^{i\theta})} = \frac{1}{|H_n(e^{i\theta})|^2} = \frac{1}{|h_n(\theta)|^2}. \tag{3.1}$$

For ease of the index notation below, we shall assume that $g_n \in \mathcal{R}_{n-1}$ (*not in* $\mathcal{R}_n$), so that the outer spectral factor H_n has the form $H_n(z) = \frac{q_{n-1}(z)}{\pi_{n-1}(z)} \in \mathcal{L}_{n-1}$. Assume that we denote by $\psi_{nk} \in \mathcal{L}_n$ the kth orthogonal rational function for the weight $w_n(\theta)$. To simplify the notation, we shall denote ψ_{nn} as ψ_n.

We first give an explicit expression for the functions $\psi_{nk}(z)$.

Theorem 3.1. Suppose $H_n \in \mathcal{L}_{n-1}$ is given and $w_n(\theta) = 1/|h_n(\theta)|^2$ with $h_n(\theta) = H_n(e^{i\theta})$ as defined above. Then the kth orthonormal rational function ψ_{nk}, orthogonal with respect to the weight function w_n is given by

$$\psi_{nk}(z) = \rho_{nk} z_k \frac{\sqrt{1-|\alpha_k|^2}}{1-\bar{\alpha}_k z} z B_{k-1}(z) H_{n*}(z), \quad k \geq n.$$

Recall that B_k denotes the kth Blaschke product, and that $z_k = \bar{\alpha}_k/|\alpha_k|$ if $\alpha_k \neq 0$ and $z_k = 1$ otherwise. The constant $\rho_{nk} \in \mathbb{T}$ is for normalization. It is chosen to be $H_n(\alpha_k)/|H_n(\alpha_k)|$.

Proof. First, it is clear that $\psi_{nk} \in \mathcal{L}_k$. We have to show that $\psi_{nk} \perp \mathcal{L}_{k-1}$ and that $\|\psi_{nk}\|_{w_n} = 1$.

Suppose $l < k$. Any function $f_l \in \mathcal{L}_l$ can be written as $f_l(z) = p_l(z)/\pi_l(z)$, with $p_l \in \Pi_l$. Furthermore, we set $p_n^*(z) = z^n p_{n*}(z)$ if $p_n \in \Pi_n$ and we assume that $H_n = q_{n-1}/\pi_{n-1}$. Also recall the definition of $z_k = \bar{\alpha}_k/|\alpha_k|$ if $\alpha_k \neq 0$ and $z_k = 1$ otherwise. We set $\eta_k = \prod_{i=1}^k z_i$.

- According to Cauchy's theorem, we get ($t = e^{i\theta}$)

$$\begin{aligned}
\langle \psi_{nk}, f_l \rangle_{w_n} &= \rho_{nk} z_k \frac{\sqrt{1-|\alpha_k|^2}}{2\pi} \int_{-\pi}^{\pi} H_{n*}(t) \frac{tB_{k-1}(t)}{1-\bar{\alpha}_k t} \frac{p_l^*(t)}{\pi_l^*(t)} \frac{\mathrm{d}\theta}{H_n(t)H_{n*}(t)} \\
&= \rho_{nk}\eta_k \frac{\sqrt{1-|\alpha_k|^2}}{2\pi} \int_{-\pi}^{\pi} t \frac{\prod_{i=l+1}^{k-1}(t-\alpha_i)}{\prod_{i=n}^{k}(1-\bar{\alpha}_i t)} \frac{p_l^*(t)}{q_{n-1}(t)} \mathrm{d}\theta \\
&= \rho_{nk}\eta_k \sqrt{1-|\alpha_k|^2} \frac{1}{2\pi i} \oint_{|t|=1} t \frac{\prod_{i=l+1}^{k-1}(t-\alpha_i)}{\prod_{i=n}^{k}(1-\bar{\alpha}_i t)} \frac{p_l^*(t)}{q_{n-1}(t)} \frac{\mathrm{d}t}{t} = 0.
\end{aligned}$$

- We now show that ψ_{nk} is normalized.

$$\begin{aligned}
\langle \psi_{nk}, \psi_{nk} \rangle_{w_n} &= \frac{1-|\alpha_k|^2}{2\pi} \int_{-\pi}^{\pi} H_{n*}(t) \frac{tB_{k-1}(t)}{1-\bar{\alpha}_k t} H_n(z) \frac{1}{(t-\alpha_k)B_{k-1}(t)} \frac{\mathrm{d}\theta}{H_n(t)H_{n*}(t)} \\
&= (1-|\alpha_k|^2) \frac{1}{2\pi i} \oint_{|t|=1} \frac{t}{(1-\bar{\alpha}_k t)(t-\alpha_k)} \frac{\mathrm{d}t}{t} = \frac{1-|\alpha_k|^2}{1-|\alpha_k|^2} = 1.
\end{aligned}$$

□

We have now the **ORF** $\{\phi_n\}$ with respect to the weight w and we have the **ORF** $\{\psi_n = \psi_{nn}\}$ with respect to the weight w_n. Our next step is to compare ψ_n and ϕ_n.

We introduce the notation $\kappa_n = \overline{\phi_n^*(\alpha_n)}$ and $\kappa_n' = \overline{\psi_n^*(\alpha_n)}$ for the leading coefficients in their expansions with respect to the basis $\{B_k\}$. We can find the explicit form of κ_n'

$$\overline{\kappa_n'} = \psi_n^*(\alpha_n) = \overline{\rho}_n \frac{H_n(\alpha_n)}{\sqrt{1-|\alpha_n|^2}} = \frac{|H_n(\alpha_n)|}{\sqrt{1-|\alpha_n|^2}} > 0. \tag{3.2}$$

Where $\rho_n = \rho_{nn}$. By the choice of ρ_{nk}, we find that $\kappa_n' > 0$. We now express ϕ_n in the basis $\{\psi_{nk}\}$.

Lemma 3.2. The **ORF** ϕ_n with respect to w is supposed to be normalized by making their leading coefficient κ_n positive. The **ORF** ψ_{nk} with respect to the weight w_n are normalized as above, i.e., such that their leading coefficient κ_n' is positive. Then ϕ_n can be expressed in terms of the ψ_{nk} by

$$\phi_n(z) = a_n\psi_n(z) + \sum_{k=0}^{n-1} a_{nk}\psi_{nk}(z) \tag{3.3}$$

where

$$a_n = \kappa_n\sqrt{1-|\alpha_n|^2}/|H_n(\alpha_n)| \tag{3.4}$$

and

$$a_{nk} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(t)\overline{\psi_k(t)}(w_n(\theta) - w(\theta))\mathrm{d}\theta, \quad k = 0, \ldots, n-1.$$

Proof. This is easy to work out since by orthogonality we have for $k = 0, \ldots, n-1$

$$\begin{aligned} a_{nk} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(t)\overline{\psi_{nk}(t)}w_n(\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(t)\overline{\psi_{nk}(t)}\left(w_n(\theta) - w(\theta)\right)\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(t)\overline{\psi_{nk}(t)}w(\theta)\mathrm{d}\theta. \end{aligned}$$

The last term vanishes because $\phi_n \perp_w \mathcal{L}_{n-1}$. So we get the expressions for a_{nk}, $k = 0, \ldots, n-1$. The form of a_n follows from $a_n = \kappa_n/\kappa_n'$ and (3.2). □

4. The asymptotics

Our strategy is now to develop the asymptotics for ψ_n, and then show that the asymptotics for ϕ_n and ψ_n are the same, thus we shall have to show that $\sum_{k=0}^{n-1} a_{nk}\psi_{nk} \to 0$ and $a_n \to 1$. Note that the expression for ψ_n contains H_{n*} and the asymptotics for the latter are already known (see Lemma 2.5).

We need to introduce first the reproducing kernels (or equivalently the Chistoffel functions) for $\mathcal{L}_n$. For the weight w, the reproducing kernels are $k_n(z, w) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)}$. The kernels $k_n(z, w)$ feature in the following optimization result that holds in any reproducing kernel Hilbert space.

Lemma 4.1. For a positive measure μ on $\mathbb{T}$, consider the rational function spaces $\mathcal{L}_n$ as subspaces of $L^2(\mathbb{T}, \mu)$, let $k_n(z, w)$ be the reproducing kernel for $\mathcal{L}_n$, then for fixed $w \in \mathbb{D} \cup \mathbb{T}$

$$\min_{f_n \in \mathcal{L}_n,\ f_n(w)=1} ||f_n||_\mu^2 = \frac{1}{k_n(w, w)}. \tag{4.1}$$

The minimum is reached for $f_n(z) = k_n(z, w)/k_n(w, w)$.

The next lemma is from [2, Theorem 9.6.4]. Under very mild conditions on α (which are satisfied when (CI) holds) and when $\log w \in L^1$, it gives the asymptotics for the reproducing kernels.

Lemma 4.2. Let $k_n(z,w)$ denote the reproducing kernel for $\mathcal{L}_n$ for the measure μ with $\log \mu' \in L^1$ and S the spectral factor for μ. Then if (CI) holds, we have for $z, w \in \mathbb{D}$

$$\lim_{n\to\infty} k_n(z,w) = \frac{1}{(1-\bar{w}z)S(z)\overline{S(w)}}.$$

This convergence is uniform for z and w in compact subsets of $\mathbb{D}$.

Moreover it holds that [2, Theorem 2.2.3] $k_n(z,\alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n,\alpha_n) = |\kappa_n|^2$. Thus, the previous lemma gives

$$\lim_{n\to\infty} |\kappa_n|^2(1-|\alpha_n|^2)|S(\alpha_n)|^2 = 1. \tag{4.2}$$

We are now ready to bound $|\kappa_n|^{-2}$.

Lemma 4.3. Under the conditions (AC), (LD), and (CI), assume that S is the outer spectral factor of w. Let $k_n(z,w)$ be the reproducing kernel for $\mathcal{L}_n$ and let $\phi_k(z) = \kappa_k B_k(z) + \cdots \in \mathcal{L}_k$ be the **ORF**. On the other hand, let H_n be the outer spectral factor of g_n, the approximant of w^{-1} as defined by the Jackson III theorem. Then

$$(1-|\alpha_n|^2)|S(\alpha_n)|^2 \le \frac{1}{|\kappa_n|^2} \le \frac{1-|\alpha_n|^2}{|H_n(\alpha_n)|^2}\left(1+O(\log n)^{-1-\lambda}\right). \tag{4.3}$$

Proof. Because $\mathcal{L}_n \subset \mathcal{L}_{n+1}$, the minimum of Lemma 4.1 does not increase if n increases

$$\frac{1}{k_n(w,w)} \ge \frac{1}{k_{n+1}(w,w)} \ge \ldots \ge (1-|w|^2)|S(w)|^2.$$

The last bound is by Lemma 4.2. If we put $w = \alpha_n$, we get the lower bound.

For the upper bound, we consider the **ORF** ψ_n for the weight function w_n. Let us define

$$f_n(z) := \frac{\psi_n^*(z)}{\psi_n^*(\alpha_n)} = \frac{\psi_n^*(z)}{\overline{\kappa_n'}} = \frac{(1-|\alpha_n|^2)H_n(z)}{(1-\bar{\alpha}_n z)H_n(\alpha_n)} \in \mathcal{L}_n. \tag{4.4}$$

This follows because of Theorem 3.1 and the fact that $H_n \in \mathcal{L}_{n-1}$. Because $f_n(z)$ is in $\mathcal{L}_n$ and satisfies $f_n(\alpha_n) = 1$, we find from (2.2), (4.1) and (4.4)

$$\begin{aligned}
\frac{1}{|\kappa_n|^2} &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\theta)|f_n(z)|^2 \mathrm{d}\theta \\
&= \frac{(1-|\alpha_n|^2)^2}{|H_n(\alpha_n)|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{S(t)H_n(t)}{t-\alpha_n}\right|^2 \mathrm{d}\theta, \quad t = e^{i\theta} \\
&= \frac{(1-|\alpha_n|^2)^2}{|H_n(\alpha_n)|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{|t-\alpha_n|^2}\left(1+O(\log n)^{-1-\lambda}\right) \\
&= \frac{(1-|\alpha_n|^2)^2}{|H_n(\alpha_n)|^2}\frac{1}{2\pi i}\oint_{|t|=1}\frac{t\mathrm{d}t}{(t-\alpha_n)(1-\bar{\alpha}_n t)t}\left(1+O(\log n)^{-1-\lambda}\right) \\
&= \frac{1-|\alpha_n|^2}{|H_n(\alpha_n)|^2}\left(1+O(\log n)^{-1-\lambda}\right).
\end{aligned}$$

So we find the result. □

It now follows that

Lemma 4.4. Let (CA), (LD), and (CI) hold, then the coefficient a_n from Lemma 3.2 satisfies

$$a_n = 1 + O(\log n)^{-1-\lambda}.$$

(Recall the normalization of the ϕ_n and the ψ_n.)

Proof. Because of (2.2) and (4.3)

$$|\kappa_n| = \frac{1}{|S(\alpha_n)|\sqrt{1-|\alpha_n|^2}} + O(\log n)^{-1-\lambda}.$$

From (3.4) we find

$$|a_n| = 1 + O(\log n)^{-1-\lambda}.$$

Because of the normalization of ϕ_n and ψ_n, this holds without the modulus bars. □

We now try to find a bound for the second term in (3.3). The idea is to find a uniform bound for $|\phi_n(t)|$. The remaining integral to be bounded is then of the form $\frac{1}{2\pi}\int_{-\pi}^{\pi}|k_{n-1}(w_n,t,z)|\mathrm{d}\theta$ where $t = e^{i\theta}$ and $k_{n-1}(w_n,t,z) = \sum_{i=0}^{n-1}\overline{\psi_{ni}(t)}\psi_{ni}(z)$ is the reproducing kernel for $\mathcal{L}_{n-1}$ with respect to the weight w_n.

We first search for an upper bound for $\psi_n(t)$ if $t = e^{i\theta} \in \mathbb{T}$.

$$\begin{aligned} |\psi_n(t)| = |\psi_n^*(t)| &= \sqrt{1-|\alpha_n|^2}\left|\frac{H_n(t)}{1-\bar{\alpha}_n t}\right| \\ &= \sqrt{1-|\alpha_n|^2}\left|\frac{h_n(\theta)}{t-\alpha_n}\right| \\ &\le \frac{\sqrt{1-|\alpha_n|^2}}{|t-\alpha_n|}\sqrt{G} = O(1). \end{aligned} \tag{4.5}$$

The last equality follows from the fact that we choose $\{\alpha_i\}$ to lie in a compact subset of $\mathbb{D}$ and that h_n is bounded on $\mathbb{T}$ because g_n is bounded, and because $g_n = |h_n|^2$ (see Lemma 2.3).

We now search for a bound for the integral in the second term.

Lemma 4.5. If we denote the kernel with respect to w_n as

$$k_n(w_n,t,z) = \sum_{i=0}^{n}\overline{\psi_{ni}(t)}\psi_{ni}(z)$$

and if the $\{\alpha_i\}$ lie in a compact subset of $\mathbb{D}$ so that $1-|\alpha_i| \ge d > 0$, and if there is a uniform constant G such that $0 < G^{-1} \le w_n$, then the following uniform upper bound is valid for $z \in \mathbb{D}\cup\mathbb{T}$

$$k_n(w_n,z,z) \le \frac{(n+1)G}{d^2}.$$

Proof. We know from Lemma 2.3 and (4.1) that for $z \in \mathbb{D}\cup\mathbb{T}$ fixed

$$\begin{aligned} \frac{1}{k_n(w_n,z,z)} &= \min_{f_n\in\mathcal{L}_n,\ f_n(z)=1} ||f_n||_{w_n}^2 \\ &= \min_{f_n\in\mathcal{L}_n,\ f_n(z)=1} \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_n(t)|^2 w_n(\theta)d\theta \\ &\ge \frac{1}{G}\min_{f_n\in\mathcal{L}_n,\ f_n(z)=1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f_n(t)|^2 d\theta. \end{aligned}$$

The latter is an optimization problem for the weight function $w \equiv 1$. The **ORF** for this weight function are (see Theorem 3.1)

$$\varphi_0 = 1, \quad \varphi_k(z) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\bar{\alpha}_k z} z B_{k-1}(z), \quad k \geq 1.$$

We removed the normalizing constants since they are irrelevant here. So for the kernel $k_n(1,t,t)$ we find

$$k_n(1,z,z) = 1 + \sum_{i=1}^{n} \frac{1-|\alpha_i|^2}{|1-\bar{\alpha}_i z|^2} |z|^2 |B_{i-1}(z)|^2 \leq 1 + \sum_{i=1}^{n} \frac{1-|\alpha_i|^2}{|1-\bar{\alpha}_i z|^2} \leq 1 + \frac{n}{d^2}.$$

Therefore

$$\frac{1}{k_n(w_n,z,z)} \geq \frac{1}{G k_n(1,z,z)} \geq \frac{1}{G(1+\frac{n}{d^2})} \geq \frac{d^2}{G(n+1)}.$$

So we proved the lemma. □

We can now prove the following lemma.

Lemma 4.6. Assume (AC), (LD) and (CI) hold. Then, with the notations as in the previous lemma, we can find the following upper bounds for the kernel $k_n(w_n,t,z)$.

$$|k_n(w_n,t,z)| \leq \frac{(n+1)G}{d^2}, \quad t,z \in \mathbb{D} \cup \mathbb{T}$$

and

$$|k_n(w_n,t,z)| \leq \frac{\pi G}{|\theta-\gamma|}, \quad t = e^{i\theta}, z = e^{i\gamma} \in \mathbb{T}.$$

Proof. For the first bound we use the Cauchy-Schwarz inequality and Lemma 4.5

$$|k_n(w_n,t,z)| \leq \sqrt{k_n(w_n,t,t) k_n(w_n,z,z)} \leq \frac{(n+1)G}{d^2}.$$

For the second bound we use the Christoffel-Darboux-relation (see [2, Theorem 3.1.3])

$$\begin{aligned} |k_n(w_n,t,z)| &= \left| \frac{\overline{\psi_n^*(t)}\psi_n^*(z) - \overline{\psi_n(t)}\psi_n(z)}{1-\overline{\zeta_n(t)}\zeta_n(z)} \right| \\ &\leq \frac{2\sup_{t\in\mathbb{T}}|\psi_n(t)| \sup_{z\in\mathbb{T}}|\psi_n(z)||t-\alpha_n||z-\alpha_n|}{|t-z|(1-|\alpha_n|^2)}. \end{aligned}$$

So we find from (4.5)

$$|k_n(w_n,t,z)| \leq \frac{2G}{|t-z|} = \frac{2G}{|1-e^{i(\gamma-\theta)}|} = \frac{G}{|\sin(\frac{\theta-\gamma}{2})|} \leq \frac{\pi G}{|\theta-\gamma|}.$$

This proves the lemma. □

Now we can derive an upper bound for the second part of (3.3). We shall prove below that $\phi_n(t)$ is uniformly bounded. The remaining integral can be bounded as follows.

Lemma 4.7. Assume that (AC), (LD) and (CI) hold. Then for the integral of the modulus of the kernel $k_{n-1}(w_n,t,z)$, we get the bound

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |k_{n-1}(w_n,t,z)| \mathrm{d}\theta \leq O(\log n), \quad z \in \mathbb{D} \cup \mathbb{T}.$$

Proof. Since $k_{n-1}(w_n, t, z)$ is analytic in $z \in \mathbb{D}$, it reaches its maximum modulus on the boundary. Thus if we have $\sup_{s\in\mathbb{T}} |k_{n-1}(w_n, t, s)| = |k_{n-1}(w_n, t, e^{i\gamma})|$, we can bound $\int_{-\pi}^{\pi} |k_{n-1}(w_n, t, z)| \mathrm{d}\theta \leq \int_{-\pi}^{\pi} |k_{n-1}(w_n, t, e^{i\gamma})| \mathrm{d}\theta$. Therefore, it is sufficient to prove the theorem for $z \in \mathbb{T}$. For $z \in \mathbb{T}$, we split the integral in two parts

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(w_n, t, z)| \mathrm{d}\theta = \frac{1}{2\pi} \int_{E_n} |k_n(w_n, t, z)| \mathrm{d}\theta + \frac{1}{2\pi} \int_{E_n'} |k_n(w_n, t, z)| \mathrm{d}\theta.$$

Where E_n refers to the part where $|\theta - \gamma| \leq n^{-1}$ and $E_n' = [-\pi, \pi] \setminus E_n$. For the first integral we use the first upper bound from Lemma 4.6 which gives an upper bound of the form $O(1)$. For the second we use the second upper bound from Lemma 4.6. This gives

$$O(1) \int_{E_n'} \frac{\mathrm{d}\theta}{|\theta - \gamma|} = O(\log n).$$

□

We now try to find a bound for $|\phi_n(z)|$ if $z \in \mathbb{D} \cup \mathbb{T}$.

Lemma 4.8. Suppose (AC), (LD) and (CI) hold. Then there is an absolute constant U such that $|\phi_n(z)| \leq U$ uniformly in n and $z \in \mathbb{D} \cup \mathbb{T}$.

Proof. Since ϕ_n has all its poles in $\mathbb{E}$, it is sufficient to prove this for $z = t \in \mathbb{T}$. Suppose

$$\max_{|t|=1} |\phi_n(t)| = U = U(n).$$

From Lemma 2.4 and Lemma 3.2, we find $(\tau = e^{i\theta})$

$$U \leq |a_n| \max_{|t|=1} |\psi_n(t)| + UK_3(\log n)^{-1-\lambda} \max_{|t|=1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{i=0}^{n-1} \overline{\psi_{ni}(\tau)} \psi_{ni}(t) \right| \mathrm{d}\theta. \tag{4.6}$$

Using (4.5) and Lemma 4.7, we can bound (4.6) as

$$U \leq O(1) + UO(\log n)^{-1-\lambda}(\log n). \tag{4.7}$$

This implies $U = O(1)$. □

5. Proof of the main theorem

We now have all the ingredients to prove our main result

Proof of Theorem 1.1. We write the superstar conjugate of (3.3) as

$$\phi_n^*(z) = a_n \psi_n^* + B_n(z) E_{n*}(z), \quad E_n(z) = \sum_{k=0}^{n-1} a_{nk} \psi_{nk}(z).$$

Recall that $a_n > 0$ by the normalization of the **ORF**. Substitute the expression for $\psi_n = \psi_{nn}$ from Theorem 3.1 and multiply with $\rho_n(1 - \overline{\alpha}_n z)/\sqrt{1 - |\alpha_n|^2}$ to get

$$\rho_n \frac{1 - \overline{\alpha}_n z}{\sqrt{1 - |\alpha_n|^2}} \phi_n^*(z) = a_n H_n(z) + \rho_n \frac{1 - \overline{\alpha}_n z}{\sqrt{1 - |\alpha_n|^2}} B_n(z) E_{n*}(z).$$

Let us call the second term in this expression $r_n(z)$. Because of the (CI) condition and because of Lemmas 4.7 and 4.8 we find that $r_n(z) = O(\log n)^{-\lambda}$.

Furthermore, we know that $a_n = 1 + O(\log n)^{-1-\lambda}$ by Lemma 4.4 and by Lemma 2.5, $H_n(z) = S(z) + O(\log n)^{-\lambda}$ uniformly in $\mathbb{D} \cup \mathbb{T}$. Thus

$$\rho_n \frac{1 - \overline{\alpha}_n z}{\sqrt{1 - |\alpha_n|^2}} \phi_n^*(z) = (1 + O(\log n)^{-1-\lambda})(1/S(z) + O(\log n)^{-\lambda}) + \varepsilon_n(z)$$
$$= 1/S(z) + \varepsilon_n(z), \quad \varepsilon_n(z) = O(\log n)^{-\lambda}.$$

The unimodular constants ρ_n were defined as $H_n(\alpha_n)/|H_n(\alpha_n)|$, which can not be defined in terms of ϕ_n. However, since we know that a sequence of unimodular constants exists, we can as well take any other sequence of unimodular constants, as long as the left-hand side converges to a positive constant for $z = 0$ because that is how $S(z)$ is normalized: $S(0) > 0$. Thus we can choose $\rho_n = |\phi_n^*(0)|/\phi_n^*(0)$. This proves the theorem, since the second formula for the modulus square is trivial.

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