Sensitivity to infinitesimal delays in neutral equations

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Abstract

In this paper we investigate the sensitivity of the stability of neutral functional differential equations with respect to changes in the delays. This sensitivity is caused by the behaviour of the essential spectrum which, in turn, is determined by the roots of an exponential polynomial. In [1], Avellar and Hale considered the case of multiple fixed and nonzero delays. In a first part of this paper we visualize and interpret their results by means of computed spectral plots. In a second part we extend the theory of [1] to the important limit case of arbitrarily small delays and show that this can lead to eigenvalues with arbitrary large positive real part. Necessary and sufficient conditions are provided. We conclude with two illustrative examples. The first of these is the analysis of the robustness of a boundary controlled partial differential equation in the presence of small feedback delays.

Keywords : neutral equation, sensitivity, boundary control
AMS(MOS) Classification : 34K40,34K35

1 Introduction

In this paper we study the behaviour of the zeros of the exponential polynomial

$$H(\lambda) \triangleq 1 - \sum_{j=1}^{N} a_j e^{-\lambda\tau_j}, \quad \tau_j \in \mathbb{R}^+, \quad a_j \in \mathbb{R}, \quad j = 1, \dots, N \tag{1}$$

in the complex plane. $H(\lambda) = 0$ is the characteristic equation of the functional difference equation $x(t) = \sum_{j=1}^{N} a_j x(t-\tau_j)$ which determines the essential spectrum of the solution operator of the neutral functional differential equation (NFDE),

$$\frac{d}{dt}\left(x(t) - \sum_{j=1}^{N} a_j x(t-\tau_j)\right) = b_0 x(t) - \sum_{j=1}^{N} b_j x(t-\tau_j)$$

for which it was proven that the sensitivity to infinitesimal changes of the delays is caused by the behaviour of the essential spectrum (see e.g. [7]). More specifically, it was shown that the smallest upper bound $c = \sup\{\Re(\lambda) : H(\lambda) = 0\}$ is not continuous w.r.t. the delays τ_j, $j = 1, \dots, N$. Consequently it is possible that arbitrary small changes in the delays destabilize the system [1, 7].

NFDEs arise for example in models of distributed networks [10, 9], combustion [13] and the control of structures through delayed forcing depending on the acceleration [2].

The lack of robustness w.r.t. small changes in the delays is also observed for boundary controlled hyperbolic partial differential equations [7, 6, 5, 4, 11, 12]: small delays in the application of the boundary forces, which are inevitable in practise due to measurement delays, AD-DA conversion, ..., can lead to instability of the stable undelayed system and therefore it is of extreme importance to include all possible delays in the model. In [11, 12] frequency domain techniques are used to analyse the problem: the system is rewritten as an input-output mapping with delayed unity feedback and conditions are formulated on the open loop transfer function for robustness w.r.t. the delays. For a class of these problems the characteristic equation (of the closed loop system) corresponds to the characteristic equation of a NFDE or is of the form (1).

In [1] a theoretical framework is developed for the analysis of the solutions of (1) and their dependency on the delays. This theory starts from the assumption that all the delays are fixed and nonzero. In §2 we repeat the important results of [1]. In §3 we visualize and interpret these results by means of plots of the solutions. We show that the sensitivity of c to arbitrary small changes of the delays is caused by zeros with large imaginary part. In §4 we treat the new limit case of vanishing delays. We show that this may lead to solutions with real part going to $+\infty$ and we prove necessary and sufficient conditions. We conclude in §6 with two illustrative examples of the theory developed throughout the paper.

2 Analysis with fixed delays

In this section we describe the main results of [1].

2.1 Definitions and notation

We study the zeros of exponential polynomials of the form,

$$H(\lambda) \triangleq 1 - \sum_{j=1}^{N} a_j e^{-\lambda \tau_j}, \tag{2}$$

where the delays τ_j are fixed and satisfy $0 < \tau_1 < \tau_2 < \ldots < \tau_N$.

Define the collection of the real parts of all the solution of (2) as Z,

$$Z = \{\Re(\lambda) : H(\lambda) = 0\},$$

and denote its closure by $\bar{Z}$. The smallest upper bound of $\bar{Z}$, which is important for stability considerations, is

$$c = \sup\{\Re(\lambda) : H(\lambda) = 0\}.$$

Assume that the N delays τ_j, $j = 1, \ldots, N$, depend on $M \leq N$ so called independent delays $r_1, \ldots, r_M$:

$$\tau_j = \sum_{k=1}^{M} \gamma_{j,k} r_k = \gamma_j \cdot r \tag{3}$$

whereby $\gamma_j = (\gamma_{j,1}, \ldots, \gamma_{j,M}) \in \mathbb{N}^M$ are nonzero vectors with positive integer coefficients and $r \in \mathbb{R}^M$ with $r > 0$. Dependency of the kind (3) often appears in difference equations arising from practical applications, as for example in (delayed) boundary controlled wave equations (see §6). The same holds when dealing with vector valued difference equations. Indeed, the characteristic equation of

$$x(t) = \textstyle\sum_{k=1}^{M} A_k x(t - r_k) \quad x \in \mathbb{R}^n, \quad A_k \in \mathbb{R}^{n \times n}$$

is given by

$$\det\left(I - \sum_{k=1}^{M} A_k e^{-\lambda r_k}\right) = 0$$

which is seen, using an explicit formula for the determinant, to be an exponential polynomial with dependent delays.

2.2 Rationally dependent and rationally independent delays

The numbers $r_1, r_2, \ldots, r_M$ are rationally independent if and only if,

$$\sum_{k=1}^{M} n_k r_k = 0, \quad n_k \in \mathbb{N}$$

implies $n_k = 0, \quad k = 1, \ldots, M$. For example two numbers are rationally independent if their ratio is irrational.

In order to provide useful characterizations of $\bar{Z}$ we need the Hausdorff metric which is defined as follows: for any two sets E and $F \subset \mathbb{R}$ and any $\rho \in \mathbb{R}$, let

$$\begin{aligned} d(\rho, E) &= \inf_{t \in E} |\rho - t|, \\ \delta(E, F) &= \sup_{\rho \in E} d(\rho, F) \text{ and} \\ D(E, F) &= \max\{\delta(E, F), \delta(F, E)\}. \end{aligned}$$

The number $D(E, F)$ is the Hausdorff distance between the sets E and F.

Theorem 2.1 *If the components of r are rationally independent then the following statements are equivalent:*

$$\alpha \in \bar{Z}$$
$$\Updownarrow$$
$$\exists \theta = (\theta_1, \ldots, \theta_M) \text{ with } \theta_k \in [0, 2\pi],\ k = 1 \ldots M, \text{ such that}$$
$$1 - \sum_{j=1}^{N} a_j e^{-\alpha \gamma_j \cdot r} e^{-i \gamma_j \cdot \theta} = 0$$

Corollary 2.1 *$\bar{Z}$ is the union of a finite number of intervals.*

This means that the real parts of all the eigenvalues fill up these intervals in a dense way.

Corollary 2.2 *$\bar{Z}(r)$ is continuous in the Hausdorff metric w.r.t. the delays r when they are rationally independent.*

This corollary is important because it implies the continuity of the supremum $c(r)$ of $\bar{Z}(r)$ w.r.t. r. In many examples in this paper it will be shown that $\bar{Z}$ is not continuous w.r.t. the delays when one allows both rationally independent and rationally dependent delays. However in that case the following theorem holds:

Theorem 2.2 *$\bar{Z}(r)$ is lower semicontinuous in r, that is, for each r_0,*

$$\lim_{r \to r_0} \delta(\bar{Z}(r_0), \bar{Z}(r)) = 0.$$

The combination of corollary 2.2 and theorem 2.2 is very important for control problems because the rightmost eigenvalues determine stability: suppose for example that r_0 is given with rationally dependent components and denote the maximum of $\bar{Z}(r_0)$ by $c(r_0)$. On the other hand, consider a sequence of rationally independent delays $\{r_n\}_{n \geq 1}$ with limit r_0 and denote by $c(r_n)$ the maximum of $\bar{Z}(r_n)$. Then from corollary 2.2 and theorem 2.2 it follows that

$$c(r_0) \leq \lim_{n \to \infty} c(r_n).$$

In other words, the supremum of $\bar{Z}$ is always higher when one considers the given delays as independent. This means that when the delays in the characteristic equation modelling a physical system are results of independent phenomena (for example independent measurements) one has always to consider the delays as rationally independent in order to obtain a reliable upper bound on the real parts of the spectrum. In §3 it will be explained what happens with the individual eigenvalues when one deals with rationally independent delays close to rationally dependent delays.

2.3 Special cases

Fully independent delays This corresponds to the case where $M = N$, $\gamma_j = e_j$, the j-th unity vector in $\mathbb{R}^N$ and the delays $\tau_1, \tau_2, \ldots, \tau_N$ are rationally independent. Theorem 2.1 can be rewritten as:

Theorem 2.3 *When the delays are rationally independent* $c = \sup\{\Re(\lambda) : H(\lambda) = 0\}$ *satisfies*

$$1 - \sum_{j=1}^{N} |a_j| e^{-c\tau_j} = 0. \tag{4}$$

The solution c of (4) also serves as a (non-strict) upper bound in the case of rationally dependent delays.

Commensurate delays This is the case when $M = 1$. Thus delays $\tau_1, \ldots, \tau_n$ are commensurate if and only if there exists a real number r such that $\tau_j = n_j r$ with $n_j \in \mathbb{N}$, $j = 1, \ldots, N$. I.e. all the delays are integer multiples of a same number. In this case equation (2) can be rewritten as a polynomial in $e^{-\lambda r}$. As a consequence $\bar{Z}(r)$ consists of a finite numbers of points and the spectrum is periodic with period $\frac{2\pi}{r} i$.

3 Visualization and interpretation

In the previous section the delays r were considered fixed. When one approaches rationally dependent delays r_0 with independent delays, the supremum of the real parts of the solutions can have a discontinuity at $r = r_0$. First, we illustrate how this discontinuity is compatible with the continuous movement of individual eigenvalues as r approaches r_0. Secondly, we discuss the consequences for control applications.

3.1 Non-uniform convergence

Consider, as an example, the characteristic equation

$$H(\lambda, h) \triangleq 1 + 1.1e^{-\lambda} + 0.2e^{-\lambda(2+h)} = 0, \tag{5}$$

with delays 1 and $2 + h$. When h is zero, (5) is a quadratic equation in $e^{-\lambda}$ and the eigenvalues are: $\lambda \approx -1.4704 + i(2l+1)\pi,\ l \in \mathbb{Z}$, and $\lambda \approx -0.1391 + i(2l+1)\pi,\ l \in \mathbb{Z}$. When $h > 0$ and irrational (and therefore the two delays rationally independent), the supremum $c(h) = \sup\{\Re(\lambda) \mid H(\lambda) = 0\}$ satisfies:

$$1 - 1.1e^{-c(h)} - 0.2e^{-c(h)(2+h)} = 0, \tag{6}$$

which yields $\lim_{h\to 0} c(h) \approx 0.2302 > c(0) \approx -0.1391$. Hence $c(h)$, and the corresponding stability of the associated essential spectrum, changes discontinuously with respect to h. Individual (single) eigenvalues however move continuously with respect to the delays. From

$$1 + 1.1e^{-\lambda} + 0.2e^{-\lambda(2+h)} = 0,$$

one derives

$$\frac{d\lambda}{dh} = \frac{-0.2\lambda e^{-\lambda(2+h)}}{1.1e^{-\lambda} + 0.2(2+h)e^{-\lambda(2+h)}}.$$

But this 'sensitivity' of the individual eigenvalues increases to infinity as their modulus $|\lambda| \to \infty$. Figure 1 shows part of the spectrum of (5) on two different scales when $h = 0.01$. When h is reduced to zero, the spectrum converges *point-wise* and non-uniformly to the limit case $h = 0$, as shown in figure 2.

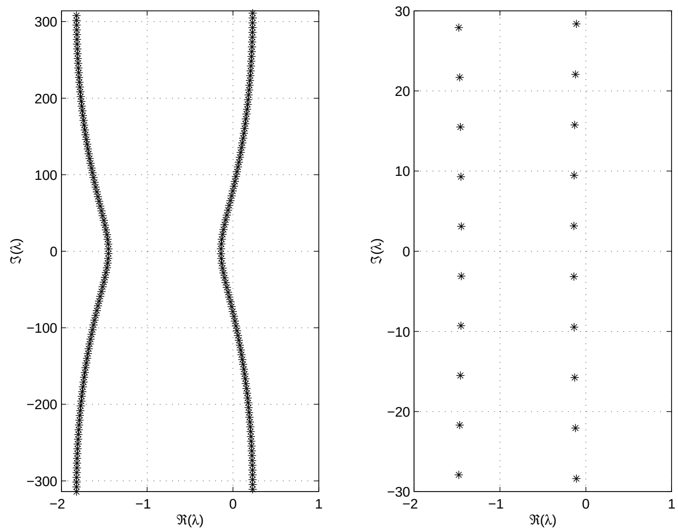

Figure 1: Part of the spectrum of (5) on two different scales for $h = 0.01$.

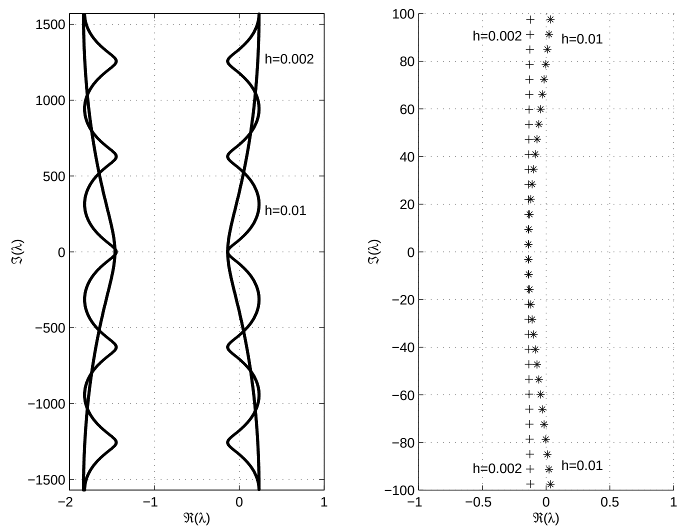

Figure 2: Part of the spectrum of (5) for $h = 0.01$ and $h = 0.002$.

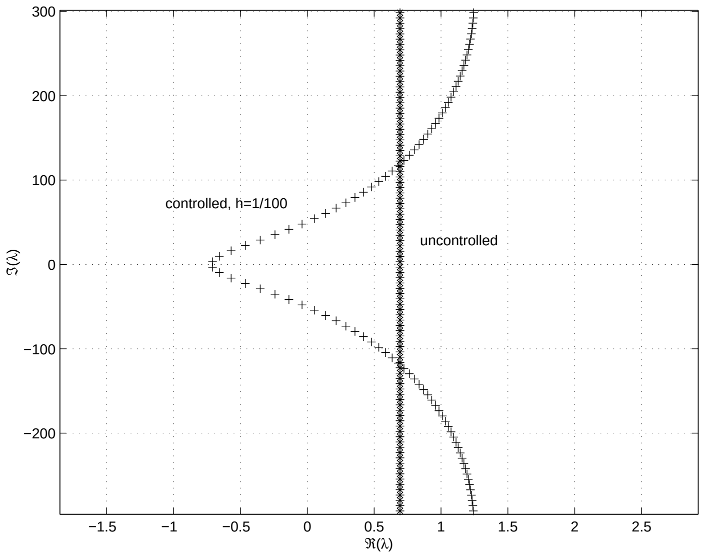

Figure 3: Part of the spectrum of the uncontrolled system (8) and the controlled system (9) for $h = 0.01$.

3.2 Unstable difference equations can not be stabilized

Consider the neutral functional differential equation

$$\frac{d}{dt}(x(t) + 2x(t-1)) = ax(t) + bx(t-\tau) + u(t), \tag{7}$$

obtained from linearizing a control system with input $u(t)$. When $u(t) \equiv 0$ the difference equation

$$x(t) + 2x(t-1) = 0 \tag{8}$$

determines the essential spectrum of the semigroup associated with (7). The zero solution of (8) is clearly unstable: all eigenvalues have real part $\log(2)$.

When applying the velocity feedback $u(t) = \frac{3}{2}\dot{x}(t-1-h)$, the difference equation is modified to

$$x(t) + 2x(t-1) - \frac{3}{2}x(t-1-h) = 0 \tag{9}$$

where h models the estimation error of the delay. For $h = 0$ the difference equation is clearly stabilized: all eigenvalues have as real part $-\log(2)$. However for irrational h the supremum $c(h)$ of the real parts of the spectrum can be calculated from

$$1 - 2e^{-c(h)} - \frac{3}{2}e^{-c(h)(1+h)} = 0$$

From which follows $\lim_{h\to 0} c(h) = \log(3.5) > \log(2)$. Thus the feedback destabilizes the original system even more. This is shown in figure 3.

Because sensitivity to small changes of the delays is caused by roots of (2) with large modulus and because the set of the real parts of the roots of (2) is contained in a finite number of intervals, such roots have large imaginary part. Thus sensitivity to

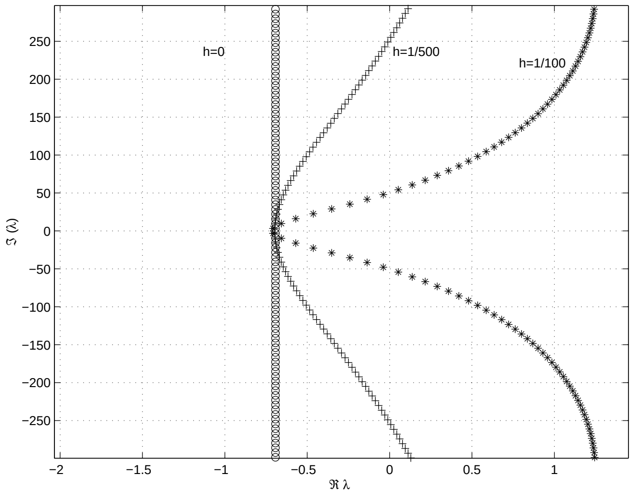

Figure 4: When $h \to 0$, the spectrum of (9) converges pointwise to the spectrum of $x(t) + \frac{1}{2}x(t-1) = 0$.

infinitesimal changes in the delays is caused by modes of very high frequency. This is shown in figure 4. Note, that the control of (7) works for low frequency modes while it does not for high frequency modes (see figure 3). We remark that the question arises whether the model used is a valid description of the modelled reality for such frequencies. In reality one (usually) expects larger damping for larger frequencies. Whether this damping occurs strong and soon enough depends on the particular application. In section 4 we will see that our generalisation leads to situations where sensitivity is *not necessarily* caused by high-frequency modes.

4 Vanishing delays

The analysis in §2 is valid under the assumption that all the delays are fixed, different and nonzero. These assumptions can be relaxed to the requirements that firstly the smallest delays are not arbitrary close to zero and secondly that the largest delays are not arbitrary close to each other. In this section we explicitly deal with these limit cases. We show that vanishing delays can give rise to solutions with unbounded positive real parts, and, that, in a similar way, coinciding largest delays can give rise to solutions with unbounded negative real part. Since the latter is of less importance for applications we only briefly mention the occurrence of this phenomenon.

4.1 Introductory example

As an introductory example we investigate the zeros of,

$$H(\lambda, h) = 1 + 2e^{-\lambda h} - \frac{1}{2}e^{-\lambda}, \tag{10}$$

as $h \to 0+$.

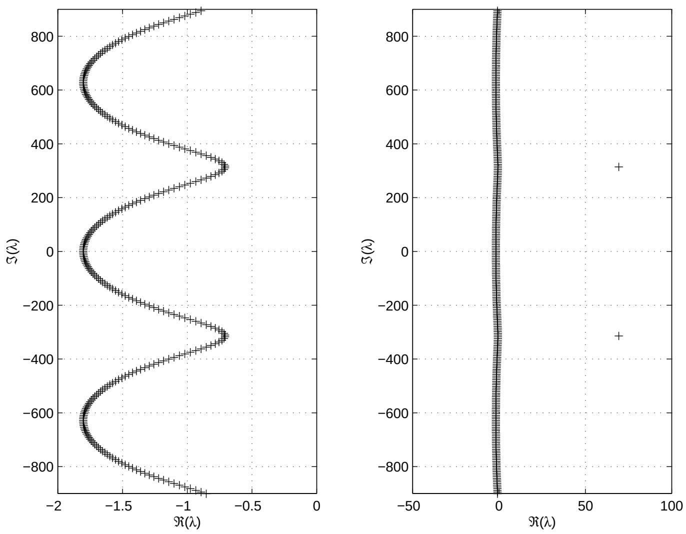

Figure 5: Zeros of (10) for $h = 0.01$ in two different regions of the complex plane.

If we set h to 0 in (10) all zeros of $H(\lambda, 0)$ are of the form $\lambda = -\log(6) + i2\pi l$, $l \in \mathbb{Z}$ and the collection of real parts of the zeros of $H(\lambda, 0)$ is $\bar{Z}(0) = \{-\log(6)\}$. However, from the analysis of section 2, we know that for h and 1 rationally independent (i.e. h irrational), $\bar{Z}(h)$ consists of all solutions α of

$$1 + 2e^{-\alpha h}e^{-i\theta_1} - \frac{1}{2}e^{-\alpha}e^{-i\theta_2} = 0. \tag{11}$$

Letting h go to 0 in (11) we are led to the conclusion that

$$\lim_{h \to 0+} \bar{Z}(h) = [-\log(6), -\log(2)], \tag{12}$$

i.e., that, although the real part of each individual zero of $H(\lambda, h)$ approaches $-\log(6)$ as h goes to zero, at the same time the collection of all the real parts of all zeros converges to (12).

Figure 5 (a) shows part of the zeros of equation (10) for $h = 0.01$. At first sight this confirms the above conclusions. However, if we look at a larger region in the complex plane (see figure 5 (b)) we see that there exist additional zeros of $H(\lambda, h)$ with quite different behaviour. When h is further reduced the real part of the zeros at $\Re(\lambda) \approx 69.3$ move off to $+\infty$, approximately as the solutions of

$$1 + 2e^{-\lambda h} = 0. \tag{13}$$

Indeed, if the real part of λ is large we cannot set λh to zero. Rather we can neglect $\frac{1}{2}e^{-\lambda}$ leading to (13) and

$$\lambda \approx \frac{1}{h}\left(\log 2 + i(2l+1)\pi\right), \; l \in \mathbb{Z}. \tag{14}$$

Formula (14) clearly illustrates that arbitrary small delays ($0 < h \ll 1$) can lead to arbitrary unstable eigenvalues ($\Re(\lambda) \gg 1$).

The situation can be summarised as follows. When h tends to zero the spectrum consists partly of eigenvalues with bounded real part, which can be analysed along the lines of §2, and partly of 'diverging' eigenvalues whose real part grows without bound as the solutions of the 'small-delay part' (13) of equation (10). In the rest of this section these properties will be generalized to the multiple delay case.

4.2 Notation

The general form of the exponential polynomial studied within this section is,

$$H(\lambda, r, s) = 1 - \sum_{i \in I} a_i e^{-\lambda \tau_i} - \sum_{j \in J} b_j e^{-\lambda \tau_j}, \tag{15}$$

where

$$\forall i \in I : \tau_i = \gamma_i \cdot r, \ \forall j \in J : \tau_j = \gamma_j \cdot r + \nu_j \cdot s.$$

and where

$$I = \{1, 2, \ldots, N_1\}, \ J = \{N_1 + 1, N_1 + 2, \ldots, N_1 + N_2\}$$

are used for notational convenience. The components of $r \in [0, +\infty)^M$ and $s \in [0, +\infty)^L$ are the independent delays; $\gamma_i \in \mathbb{N}^M$, $\gamma_j \in \mathbb{N}^M$ and $\nu_j \in \mathbb{N}^L$ are vectors with positive integer coefficients. γ_i and ν_j are nonzero vectors, that is both have at least one nonzero element for all i and j. Splitting the independent delays into r and s opens the possibility to deal with a combination of 'normal' and arbitrary small delays by letting $r \to 0$ combined with constant $s > 0$.

We also extend the definition of the inner product '$\cdot$' to the situation with $\gamma_i \in \mathbb{N}^M$ and $R \in [0, +\infty]^M$. Then

$$\gamma_i \cdot R = \sum_{j=1}^{M} \gamma_{i,j} R_j,$$

has the usual meaning, except that $\gamma_{i,j}$ times $R_j = +\infty$ is taken to be 0 when $\gamma_{i,j} = 0$ and $+\infty$ otherwise. The underlying rationale for this is that $R_j = +\infty$ will be the result of some limit while the $\gamma_{i,j}$ are fixed.

4.3 Arbitrary unstable eigenvalues

The 'small-delay part' of (15) is

$$1 - \sum_{i \in I} a_i e^{-\lambda \tau_i}.$$

We now prove how its solutions determine when arbitrary small delays can lead to arbitrary unstable eigenvalues.

Theorem 4.1 *The following statements are equivalent:*

$$\exists \theta \in [0, 2\pi]^M, \ \exists R \in [0, +\infty]^M \text{ such that } 1 - \textstyle\sum_{i \in I} a_i e^{-\gamma_i \cdot R} e^{-i \gamma_i \cdot \theta} = 0$$
$$\Updownarrow$$
$$\exists \{r_n\}_{n \geq 1}, \ \{c_n\}_{n \geq 1}, \ \{d_n\}_{n \geq 1}$$
$$\text{with } \lim_{n \to \infty} c_n = \infty, \ r_n \geq 0 \text{ and } \lim_{n \to \infty} \|r_n\| = 0 \text{ and such that}$$
$$\lim_{n \to \infty} H(c_n + i d_n, r_n, s) = 0 \text{ for fixed } s > 0.$$

Proof of $\Downarrow$: Consider a (re)ordered partition of $R = (R_1, \ldots, R_K, R_{K+1}, \ldots, R_M)$ such that $R_1, \ldots, R_K$ are finite and $R_{K+1}, \ldots, R_M$ are infinite. That is, let $R = (R^{[1]}, R^{[2]})$ with $R^{[1]} = (R_1, R_2, \ldots, R_K) \in \mathbb{R}^K$ and $R^{[2]} = (\infty, \infty, \ldots, \infty) = \infty^{M-K}$. Accordingly, consider the following partition for θ and γ_i: $\theta = (\theta^{[1]}, \theta^{[2]})$ and $\gamma_i = (\gamma_i^{[1]}, \gamma_i^{[2]})$ with $\theta^{[1]} = (\theta_1, \theta_2, \ldots, \theta_K)$ the first K components and $\theta^{[2]} = (\theta_{K+1}, \theta_{K+2}, \ldots, \theta_M)$ the remaining $M-K$ components of θ and similar for $\gamma_i^{[1]}$ and $\gamma_i^{[2]}$, $i \in I$. Define the set of indices $I_1 \subseteq I$ whereby for $i \in I_1$ the last $M-K$ components of γ_i are zero, that is, where $\gamma_i^{[2]} = 0^{M-K}$; and set $I_2 = I \setminus I_1$. Obviously

$$1 - \sum_{i \in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i \cdot \theta} = 0$$

can be written as:

$$1 - \sum_{i \in I_1} a_i e^{-\gamma_i^{[1]} \cdot R^{[1]}} e^{-i\gamma_i^{[1]} \cdot \theta^{[1]}} = 0.$$

Because the components of $R^{[1]}$ may be rationally dependent, consider a sequence $\{u_n^{[1]}\}_{n \geq 1}$ that converges to $R^{[1]}$ but whereby the components of $u_n^{[1]} \in (0, +\infty)^K$ are rationally independent for each n. Choose a (strictly positive) sequence of real numbers $\{\epsilon_n\}_{n \geq 1}$ with $\lim_{n \to \infty} \epsilon_n = 0$, such that

$$\|u_n^{[1]} - R^{[1]}\| < \epsilon_n.$$

Because $u_n^{[1]}$ has rationally independent coefficients, due to Kronecker's theorem, there exists, for each n, a sequence of real numbers $\{v_{n,m}\}_{m \geq 1}$ such that

$$\lim_{m \to \infty} e^{i\gamma_i^{[1]} \cdot (v_{n,m} u_n^{[1]} - \theta^{[1]})} = 1, \ \forall i \in I_1,$$

hence $\exists m^*(n)$ such that $|e^{i\gamma_i^{[1]} \cdot (v_{n,m^*(n)} u_n^{[1]} - \theta^{[1]})} - 1| < \epsilon_n$, $\forall i \in I_1$. Set $v_n = v_{n,m^*(n)}$. We have created $\left\{u_n^{[1]}\right\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ with

$$\begin{gathered} \|u_n^{[1]} - R^{[1]}\| < \epsilon_n, \\ |e^{i\gamma_i^{[1]} \cdot (v_n u_n^{[1]} - \theta^{[1]})} - 1| < \epsilon_n, \ \forall i \in I_1, \text{ and} \\ \lim_{n \to \infty} \epsilon_n = 0. \end{gathered}$$

Choose further $\left\{u_n^{[2]}\right\}_{n \geq 1}$ with $u_n^{[2]} \in (0, +\infty)^{M-K}$ and with $\lim_{n \to \infty} u_n^{[2]} = \infty^{M-K}$; and define $\{u_n\}_{n \geq 1}$ as $u_n = (u_n^{[1]}, u_n^{[2]}) \in (0, +\infty)^M$.

We are now in a position to choose a sequence of real parts c_n. Choose $\{c_n\}_{n \geq 1}$ with $c_n \in (0, +\infty)$ such that c_n goes to infinity faster than every component of u_n, that is, such that $\lim_{n \to \infty} c_n = +\infty$ and $\lim_{n \to \infty} \frac{1}{c_n} u_n = 0^M$. Secondly define a sequence of imaginary parts $\{d_n\}_{n \geq 1}$ as $d_n = c_n v_n$, and a sequence of vanishing delays, $\{r_n\}_{n \geq 1}$ as $r_n = \frac{1}{c_n} u_n$.

We now have

$$\begin{aligned} & H(c_n + id_n, r_n, s) \\ & = 1 - \textstyle\sum_{i \in I} a_i e^{-c_n \gamma_i \cdot r_n} e^{-id_n \gamma_i \cdot r_n} - \sum_{j \in J} b_j e^{-c_n(\gamma_j \cdot r_n + \nu_j \cdot s)} e^{-id_n(\gamma_j \cdot r_n + \nu_j \cdot s)} \\ & = 1 - \textstyle\sum_{i \in I} a_i e^{-\gamma_i \cdot u_n} e^{-i\gamma_i \cdot v_n u_n} - \sum_{j \in J} b_j e^{-c_n(\gamma_j \cdot r_n + \nu_j \cdot s)} e^{-id_n(\gamma_j \cdot r_n + \nu_j \cdot s)}. \end{aligned}$$

The second term can be split in firstly $\sum_{i\in I_1} a_i e^{-\gamma_i\cdot u_n} e^{-i\gamma_i\cdot v_n u_n}$ whereby the last $M-K$ components of γ_i are zero and thus $\gamma_i\cdot u_n = \gamma_i^{[1]}\cdot u_n^{[1]}$ and secondly $\sum_{i\in I_2} a_i e^{-\gamma_i\cdot u_n} e^{-i\gamma_i\cdot v_n u_n}$ whereby $\lim_{n\to\infty} \gamma_i \cdot u_n = +\infty$.

Hence $H(c_n + id_n, r_n, s)$

$$\begin{aligned}
&= 1 - \textstyle\sum_{i\in I_1} a_i e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}} e^{-\gamma_i^{[1]}\cdot(u_n^{[1]}-R^{[1]})} e^{-i\gamma_i^{[1]}\cdot(v_n u_n^{[1]}-\theta^{[1]})} \\
&\quad - \textstyle\sum_{i\in I_2} a_i e^{-\gamma_i\cdot u_n} e^{-i\gamma_i\cdot v_n u_n} - \sum_{j\in J} b_j e^{-c_n(\gamma_j\cdot r_n+\nu_j\cdot s)} e^{-id_n(\gamma_j\cdot r_n+\nu_j\cdot s)} \\
&= 1 - \textstyle\sum_{i\in I_1} a_i e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}} \underbrace{e^{-\gamma_i^{[1]}\cdot(u_n^{[1]}-R^{[1]})}}_{\to 1} \underbrace{e^{-i\gamma_i^{[1]}\cdot(v_n u_n^{[1]}-\theta^{[1]})}}_{\to 1} \\
&\quad - \textstyle\sum_{i\in I_2} a_i e^{-\overbrace{\gamma_i\cdot u_n}^{\to\infty}} e^{-i\gamma_i\cdot v_n \bar{u}_n} - \sum_{j\in J} b_j e^{-\overbrace{c_n \overbrace{\nu_j\cdot s}^{\neq 0}}^{\to\infty}} e^{-c_n\gamma_j\cdot r_n} e^{-id_n(\gamma_j\cdot r_n+\nu_j\cdot s)}
\end{aligned}$$

tends to zero as n approaches infinity which completes this part of the proof.

Proof of $\Uparrow$: The $\lim_{n\to\infty} H(c_n + id_n, r_n, s) = 0$ implies

$$\lim_{n\to\infty} 1 - \sum_{i\in I} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot d_n r_n} = 0 \tag{16}$$

because the other term vanishes at infinity.

Consider the sequence $\{c_n r_n\}_{n\geq 1}$ with elements $c_n r_n = (c_n r_{n,1}, \ldots, c_n r_{n,M})$. For each sequence of the k-th component, $\{c_n r_{n,k}\}_{n\geq 1}$, there are two possibilities as n tends to infinity: the sequence is unbounded (with or without limit), or the sequence is bounded (with or without limit). Suppose that $\{c_n r_{n,k}\}_{n\geq 1}$ is unbounded. Then there exists a subsequence with limit infinity. When, on the other hand, $\{c_n r_{n,k}\}_{n\geq 1}$ is bounded, a subsequence with finite limit exists.

This way one can recursively construct subsequences to obtain an infinite set of indices S_1 such that for each $k \in \{1, 2 \ldots, M\}$, $\lim_{n\to\infty, n\in S_1} c_n r_{n,k}$ exists in $[0, +\infty]$. We still have,

$$\lim_{n\to\infty, n\in S_1} 1 - \sum_{i\in I} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot d_n r_n} = 0, \tag{17}$$

and because each γ_i is a vector of integer coefficients, $e^{-i\gamma_i\cdot d_n r_n}$ equals $e^{-i\gamma_i\cdot((d_n r_n) \bmod 2\pi)}$ and thus

$$\lim_{n\to\infty, n\in S_1} 1 - \sum_{i\in I} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot((d_n r_n) \bmod 2\pi)} = 0.$$

Consider the sequence of vectors $\{(d_n r_n) \bmod 2\pi\}_{n\geq 1, n\in S_1}$ which is bounded and thus contained in a compact set (w.r.t. some norm). Consequently it must have a converging subsequence, denoted by the indices $S_2 \subset S_1$. Finally

$$\lim_{n\to\infty, n\in S_2} 1 - \sum_{i\in I} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot((d_n r_n) \bmod 2\pi)} = 0,$$

and we define

$$R_k = \lim_{n\to\infty, n\in S_2} c_n r_{n,k} \in [0, +\infty], \ k = 1, 2, \ldots, M,$$

and

$$\theta_k = \lim_{n\to\infty, n\in S_2} (d_n r_{n,k}) \bmod 2\pi \in [0, 2\pi], \;\; k = 1, 2, \ldots, M.$$

Hence we have defined every component of R and θ and by its construction it follows

$$1 - \sum_{i\in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i \cdot \theta} = 0,$$

which completes the proof. □

In the following theorem we make the results of the previous theorem stronger: we prove the existence of a sequence of *solutions* λ_n of $H(\lambda, r_n, s) = 0$ with real part tending to $+\infty$, in other words, that arbitrary small delays cause eigenvalues with arbitrary large real part.

Theorem 4.2 *Consider the statements:*

(a) $\exists \theta \in [0, 2\pi]^M, \; \exists R \in [0, +\infty]^M$ such that $1 - \sum_{i\in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i\cdot\theta} = 0,$

(b) $\exists i \in I$ such that $\gamma_i \cdot R \neq 0$ and $\gamma_i \cdot R \neq +\infty,$

(c) $\exists \left\{r_n\right\}_{n\geq 1}, \; \left\{\lambda_n = e_n + i f_n\right\}_{n\geq 1}$ with $\lim_{n\to\infty} e_n = +\infty, \; r_n \geq 0$ and $\lim_{n\to\infty} r_n = 0^M$such that$H(\lambda, r_n, s) = 0.$

Then the following holds:

1. *(a) and (b)* $\Rightarrow$ *(c)*
2. *(c)* $\Rightarrow$ *(a)*

Proof of 2. Trivial (theorem 4.1).

Proof of 1. Following theorem 4.1 there exist sequences $\left\{r_n\right\}_{n\geq 1}, \left\{c_n\right\}_{n\geq 1}, \left\{d_n\right\}_{n\geq 1}$ with $\lim_{n\to\infty} c_n = +\infty, \; r_n \geq 0$ and $\lim_{n\to\infty} r_n = 0^M$ such that

$$\lim_{n\to\infty} H(c_n + i d_n, r_n, s) = 0$$

and the proof provided us with a way to construct such sequences.

Choose $\left\{\epsilon_n\right\}_{n\geq 1}, \left\{u_n^{[1]}\right\}_{n\geq 1}$ and $\left\{v_n\right\}_{n\geq 1}$ as in theorem 4.1 and such that

$$\left| 1 - \sum_{i\in I_1} a_i e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}} e^{-\gamma_i^{[1]}\cdot(u_n^{[1]} - R^{[1]})} e^{-i\gamma_i^{[1]}\cdot(v_n u_n^{[1]} - \theta^{[1]})} \right| < e^{-n}.$$

Further requirements on the decay-rate, which can be chosen arbitrary fast, will be given later in the proof (see formulae (18) and (19)). Choose $\left\{c_n\right\}_{n\geq 1}$ as $c_n = n$, $\left\{u_n^{[2]}\right\}_{n\geq 1}$ as $u_n^{[2]} = \sqrt{n}(1, 1, \ldots, 1)$, $\left\{d_n\right\}_{n\geq 1}$ as $d_n = v_n c_n$, $\left\{u_n\right\}_{n\geq 1}$ as $u_n = (u_n^{[1]}, u_n^{[2]})$, and $\left\{r_n\right\}_{n\geq 1}$ as $r_n = \frac{1}{c_n} u_n$.

Consider the functions $\bar{H}_n(\lambda) = c_n H(\lambda, r_n, s)$. Using the particular choices above it is straightforward to show that

$$\lim_{n\to\infty} \bar{H}_n(c_n + id_n) = 0, \ \lim_{n\to\infty} \frac{d\bar{H}_n}{d\lambda}(c_n + id_n) = Q,$$

where

$$Q = \sum_{i\in I_1} a_i \gamma_i^{[1]} \cdot R^{[1]} e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}} \in \mathbb{C},$$

and

$$\lim_{n\to\infty} \frac{d^k\bar{H}_n}{d\lambda^k}(c_n + id_n) = 0, \ k \geq 2.$$

Around $c_n + id_n$ the analytic function $\bar{H}_n$ can be expanded as

$$\bar{H}_n(\lambda) = \sum_{k=0}^{\infty} \left.\frac{d^k\bar{H}_n}{d\lambda^k}\right|_{c_n+id_n} (\lambda - (c_n + id_n))^k$$

and when defining $\mu = \lambda - (c_n + id_n)$,

$$\bar{H}_n(\mu) = \sum_{k=0}^{\infty} \left.\frac{d^k\bar{H}_n}{d\lambda^k}\right|_{c_n+id_n} \mu^k$$

We will now show that $\bar{H}_n(\mu)$ converges uniformly on $|\mu| \leq 1$ to the function $\bar{H}(\mu) = Q\mu$ or equivalently that the function

$$D_n(\mu) = \bar{H}_n(\mu) - Q\mu$$

converges uniformly to zero in $|\mu| \leq 1$. In the appendix it is shown (lemma A.1) that this implies that $\bar{H}_n(\mu)$ has a zero near $\mu = 0$ for sufficiently large n when $Q \neq 0$. Hereby one should emphasize that μ is in fact a local coordinate depending on n: we compare the local behaviour of each function $\bar{H}_n(\lambda)$ near $c_n + id_n$ with $\bar{H}(\mu) = Q\mu$.

To construct an upper bound on $|D_n(\mu)|$ for $|\mu| \leq 1$, first define strictly positive numbers Δ_i, Δ_j, β_i and integer P such that for $n \geq P$,

$$\begin{gathered} \gamma_j \cdot r_n \leq \Delta_j, \ j \in J, \\ \gamma_i \cdot c_n r_n \leq \gamma_i^{[1]} \cdot R^{[1]} + \Delta_i, \ i \in I_1, \\ \gamma_i \cdot r_n \leq 1, \ i \in I = I_1 \cup I_2 \text{ and} \\ \gamma_i \cdot c_n r_n \geq \beta_i \sqrt{n}, \ i \in I_2. \end{gathered}$$

Secondly the decay rate of $\{\epsilon_n\}_{n\geq 1}$ should have been chosen in such a way that for $n \geq P$

$$n \left| 1 - \sum_{i\in I_1} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot d_n r_n} \right| \leq \frac{1}{n} \sum_{i\in I_1} |a_i| \tag{18}$$

and

$$\left| \sum_{i\in I_1} a_i \gamma_i \cdot c_n r_n e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot d_n r_n} - Q \right| \leq \frac{1}{n} \sum_{i\in I_1} |a_i| (\gamma_i^{[1]} \cdot R^{[1]} + \Delta_i). \tag{19}$$

In this way we can compute bounds for $D_n(0)$ and its derivatives in $\mu = 0$, $n \geq P$.

$$\begin{array}{rcl}|D_n(0)| & \leq & n\left|1-\sum_{i\in I_1} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot d_n r_n}\right| + n\left|\sum_{i\in I_2} a_i e^{-\gamma_i\cdot c_n r_n} e^{-i\gamma_i\cdot d_n r_n}\right| \\ & & +n\sum_{j\in J}\left|b_j e^{-\nu_j\cdot s c_n} e^{-\gamma_j\cdot c_n r_n} e^{-i\gamma_j\cdot d_n r_n}\right| \\ & \leq & \frac{1}{n}\sum_{i\in I_1}|a_i| + \sum_{i\in I_2} n e^{-\beta_i\sqrt{n}}|a_i| + \sum_{j\in J} n e^{-n\nu_j\cdot s}|b_j|\end{array}$$

and concerning the derivatives:

$$\begin{array}{rcl}|\frac{dD_n}{d\mu}(0)| & \leq & \frac{1}{n}\sum_{i\in I_1}|a_i|(\gamma_i^{[1]}\cdot R + \Delta_i) + \sum_{i\in I_2} n e^{-\beta_i\sqrt{n}}|a_i| \\ & & +\sum_{j\in J} n(\Delta_j + \nu_j\cdot s)e^{-n\nu_j\cdot s}|b_j|\end{array}$$

and for $k \geq 2$,

$$\begin{array}{rcl}|\frac{d^k D_n}{d\mu^k}(0)| & \leq & \frac{1}{n}\sum_{i\in I_1}|a_i|(\gamma_i^{[1]}\cdot R^{[1]} + \Delta_i) + \sum_{i\in I_2} n e^{-\beta_i\sqrt{n}}|a_i| \\ & & +\sum_{j\in J} n(\Delta_j + \nu_j\cdot s)^k e^{-n\nu_j\cdot s}|b_j|.\end{array}$$

Finally, for each μ with $|\mu| \leq 1$,

$$\begin{array}{rcl}|D_n(\mu)| & \leq & \sum_{k=0}^{\infty}\frac{1}{k!}|\frac{d^k D}{d\mu^k}| \\ & \leq & \sum_{i\in I_1}\frac{1}{n}|a_i|(\gamma_i^{[1]}\cdot R^{[1]} + \Delta_i)\sum_{k=0}^{\infty}\frac{1}{k!} + \sum_{i\in I_2} n e^{-\beta_i\sqrt{n}}\sum_{k=0}^{\infty}|a_i|\frac{1}{k!} \\ & & +\sum_{j\in J} n e^{-n\nu_j\cdot s}|b_j|\sum_{k=0}^{\infty}\frac{(\Delta_j+\nu_j\cdot s)^k}{k!} \\ & \leq & \sum_{i\in I_1}\frac{1}{n}|a_i|(\gamma^{[1]}\cdot R^{[1]} + \Delta_i)e + \sum_{i\in I_2} n e^{-\beta_i\sqrt{n}}|a_i|e \\ & & +\sum_{j\in J} n e^{-n\nu_j\cdot s}|b_j|e^{(\Delta_j+\nu_j\cdot s)}\end{array}$$

can thus be bounded uniformly whereby the upper bound tends to zero as $n \to \infty$.

Because the series $\{\bar{H}_n(\mu)\}_{n\geq 1}$ converges uniformly to a function $\bar{H}(\mu) = Q\mu$, due to lemma A.1, there must be a number $N \in \mathbb{N}$ such that for $\forall n \geq N$, $\bar{H}_n(\mu)$ has a zero which converges to zero as $n \to \infty$, whenever $Q \neq 0$. Returning to the original problem, this means that $c_n + id_n$ asymptotically coincides with a zero $e_n + if_n$ of $\bar{H}_n(\lambda)$ which is of course a zero of $H(\lambda, r_n, s)$. Thus $\forall n \geq N, \exists e_n + if_n : H(e_n + if_n, r_n, s) = 0$ and $\lim_{n\to\infty}(c_n - e_n) + i(d_n - f_n) = 0$. By renumbering the sequences starting from $n = N$ the proof is complete in the case $Q \neq 0$.

We will now show that when $Q = 0$, there always exists an integer k such that $\sum_{i\in I_1} a_i(\gamma_i^{[1]}\cdot R^{[1]})^l e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}} = 0$ for $l < k$ and $Q' = \sum_{i\in I_1} a_i(\gamma_i^{[1]}\cdot R^{[1]})^k e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}} \neq 0$. In this case $c_n^k H(\lambda, r_n, s)$ will converge locally to $Q'(\lambda - (c_n + id_n))^k$.

We now prove that if Q' does not exist, statement (b) is violated. For notational convenience, define $\hat{a}_i = a_i e^{-\gamma_i^{[1]}\cdot R^{[1]}} e^{-i\gamma_i^{[1]}\cdot\theta^{[1]}}$. Given $\sum_{i\in I_1}\hat{a}_i = 1$ and $\sum_{i\in I_1}(\gamma_i^{[1]}\cdot R^{[1]})\hat{a}_i = 0$, $\sum_{i\in I_1}(\gamma_i^{[1]}\cdot R^{[1]})^2\hat{a}_i = 0, \ldots$ (because Q' does not exist), multiply the above equations with scalars and add them together to obtain:

$$p(0) = \sum_{i\in I_1}\hat{a}_i p(\gamma_i^{[1]}\cdot R^{[1]}) \quad (20)$$

for all polynomials $p(\lambda)$. In (20) it is possible that for some indices $k, l \in I_1$, $\gamma_k^{[1]}\cdot R^{[1]} = \gamma_l^{[1]}\cdot R^{[1]}$ or $\gamma_k^{[1]}\cdot R^{[1]} = 0$. We group these factors:

$$g_0 p(0) + \sum g_j p(\gamma_j^{[1]}\cdot R^{[1]}) = 0.$$

If we choose for $p(\lambda)$ an interpolating (complex) polynomial such that $p(0) = \bar{g}_0$ and $p(\gamma_j^{[1]} \cdot R^{[1]}) = \bar{g}_j, \ \ \forall j$, this leads to

$$|g_0|^2 + \sum |g_j|^2 = 0 \rightarrow g_0 = 0, \ g_j = 0, \ \forall j \rightarrow g_0 + \sum g_j e^{\gamma_j^{[1]} \cdot R^{[1]}} = 0$$

or

$$1 - \sum_{i \in I_1} a_i e^{-i\gamma_i^{[1]} \cdot \theta^{[1]}} = 0.$$

This means that in $1 - \sum_{i \in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i \cdot \theta} = 0$, for all $i \in I$, $\gamma_i \cdot R$ is zero or infinity, which contradicts statement (b). □

For control applications the following theorem is important. It provides sufficient and necessary conditions for equation (15) to have eigenvalues with arbitrary large real part but small imaginary part in the presence of vanishing delays.

Theorem 4.3 *Consider the statements:*

(a) $\exists R \in [0, +\infty]^M$ such that $1 - \sum_{i \in I} a_i e^{-\gamma_i \cdot R} = 0$,

(b) $\exists i \in I$ such that $\gamma_i \cdot R \neq 0$ and $\gamma_i \cdot R \neq +\infty$,

(c) $\exists \{r_n\}_{n \geq 1}, \ \{\lambda_n\}_{n \geq 1}$ with $\lambda_n \in \mathbb{C}$, $\lim_{n \to \infty} \Re(\lambda_n) = +\infty$, $\lim_{n \to \infty} \Im(\lambda_n) = 0$, $r_n \geq 0$ and $\lim_{n \to \infty} \|r_n\| = 0$ and such that $H(\lambda_n, r_n, s) = 0$.

Then the following holds:

1. *(a) and (b) ⇒ (c)*
2. *(c) ⇒ (a)*

The above result can be shown by following the lines of the proof of theorem 4.1 and 4.2.

4.4 Interpretation and illustration

When the delays r in equation (15) approach zero, the real part of each eigenvalue remains bounded or moves to $+\infty$. We call the set of these eigenvalues the *finite* respectively the *infinite* part of the spectrum.

The supremum of the real parts of the finite part of the spectrum, $c_{\rm f}$, can be calculated as the rightmost solution α of

$$1 - \sum_{i \in I} a_i e^{-i\gamma_i \cdot \theta_1} - \sum_{j \in J} b_j e^{-\alpha \nu_j \cdot s} e^{-i(\gamma_j \cdot \theta_1 + \nu_j \cdot \theta_2)} = 0 \qquad (21)$$

which corresponds to applying theorem 2.1 and putting αr to zero. This last step is allowed because the delays r are arbitrary small and α finite.

The infinite part of the spectrum is empty if

$$1 - \sum_{i \in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i \cdot \theta} = 0 \qquad (22)$$

has no solution with $R \in [0, +\infty]^M$. In the other case we can conclude arbitrary unstable eigenvalues except in degenerate cases whereby all the positive solutions of (22) (w.r.t. R) satisfy that $\forall i \in I, \gamma_i \cdot R$ is zero or infinite. For these degenerate cases, the small-delay part of the exponential polynomial does not provide enough information about the existence of the infinite spectrum, as will be shown in 4.5.

We now illustrate how the solutions of (22) determine the structure of the infinite part of the spectrum and raise and partially answer an important open question which naturally arises from our analysis.

Consider the following example,

$$1 - 2e^{-\lambda(r_1+r_2)} + 4e^{-\lambda 8 r_2} - \frac{1}{2}e^{-\lambda 2} = 0, \tag{23}$$

which has, when r_1 and $r_2 \to 0+$, the following small-delay part:

$$1 - 2e^{-\lambda(r_1+r_2)} + 4e^{-\lambda 8 r_2} = 0. \tag{24}$$

The supremum of the real parts of the finite part of the spectrum is $c_{\mathrm{f}} = -\frac{1}{2}\log 2 < 0$. The infinite part of the spectrum is nonempty if and only if

$$1 - 2e^{-(R_1+R_2)}e^{-i(\theta_1+\theta_2)} + 4e^{-8R_2}e^{-i8\theta_2} = 0, \tag{25}$$

has solutions with R_1 and $R_2 \in [0, +\infty]$. In figure 6 the *area* between curves A, B and C is the projection on the (R_1, R_2)-plane of the solutions of (25). Using theorem 4.3, equation (23) has solutions with real part tending to $+\infty$ but small imaginary part if and only if (25) has a solution with R_1 and $R_2 \in [0, +\infty]$ and $\theta_1 = \theta_2 = 0$. These solutions form the *curve* C in figure 6.

Following theorem 4.2 the solutions of $1 - \sum_{i \in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i \cdot \theta} = 0$ determine the existence of eigenvalues with large real part. When the delays approach zero with a fixed ratio they will provide a complete characterization of the structure of the large eigenvalues. This will be illustrated for the example discussed above. We will only consider the solutions of (24), since one can show that these solutions asymptotically coincide with solutions of (23) as $(r_1, r_2) \to (0, 0)$ (proof similar to theorem 4.2).

Suppose (25) has a solution with

$$(R_1, R_2) = \alpha(R_1^*, R_2^*) \quad 0 \le \alpha_1 < \alpha \le \alpha_2$$

whereby we assume at the moment that R_1^* and R_2^* are finite and rationally independent. Then such a solution for each α satisfies

$$1 - 2e^{-\alpha k \frac{R_1^* + R_2^*}{k}} e^{-(\theta_1+\theta_2)} + 4e^{-8\alpha k \frac{R_2^*}{k}} e^{-8\theta_2} = 0$$

whereby θ_1 and θ_2 depend on α. Applying theorem 2.1 to this formula it is clear that for delays $(R_1, R_2) = (\frac{R_1^*}{k}, \frac{R_2^*}{k})$ (24) has solutions with real part arbitrary close to $k\alpha$, $\alpha_1 \le \alpha \le \alpha_2$.

This leads us to the following conclusions: the intervals in figure 6 which form the intersection of a line through the origin and the projected solution surface of (25) correspond to intervals of $\bar{Z}$ which move to infinity ($k \to \infty$) as the delays approach zero in the ratio determined by the slope of the line. Secondly the number and

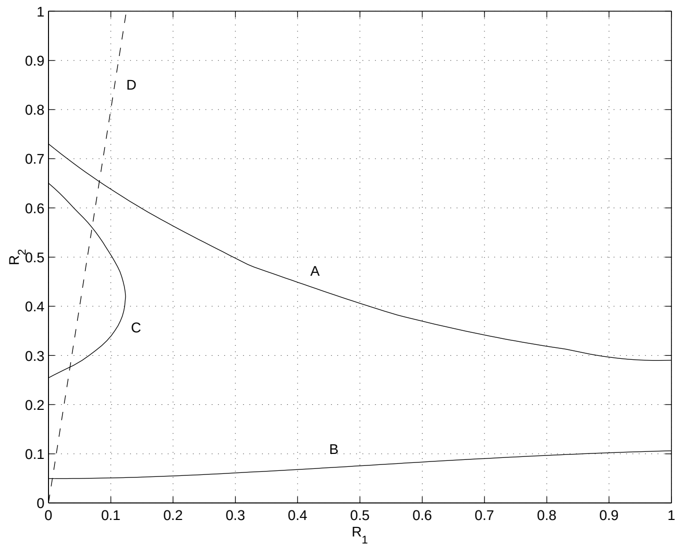

Figure 6: Projection of solutions of (25)

lengths of these intervals, when existing, depend strongly on the way (ratio) the delays approach zero.

Using the same kind of arguments, the intersection of a line through the origin with curve C in figure 6 corresponds to a real eigenvalue going to infinity. As can be seen from the picture, the *occurrence* of this phenomenon depends on the ratio of the delays.

When $\frac{R_1^*}{R_2^*}$ would be rational, the delays are commensurate and $\bar{Z}$ consists of a number of points. However when the ratio is close[1] to an irrational number (due to the lower semi-continuity of $\bar{Z}$ w.r.t. the delays) these points will fill up the intervals (rat. independent case) quite well.

Figure 7 shows some of the zeros of (24) for the delays $(R_1, R_2) = (\frac{1}{k}, \frac{8}{k})$ which corresponds to line D in figure 6. For computational convenience these delays are chosen rationally dependent but the resonance is relatively weak: the two intervals predicted by figure 6 are already visible. The two real eigenvalues correspond to the intersection of curves D and C in figure 6.

Note that when the delays approach zero, the imaginary parts of the non real eigenvalues increase to infinity. As already mentioned, when dealing with practical control problems, the question arises whether the damping at very high frequencies is underestimated in the model ($\rightarrow$ exponential polynomial) or not. In any case the large real eigenvalues are important. When one can estimate the ratio of the delays in the real system, one can predict whether such unstable behaviour occurs using diagrams like figure 6.

Important Remark

The occurrence of arbitrary large real eigenvalues does *not* depend on the fact whether the small delays are rationally (in)dependent.

[1] A (positive) rational number a is close to an irrational number iff $a = \frac{N_1}{N_2}$ whereby N_1 and $N_2 \in \mathbb{N}$ are large and coprime.

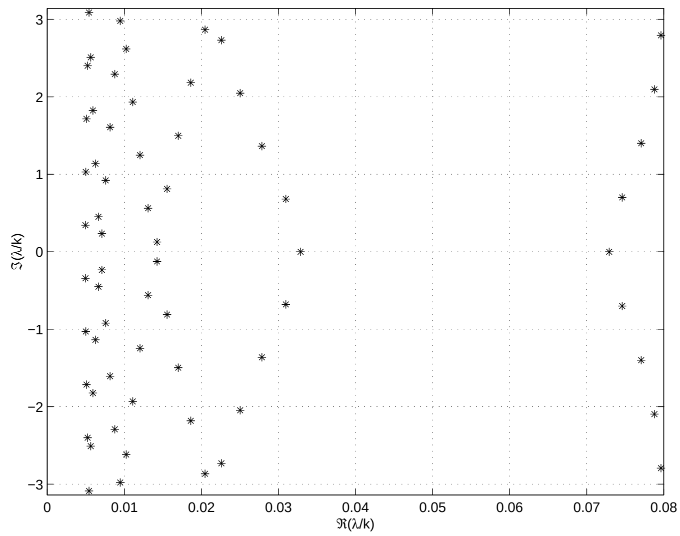

Figure 7: Part of the spectrum of (24) for the delays $(R_1, R_2) = (\frac{1}{k}, \frac{8}{k})$, corresponding to the dashed line D in figure 6

4.5 Degenerate cases

We call the exponential polynomial (15) degenerate iff all the positive solutions (w.r.t. R) of

$$1 - \sum_{i \in I} a_i e^{-\gamma_i \cdot R} e^{-i\gamma_i \cdot \theta} = 0 \tag{26}$$

satisfy

$$\gamma_i \cdot R = 0 \text{ or } \gamma_i \cdot R = +\infty, \quad \forall i \in I.$$

For example equation

$$1 - \sum_{i=1}^{N} a_i \left(e^{-\lambda . r}\right)^i - \sum_{j \in J} b_j e^{-\lambda(\gamma_j r + \nu_j \cdot s)} = 0$$

is degenerate iff the polynomial $1 - \sum_{i=1}^{N} a_i x^i$ has some zeros on and all other zeros outside the unit circle. The two degrees of freedom equation $1 - a_1 e^{-\lambda r_1} - a_2 e^{-\lambda r_2} - \sum_{j \in J} b_j e^{-\lambda(\gamma_j \cdot r + \nu_j \cdot s)} = 0$ is degenerate iff $|a_1| + |a_2| = 1$.

By means of an example we show that the large delay-part plays a role in the existence of arbitrary large eigenvalues for vanishing delays. Therefore, consider firstly

$$H(\lambda = c + id, h) \triangleq 1 + e^{-h\lambda} + e^{-2\lambda} - e^{-(2+h)\lambda} = 0, \tag{27}$$

which is clearly degenerate. With

$$h_n = \frac{1}{n + \frac{1}{2}}, \quad d_n = (n + \tfrac{1}{2})\pi,$$

we plot

$$H(c + id_n, h_n) = 1 - e^{-h_n c} - e^{-2c} - e^{-(2+h_n)c} \tag{28}$$

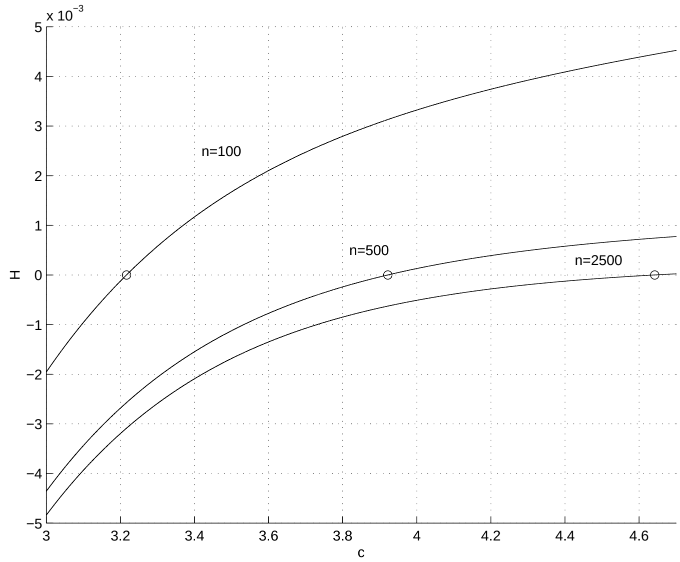

Figure 8: Plot of (28) for different values of n

in figure 8 for $n = 100, 500, 2500$. The zero, caused by the negative coefficients of the large-delay terms, tends to infinity as n does ($H(c+id_n, h_n) = 0 \Leftrightarrow e^{-h_n c} = \tanh(c)$).

However when one modifies the equation to

$$1 + e^{-h\lambda} + e^{-2\lambda} + e^{-(2+h)\lambda} = 0$$

degeneration is preserved while there are *no* solutions with arbitrary large real part: when substituting $\lambda = c + di$ the real part of the equation becomes

$$\underbrace{1 + e^{-ch}\cos(\theta_1)}_{\text{term } A} + \underbrace{e^{-2c}\cos(\theta_2) + e^{-(2+h)c}\cos(\theta_1 + \theta_2)}_{\text{term } B}, \tag{29}$$

whereby $\theta_1 = (dh)\mathrm{mod}2\pi$ and $\theta_2 = (d2)\mathrm{mod}2\pi$.

Case 1: $\theta_1 \neq \pi$

There are no zeros with arbitrary large real part for vanishing h because term A is greater than $\min(1, 1 - |\cos(\theta_1)|) > 0$, whereas term B tends to zero exponentially:

$$\begin{aligned} &\left|e^{-2c}\cos(\theta_2) + e^{-(2+h)c}\cos(\theta_1 + \theta_2)\right| \\ &\leq e^{-2c}\left|\cos(\theta_2) + e^{-hc}\cos(\theta_1 + \theta_2)\right| \\ &\leq e^{-2c}\left(|\cos(\theta_2)| + |\cos(\theta_1 + \theta_2)|\right). \end{aligned}$$

Case 2: $\theta_1 = \pi$

The equation reduces to

$$(1 - e^{-ch})(1 + e^{-2c}\cos(\theta_2)) = 0$$

which has no solutions for vanishing h.

5 Largest delays coincide

In the previous section we showed that in the presence of both normal and arbitrary small delays there may exist eigenvalues with arbitrary large real part. Now we describe an analogous phenomenon: when the largest delays are arbitrary close together there may be eigenvalues with real part moving to $-\infty$. From a control point of view this phenomenon is less important (no stability problem) and therefore we only illustrate the phenomenon with an example and formulate a theorem analogous to theorem 4.2.

5.1 Example

In the following characteristic equation,

$$1 + e^{-\lambda} - 3e^{-\lambda 2} + 2e^{-\lambda(2+h)} = 0, \tag{30}$$

the largest delays 2 and $2 + h$ coincide as $h \to 0$. Multiplying this equation with $e^{\lambda 2}$ yields

$$e^{\lambda 2} + e^{\lambda} - 3 + 2e^{-\lambda h} = 0$$

and it is clear that the solutions of $-3 + 2e^{-\lambda h}$,

$$\lambda = \frac{-\log(3/2) + i2\pi l}{h}, \; l \in \mathbb{N},$$

approximate solutions of (30) as $h \to 0+$ with $\Re(\lambda) \to -\infty$.

5.2 Theorem

In order to provide conditions for the existence of eigenvalues moving to $-\infty$ when the small delays r_n approach zero, we introduce the following definitions:

- Take the general form of equation (15) and consider a partition of the index set J into $J_{\max}$ and $J \setminus J_{\max}$ whereby for $j \in J_{\max}$,
$$\nu_j \cdot s = \max_{k \in J} \nu_k \cdot s.$$
- An index set $J_1 \subseteq J_2$ is dominant w.r.t. J_2 if and only if $\exists R \in [0, +\infty)^M$ such that
$$\forall i, j \in J_1, \; \forall k \in J_2 : \; \gamma_i \cdot R = \gamma_j \cdot R > \gamma_k \cdot R.$$
- An index set J^* is a nested dominant set w.r.t. J if and only if there exists a finite number of index sets, $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots \supseteq J_k$ such that J_1 is dominant w.r.t. J, J_{l+1} is dominant w.r.t. J_l, $l = 1, \ldots k-1$, and J^* is dominant w.r.t. J_k.

We can now formulate:

Theorem 5.1 *The following statements are equivalent:*

$$\exists \theta \in [0, 2\pi]^M, \; \exists R \in [0, +\infty)^M \text{ such that } \textstyle\sum_{j \in J^*} b_j e^{+\gamma_j \cdot R} e^{-i\gamma_j \cdot \theta} = 0$$
$$\text{whereby } J^* \text{ is a nested dominant set w.r.t. } J_{\max}$$
$$\Updownarrow$$
$$\exists \{r_n\}_{n \geq 1}, \; \{\lambda_n = e_n + if_n\}_{n \geq 1}$$
$$\text{with } \lim_{n \to \infty} e_n = -\infty, \; r_n \geq 0 \text{ and } \lim_{n \to \infty} \|r_n\| = 0 \text{ and such that}$$
$$H(\lambda_n, r_n, s) = 0$$

A proof can be found in appendix B.

6 Applications and examples

We illustrate the results obtained in this paper by means of two examples.

6.1 Boundary controlled PDE

This example and the phenomena which occur are also described in [6]. Consider

$$w_{xx} = w_{tt},\ 0 \le t < \infty,\ 0 \le x \le 1, \tag{31}$$

subject to the boundary conditions

$$\begin{cases} w(0,t) = 0 \\ w_x(1,t) = -kw_t(1,t-h) \end{cases} \tag{32}$$

where $h \ge 0$, $k > 0$. Formulas (31) and (32) describe the transversal movement of a beam clamped at one side stabilized by applying a force at the other side. h represents a small delay in the velocity feedback. To calculate the spectrum of (31) one looks for solutions of the form $w(x,t) = e^{\lambda t}z(x)$. Substituting such a solution in (31) and taking the boundary conditions into account, the following characteristic equation is obtained,

$$e^{\lambda h} + k\tanh(\lambda) = 0, \tag{33}$$

which can be rewritten as

$$1 + e^{-\lambda 2} + ke^{-\lambda h} - ke^{-\lambda(2+h)} = 0. \tag{34}$$

6.1.1 Analysis of the undelayed case

When $h = 0$, the eigenvalues are

$$\lambda = -\frac{1}{2}\mathrm{Log}\left(\frac{1+k}{1-k}\right) + i\pi l,\quad l \in \mathbb{Z},$$

where Log denotes the principal value of the logarithm. Because $c(0) = \Re(\mathrm{Log}\left(\frac{1+k}{1-k}\right)) < 0$ for $k > 0$ the undelayed system is stable, i.e. all eigenvalues lie in the left half plane. When k approaches 1, the real part of the eigenvalues moves to $-\infty$, which indicates superstability. This can be explained as follows: the general solution of (31) can be written as a combination of two travelling waves, a solution $\phi(x-t)$ moving to the right and a solution $\psi(x+t)$ moving to the left. When $k = 1$, $\phi(x-t)$ satisfies the second boundary condition, and thus the reflection coefficient at $x = 1$ is zero; at $x = 0$ the wave $\phi(x+t)$ is reflected completely. Consequently all perturbations disappear in a finite time (at most 2 time-units).

6.1.2 Analysis for arbitrary small delays

Equation (34) is very interesting from a theoretical point of view. The three delays h, 2 and $2+h$ are function of only two independent delays, 2 and h. When $h \to 0$ there is both an arbitrary small delay and the largest delays asymptotically coincide.

When the delays 2 and h are rationally independent, c_{f} can be calculated as the rightmost solution α of

$$1 + e^{-2\alpha} e^{-i\theta_1} + k e^{-i\theta_2} - k e^{-2\alpha} e^{-i(\theta_1+\theta_2)} = 0 \tag{35}$$

which, after multiplication with $e^{2\alpha} e^{i(\theta_1+\theta_2)}$, can be rewritten as

$$e^{2\alpha} e^{i\theta_1} = \frac{k - e^{i\theta_2}}{k + e^{i\theta_2}}$$

and can be interpreted for each α as the intersection points of two circles in the complex plane. Note that in this case the obtained upper bound will equal the upper bound calculated when all delays are considered independent:

$$|1-k| - e^{-2c_{\mathrm{f}}} - k e^{-2c_{\mathrm{f}}} = 0. \tag{36}$$

Indeed (35) is transformed in (36) when choosing $\theta_1 = \theta_2 = \pi$ if $k < 1$ and $\theta_1 = 0$, $\theta_2 = \pi$ if $k > 1$. Thus the upper bound c_{f} in the case of rationally independent delays satisfies

$$c_{\mathrm{f}} = \frac{1}{2} \mathrm{Log} \left(\frac{1+k}{1-k} \right). \tag{37}$$

Following theorem 4.2 there are solutions with arbitrary large real part (for arbitrary small delays) if $\exists R \in (0, +\infty)$ and $\theta \in [0, 2\pi]$ such that $1 + k e^{-R} e^{-i\theta} = 0$. This is the case when $k > 1$. These solutions have nonzero imaginary part (no solution with $\theta = 0$). When $k = 1$ we are dealing with a degenerate case: equation (34) becomes (27) for which the existence of an infinite spectrum is shown in 4.5.

Theorem 5.1 allows to check the possibility of having eigenvalues tending to $-\infty$. $-k e^{R} e^{-i\theta} = 0$ has no solution but $1 - k e^{R} e^{-i\theta} = 0$ has a solution with $R \in [0, +\infty)$ when $k \leq 1$.

6.1.3 Discussion

The obtained results are summarized in figure 9 and one can make the following conclusions:

- When one doesn't take the delay h into account in the model, the spectrum lies in the left half plane, although an infinitesimal delay leads to unstable eigenvalues for each value of k. This illustrates once more that a controller design based on an undelayed model is inherently unsafe.
- When $k > 1$ unstable eigenvalues with arbitrary large real part occur for arbitrary small delays.
- When $k \to 1$ the supremum of the real parts of the finite part of the spectrum, c_{f}, moves to $+\infty$. When $k \approx 1$ the predicted zeros based on a non-delayed model are very stable while the positions of the actual zeros (small delays) indicate strong instability.

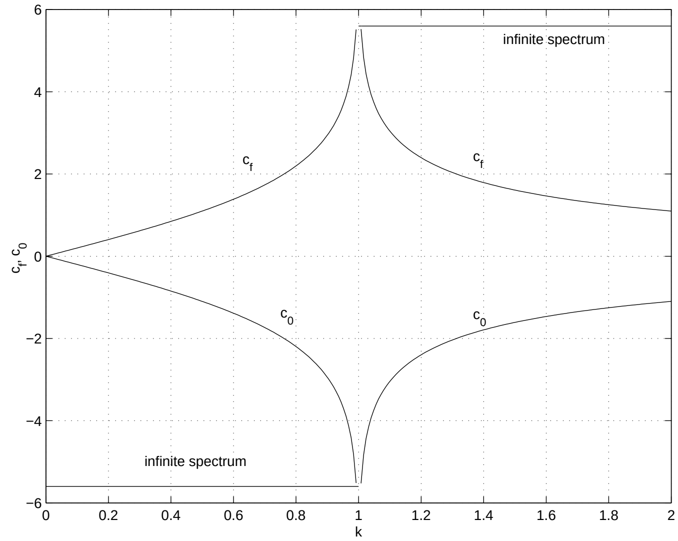

Figure 9: Upper bounds on the spectrum of (31): $c(0) = c_0$ is the real part of the eigenvalues in the undelayed case, c_{f} the upper bound on the finite part of the spectrum for an arbitrary small delay, $h = 0+$.

6.2 Bifurcation diagram: parameter dependence

As a last example we discuss the position of the eigenvalues of the characteristic equation

$$1 + ae^{-\lambda h} + be^{-\lambda 2h} \tag{38}$$

as a function of parameters a en b. Note that (5) is a special case of (38).

When h and $2h$ can be considered as dependent delays, the eigenvalues of (38) can be calculated from:

$$e^{-\lambda h} = \frac{-a \pm \sqrt{a^2 - 4b}}{2b} \tag{39}$$

Hence, when $a^2 - 4b < 0$ (> 0) the spectrum is situated on one (two) vertical lines in the complex plane. The position of these lines depend on the parameters. For $a^2 - 4b > 0$ one can show that such a line crosses the imaginary axis when $a = b + 1$ and when $a = -b - 1$, indicating a change of stability of the line under consideration. When $a^2 - 4b < 0$ the single line crosses the imaginary axis when $b = 1$. The corresponding curves in two-parameter space (a, b) are shown in figure 10.

When h and $2h$ should be considered independent. the upper bound $c(h)$ can be calculated from

$$1 - |a|e^{-c(h)h} - |b|e^{-2c(h)h} = 0,$$

and thus the system is stable if and only if $|a| + |b| \leq 1$.

Comparing the dependent with the independent case, one can see that the dangerous area in parameter space, i.e. the parameter values for which small changes in the delays destroy stability, is enclosed by the triangles $(0, 1), (1, 0), (1, 2)$ and $(0, -1), (1, 0), (1, -2)$ (see figure 10).

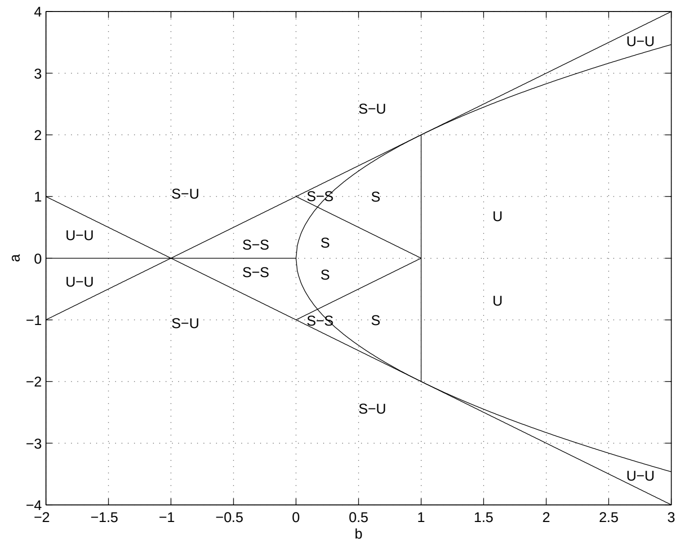

Figure 10: Stability areas of equation (38). When the delays are dependent, 'S' denotes a stable vertical line of eigenvalues in the complex plane and 'U' an unstable line. When the delays are independent, the system is stable for parameter values inside the curve $|a| + |b| = 1$

The characteristic equation of (38) corresponds to the small-delay part of the more general equation

$$1 + ae^{-\lambda h} + be^{-\lambda 2h} + de^{-\lambda} = 0 \tag{40}$$

as $h \to 0$. It is easy to see that in the case of independent delays (40) has eigenvalues with real part $\to \infty$ for parameter values a en b outside the curve $|a|+|b| = 1$ and that there are arbitrary unstable eigenvalues with small imaginary part for $(a + b + 1 \leq 0) \cap (4b - a^2 \leq 0)$. The finite part of the spectrum tends to infinity when approaching the curves $a = b+1$, $a = -b-1$ and $b = 1, -2 \leq a \leq 2$. All eigenvalues of (40) are in the left half plane if and only if $|a| + |b| \leq 1$ and $|d| \leq 1 - |a| - |b|$.

7 Conclusions

Sensitivity of neutral functional differential equations to infinitesimal changes of the delays is caused by the behaviour of the essential spectrum which is determined by the zeros of an exponential polynomial. A remarkable conclusion of the theory developed in [1, 7], concerning the zeros of exponential polynomials, is that the supremum of the real parts of the spectrum can change discontinuously w.r.t. the delays whereas the individual eigenvalues move continuously. In a first part of this paper the underlying mechanisms are interpreted and explained by means of spectral plots. For example, when rationally independent delays approach rationally dependent delays, this gives rise to a pointwise and non-uniform convergence of the spectrum, whereby the sensitivity of an eigenvalue increases as its modulus increases.

In a second part, we extend the theory developed in [1] to the case where some of the delays are arbitrary small, which can result in eigenvalues with arbitrary large real part. Sufficient and necessary conditions are provided. Thereby we also treat the special case of eigenvalues with large real part but small imaginary part. We further show that when the small delays are brought to zero in a fixed ratio, the structure of the set of eigenvalues with large real part depends strongly on this ratio. In the case of eigenvalues with small real part it even determines the occurrence of the phenomenon. Whether this is also in general the case is an interesting open question.

The paper ends with two illustrative examples. For instance the model described in [6] is treated systematically.

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A Lemma

We formulate a lemma which is used in the proof of theorem 4.2.

Lemma A.1 *Let $f(\lambda)$ and the sequence $\{f_n(\lambda)\}_{n\geq 1}$ be analytic functions on an (open) domain D. Suppose that $\{f_n(\lambda)\}_{n\geq 1}$ converges uniformly to $f(\lambda)$ on the disc $\{\lambda : |\lambda| \leq R\} \subset D$ for some $R \in \mathbb{R}^+$ and that on this disc $f(\lambda)$ only has a zero in $\lambda = 0$ with multiplicity k.*
Then there exists a number $N \in \mathbb{N}$ such that $\forall n \geq N$,
$f_n(\lambda)$ has exactly k zeros $\lambda_{n,1}, \dots, \lambda_{n,k}$ in $|\lambda| \leq R$ whereby $\lim_{n\to\infty} \lambda_{n,j} = 0$, $\forall j \in \{1, \dots, k\}$.

This lemma is a modification of Hurwitz's theorem, see e.g. [3].

Proof Consider for some $0 < r \leq R$ the curve $\Gamma : [0, 2\pi] \to \mathbb{C} : t \to \Gamma(t) = re^{it}$.
The function $|f(\lambda)|$ attains a minimum M on Γ whereby $M > 0$.
Because the sequence of functions $f_n(\lambda)$ is uniformly converging,

$$\exists N \text{ such that } \forall n \geq N : |f_n(\lambda) - f(\lambda)| < M \leq |f(\lambda)|$$

on Γ.
Consequently

$$|1 - \frac{f_n(\lambda)}{f(\lambda)}| < 1$$

on Γ.
For each $n \geq N$ the curve $\gamma(t) = \frac{f_n(re^{it})}{f(re^{it})}, \quad t \in [0, 2\pi]$ satisfies

$$|1 - \gamma_n(t)| < 1$$

and because it can be embedded in a closed disc not containing the origin the winding number of the curve w.r.t. the origin, $n(\gamma_n, 0)$, is zero:

$$n(\gamma_n, 0) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{d\lambda}{\lambda} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma_n'(t)}{\gamma_n(t)} dt = 0.$$

Using the definition of $\gamma_n(t)$, $\frac{1}{2\pi i}\int_0^{2\pi}\frac{\gamma_n'(t)}{\gamma_n(t)}dt$ can be written as $\frac{1}{2\pi i}\int_\Gamma\left(\frac{f_n'(\lambda)}{f_n(\lambda)}-\frac{f'(\lambda)}{f(\lambda)}\right)d\lambda$. This integral is well defined because from the previous it follows that neither f_n nor f are zero in $\{\lambda : r-\epsilon_n<|\lambda|<r+\epsilon_n\}$ for some $\epsilon_n\in\mathbb{R}^+$.

But this implies

$$\frac{1}{2\pi i}\int_\Gamma\frac{f'}{f}d\lambda=\frac{1}{2\pi i}\int_\Gamma\frac{f_n'}{f_n}d\lambda,\quad \forall n\geq N.$$

Following the theorem of the principle of the argument (see for example [14]) , these integrals correspond to the zero-pole excess of f and f_n (number of zeros-number of poles) inside Γ and in this case to the number of zeros. When taking $r=R$ the first statement of the lemma is proven. The second statement follows from the fact that r can be chosen arbitrary small. □

B Proof of theorem 5.1

Proof of ⇓ Suppose

$$\sum_{j\in J^*}b_je^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}=0$$

whereby for simplicity we assume the following (defining) structure for J^*: J_1 is dominant w.r.t. $J_{\max}$ with R_1, $J^*=J_2$ is dominant w.r.t. J_1 with R_2. As will follow from the proof this involves no loss of generality.

Define a sequence $\{u_n\}_{n\geq1}$ with

$$u_n=\frac{R_1}{n}+\frac{R_2}{n^2}+\frac{R}{n^3}$$

and a strictly positive sequence $\{\epsilon_n\}_{n\geq1}$ with $\lim_{n\to\infty}\epsilon_n=0$. Choose $\{r_n\}_{n\geq1}$ with rationally independent components such that

$$\|n^3(r_n-u_n)\|<\epsilon_n.$$

Because $\{r_n\}_{n\geq1}$ has rationally independent components, due to Kronecker's theorem:

$$\exists\,\{v_{n,m}\}_{m\geq1}\ \text{such that}\ \lim_{m\to\infty}e^{-i\gamma_j\cdot(v_{n,m}r_n-\theta)}=1,\ \forall j\in J,$$

hence $\exists m^*(n)$ such that $|e^{-i\gamma_j\cdot(v_{n,m^*(n)}r_n-\theta)}-1|<\epsilon_n$, $\forall j\in J$. Set $d_n=v_{n,m^*(n)}$. We have created $\{u_n\}_{n\geq1}$, $\{r_n\}_{n\geq1}$ and $\{d_n\}_{n\geq1}$ with

$$\begin{array}{c}\|n^3(u_n-r_n)\|<\epsilon_n,\\ |e^{i\gamma_j\cdot(d_nr_n-\theta)}-1|<\epsilon_n,\ \forall j\in J,\\ \lim_{n\to\infty}\epsilon_n=0.\end{array}$$

The decay rate of ϵ_n to zero, which can be chosen arbitrary fast, will be determined later in the proof.

Define a sequence of real parts $\{c_n\}_{n\geq1}$ with $c_n=-n^3$. Let's now focus on the behaviour of the residual:

$$H(\lambda,r_n,s)=1-\sum_{i\in I}a_ie^{-\lambda\gamma_i\cdot r_n}-\sum_{j\in J_{\max}}b_je^{-\lambda(\gamma_j\cdot r_n+\nu_{\max}\cdot s)}-\sum_{j\in J\setminus J_{\max}}b_je^{-\lambda(\gamma_j\cdot r_n+\nu_j\cdot s)}.$$

Obviously the scaled residual

$$\begin{aligned}&e^{\nu_{\max}\cdot s\lambda_n}H(c_n+id_n,r_n,s)=e^{-\nu_{\max}\cdot sn^3}e^{i\nu_{\max}\cdot sd_n}H(c_n+id_n,r_n,s)\\&=-\textstyle\sum_{j\in J}b_je^{-\lambda(\gamma_j\cdot r_n)}+O(e^{-(\nu_{\max}-\nu_2)\cdot sn^3}),\end{aligned}$$

whereby $\nu_2\cdot s<\nu_{\max}\cdot s$ is the largest term in $\{\nu_j\cdot s:\ j\in J\setminus J_{\max}\}$. By a suitable choice of $\{\epsilon_n\}_{n\geq 1}$ the first term which equals

$$-\sum_{j\in J_{\max}}b_je^{-\gamma_j\cdot(c_nu_n+i\theta)}\underbrace{e^{\gamma_j\cdot(c_nu_n-c_nr_n)}e^{-i\gamma_j\cdot(d_nr_n-\theta)}}_{\to 1}$$

converges arbitrary fast to

$$-\sum_{j\in J_{\max}}b_je^{\gamma_j\cdot(R_1n^2+R_2n+R)-i\gamma_j\cdot\theta}.$$

and thus $\{\epsilon_n\}$ can be chosen s.t.

$$e^{\nu_{\max}\cdot s\lambda_n}H(c_n+id_n,r_n,s)=-\sum_{j\in J_{\max}}b_je^{\gamma_j\cdot(R_1n^2+R_2n+R)-i\gamma_j\cdot\theta}+O(e^{-(\nu_{\max}-\nu_2)\cdot sn^3})$$

J_1 is dominant w.r.t. $J_{\max}$ with R_1, thus for some $\bar{\jmath}\in J_1$,

$$\begin{aligned}&e^{\nu_{\max}\cdot s\lambda_n}e^{-\gamma_{\bar{\jmath}}\cdot R_1n^2}H(c_n+id_n,r_n,s)\\&=-\textstyle\sum_{j\in J_1}e^{\gamma_j\cdot(R_2n+R)}e^{-i\gamma_j\cdot\theta}+O(e^{-\alpha n^2})\end{aligned}$$

with $\alpha>0$. We can now repeat this procedure: J_2 is dominant w.r.t J_1 with R_2, thus for some $\tilde{\jmath}\in J_2$,

$$\begin{aligned}&e^{\nu_{\max}\cdot s\lambda_n}e^{-\gamma_{\bar{\jmath}}\cdot R_1n^2}e^{-\gamma_{\tilde{\jmath}}\cdot R_2n}H(c_n+id_n,r_n,s)\\&=-\textstyle\sum_{J_2=J^*}e^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}+O(e^{-\beta n})=O(e^{-\beta n})\end{aligned}$$

with $\beta>0$.

Up to now we have proven that the function

$$\bar{H}_n(\lambda)=ne^{\nu_{\max}\cdot s\lambda_n}e^{-\gamma_{\bar{\jmath}}\cdot R_1n^2}e^{-\gamma_{\tilde{\jmath}}\cdot R_2n}H(c_n+id_n,r_n,s)$$

satisfies:

$$\bar{H}_n(c_n+id_n)=O(ne^{-\beta n^2})\to 0\text{ as }n\to\infty.$$

At this point the proof is not complete but if one can show that $\bar{H}_n(\lambda)$ has a zero near c_n+id_n for large n, so does $H(\lambda,r_n,s)$. This will be proven in the following way: we will show that the local approximation of $\bar{H}_n(\lambda)$ around c_n+id_n converges uniformly (as $n\to\infty$) to a function $\bar{H}$ with a zero in order to apply lemma A.1. Let's first define $\bar{H}$.

In the same way as we considered the asymptotic behaviour of $\bar{H}_n(c_n+id_n)$ we can show that $\frac{d^k\bar{H}_n}{d\lambda^k}$, $k\geq 1$ which equals

$$\begin{aligned}&\tfrac{d^k\bar{H}_n}{d\lambda^k}\\&=ne^{\nu_{\max}\cdot s\lambda_n-\gamma_{\bar{\jmath}}\cdot R_1n^2-\gamma_{\tilde{\jmath}}\cdot R_2n}\Big(-\textstyle\sum_{i\in I}a_i(-\gamma_i\cdot r_n)^ke^{-\gamma_i\cdot r_n}e^{-\gamma_i\cdot d_nr_n}\\&-\textstyle\sum_{j\in J}b_j(-\gamma_j\cdot r_n-\nu_j\cdot s)^ke^{-\gamma_j\cdot c_nr_n}e^{-i\gamma_j\cdot d_nr_n}\Big)\end{aligned}$$

converges to

$$\sum_{j\in J^*} b_j\gamma_j \cdot R_1 k(-\nu_{\max}\cdot s)^{k-1} e^{\gamma_j\cdot R} e^{-i\gamma_j\cdot\theta} = Qk(-\nu_{\max}\cdot s)^{k-1}, \quad k\geq 1.$$

for some complex number Q because $nr_n \to R_1$ and $n\sum_{j\in J^*} b_j(-\nu_{\max}\cdot s)^k e^{\gamma_j\cdot R} e^{-i\gamma_j\cdot\theta} = 0$.

Important remark
Hereby we implicitly assumed that $Q\neq 0$. (In the non generic case where $Q=0$ one can consider the behaviour $n\bar{H}_n(\lambda)$. When this gives rise to derivatives tending to zero one can study $n^2\bar{H}_n(\lambda)\ldots$)

Because for each n, $\bar{H}_n(\lambda)$ is an analytic function it has a Taylor-expansion around $\bar{\lambda}_n = c_n + id_n$:

$$\bar{H}_n(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\left.\frac{d^k\bar{H}_n}{d\lambda^k}\right|_{c_n+id_n} (\lambda-\bar{\lambda}_n)^k,$$

and when defining $\mu \triangleq \lambda - \bar{\lambda}_n$ this can be written as:

$$\bar{H}_n(\mu) = \sum_{k=0}^{\infty} \frac{1}{k!}\left.\frac{d^k\bar{H}_n}{d\lambda^k}\right|_{c_n+id_n} \mu^k$$

We define $\bar{H}(\mu)$ in such a way that

$$\left.\frac{d^k\bar{H}}{d\mu^k}\right|_0 = \lim_{n\to\infty}\left.\frac{d^k\bar{H}_n}{d\lambda^k}\right|_{c_n+id_n}, \quad k\geq 0.$$

$\bar{H}(\mu)$ is an analytic function and its Taylor expansion around $\mu=0$ is given by

$$\begin{aligned}\bar{H}(\mu) &= \textstyle\sum_{k=0}^{\infty}\frac{1}{k!}\left.\frac{d^k\bar{H}}{d\mu^k}\right|_0 \mu^k\\ &= K\textstyle\sum_{k=1}^{\infty}\frac{1}{k!}k(-\nu_{\max}\cdot s)^{k-1}\mu^k\\ &= K\mu e^{-\nu_{\max}\cdot s\mu}.\end{aligned}$$

Thus the function $\bar{H}(\mu)$ has a zero in $\mu=0$. In order to apply lemma A.1 we will now show that the series of functions $\bar{H}_n(\mu)$ converges uniformly to $\bar{H}(\mu)$ in $|\mu|\leq 1$. Hereby it should be emphasized that μ is in fact a local coordinate which depend on n; in other words we will prove that for large n the behaviour of $\bar{H}_n(\lambda)$ around $\lambda_n = c_n + id_n$ corresponds to the behaviour of $\bar{H}(\mu)$. In the non generic case $Q=0$ the functions $n\bar{H}(n)$ will converge to a function with a double zero in $\mu=0$.)

For each μ with $|\mu| \leq 1$,

$$|\bar{H}_n(\mu) - \bar{H}(\mu)|$$
$$= \left| (\bar{H}_n - \bar{H})\big|_0 + \frac{(\frac{d\bar{H}_n}{d\mu} - \frac{d\bar{H}}{d\mu})\big|_0}{1!}\mu + \dots \right|$$
$$\leq \underbrace{\left| \sum_{i\in I} n e^{\nu_{\max}\cdot s\lambda_n - \gamma_{\bar{j}}\cdot R_1 n^2 - \gamma_{\bar{j}}\cdot R_2 n} a_i e^{-\gamma_i \cdot c_n r_n} e^{-i\gamma_i \cdot d_n r_n} \sum_{k=0}^{\infty} \frac{1}{k!}(-\gamma_i \cdot r_n)^k \mu^k \right|}_{A}$$
$$+ \underbrace{\left| \sum_{j\in J\setminus J^*} n e^{\nu_{\max}\cdot s\lambda_n - \gamma_{\bar{j}}\cdot R_1 n^2 - \gamma_{\bar{j}}\cdot R_2 n} b_j e^{-c_n(\gamma_j\cdot r_n + \nu_j \cdot s)} e^{-i\gamma_j\cdot d_n r_n - i\nu_j\cdot d_n s} \sum_{k=0}^{\infty} \frac{1}{k!}(-\gamma_j \cdot r_n - \nu_j \cdot s)^k \mu^k \right|}_{B}$$
$$+ \underbrace{\left| \sum_{j\in J^*} n e^{-\nu_{\max}\cdot sn^3 - \gamma_{\bar{j}}\cdot R_1 n^2 - \gamma_{\bar{j}}\cdot R_2 n} b_j \left(e^{-c_n(\gamma_j\cdot r_n + \nu_j\cdot s)} e^{-i\gamma_j\cdot d_n r_n} - e^{-(\gamma_j\cdot c_n u_n + \nu_j\cdot c_n s)} e^{-i\gamma_j\cdot\theta}\right) \sum_{k=0}^{\infty} \frac{1}{k!}(-\gamma_j \cdot r_n - \nu_j \cdot s)^k \mu^k \right|}_{C}$$
$$+ \underbrace{\left| \sum_{j\in J^*} n b_j e^{\gamma_j\cdot R} e^{-i\gamma_j\cdot\theta} \sum_{k=0}^{\infty} \frac{1}{k!}(-\gamma_j\cdot r_n - \nu_{\max}\cdot s)^k \mu^k + \sum_{j\in J^*} b_j e^{\gamma_j\cdot R} e^{-i\gamma_j\cdot\theta} \sum_{k=0}^{\infty} \frac{1}{k!}\gamma_j \cdot R_1 k(-\nu_{\max}\cdot s)^{k-1}\mu^k \right|}_{D}$$

and we will show that all these terms can be bounded uniformly on $|\mu| \leq 1$ whereby this bound tends to zero as $n \to \infty$ which proves the uniform convergence.
Thereby we use the following notation: because $\{r_n\}_{n\geq 1}$ converges to zero, for all $i \in I$ and $J \in J$ the sequences $\{\gamma_i \cdot r_n\}$ and $\{\gamma_j \cdot r_n\}$ can be bounded uniformly for $n \geq P$ with P a fixed integer number and we define Δ_i and Δ_j such that $\forall n \geq P$,

$$\begin{array}{ll} \gamma_i \cdot r_n \leq \Delta_i & i \in I \\ \gamma_j \cdot r_n \leq \Delta_j & j \in J. \end{array}$$

<u>Term A</u> This term can be bounded by

$$\sum_{i\in I} n e^{-\nu_{\max}\cdot sn^3} |a_i| e^{-\gamma_i\cdot c_n r_n} e^{\Delta_i}$$

and this bound tends to zero as $n \to \infty$ because $c_n r_n = O(-n^2)$.
<u>Term B</u> Using the triangle inequality this term

$$\left| \sum_{j\in J\setminus J^*} \underbrace{n e^{\nu_{\max}\cdot s\lambda_n - \gamma_{\bar{j}}\cdot R_1 n^2 - \gamma_{\bar{j}}\cdot R_2 n} b_j e^{-c_n(\gamma_j\cdot r_n + \nu_j\cdot s)} e^{-i\gamma_j\cdot d_n r_n - i\nu_j\cdot d_n s} \sum_{k=0}^{\infty} \frac{1}{k!}(-\gamma_j\cdot r_n - \nu_j\cdot s)^k \mu^k}_{T} \right|$$

is smaller than

$$\left| \sum_{j\in J\setminus J_{\max}} T \right| + \left| \sum_{j\in J_{\max}\setminus J_1} T \right| + \left| \sum_{j\in J_1\setminus J_2} T \right|.$$

$\left|\sum_{j\in J\setminus J_{\max}} T\right|$ can be bounded by

$$\sum_{j\in J\setminus J_{\max}} ne^{-\nu_{\max}\cdot sn^3}|b_j|e^{-\gamma_j\cdot c_n r_n}e^{-c_n\nu_j\cdot s}e^{\nu_j\cdot s+\Delta_j}$$

and the bound approaches zero as $n\to\infty$ because for $j\in J\setminus J_{\max}$, $\nu_j\cdot s<\nu_{\max}\cdot s$. $\left|\sum_{j\in J_{\max}\setminus J_1} T\right|$ is bounded by

$$\sum_{j\in J_{\max}\setminus J_1} ne^{-\gamma_{\bar{j}}\cdot R_1 n^2}|b_j|e^{-\gamma_j\cdot c_n r_n}e^{\nu_j\cdot s+\Delta_j}$$

and this term tends to zero because $\lim_{n\to\infty}\gamma_{\bar{j}}\cdot R_1 n^2-|\gamma_j\cdot c_n r_n|=\infty$ ($J_{\max}\setminus J_1$ is not dominant w.r.t. $J_{\max}$ with R_1). $\left|\sum_{j\in J_1\setminus J_2} T\right|$ is bounded by

$$\sum_{j\in J_1\setminus J_2} ne^{-\gamma_{\bar{j}}\cdot R_1 n^2}e^{-\gamma_{\bar{j}}\cdot R_2 n}|b_j|e^{-\gamma_j\cdot c_n r_n}e^{\nu_j\cdot s+\Delta_j}$$

and because of the definition of J_1 and no dominance w.r.t. R_2, $\lim_{n\to\infty}\gamma_{\bar{j}}\cdot R_1 n^2+\gamma_{\bar{j}}\cdot R_2 n-|c_n r_n|=\infty$. Consequently also this term vanishes.
<u>Term C</u> This term is bounded by

$$\sum_{j\in J^*} ne^{-\gamma_{\bar{j}}\cdot R_1 n^2-\gamma_{\bar{j}}\cdot R_2 n}|b_j|\left|e^{-\gamma_j\cdot c_n r_n}e^{-i\gamma_j\cdot d_n r_n}-e^{-\gamma_j\cdot c_n u_n}e^{-i\gamma_j\cdot\theta}\right|e^{\nu_{\max}\cdot s+\Delta_j}$$

and by a suitable choice of $\{\epsilon_n\}_{n\geq 1}$ which determines the convergence $u_n\to r_n$ and $d_n r_n \mathrm{mod} 2\pi\to\theta$ this bound converges to zero.
<u>Term D</u>

$$\begin{aligned}
&\left|\sum_{j\in J^*} b_j e^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}n\sum_{k=0}^{\infty}\tfrac{1}{k!}(-\gamma_j\cdot r_n-\nu_{\max}\cdot s)^k\mu^k\right.\\
&\left.+\sum_{j\in J^*} b_j e^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}\sum_{k=1}^{\infty}\tfrac{1}{k!}\gamma_j\cdot R_1 k(-\nu_{\max}\cdot s)^{k-1}\mu^k\right|\\
&=\left|\sum_{j\in J^*} b_j e^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}\left(ne^{-(\gamma_j\cdot r_n+\nu_{\max}\cdot s)\mu}+\gamma_j\cdot R_1\mu e^{-\nu_{\max}\cdot s\mu}\right)\right|\\
&=\left|\sum_{j\in J^*} b_j e^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}e^{-\nu_{\max}\cdot s\mu}\left(n+n(e^{-\gamma_j\cdot r_n\mu}-1)+\gamma_j\cdot R_1\mu\right)\right|\\
&=\left|\sum_{j\in J^*} b_j e^{\gamma_j\cdot R}e^{-i\gamma_j\cdot\theta}e^{-\nu_{\max}\cdot s\mu}\left(n(e^{-\gamma_j\cdot r_n\mu}-1)+\gamma_j\cdot R_1\mu\right)\right|\\
&\leq\sum_{j\in J^*}|b_j|e^{\gamma_j\cdot R}e^{\nu_{\max}\cdot s}\left|n(e^{-\gamma_j\cdot r_n\mu}-1)+\gamma_j\cdot R_1\mu\right|
\end{aligned}$$

Once again this upperbound tends to zero as $n\to\infty$. Indeed

$$\begin{aligned}
&\lim_{n\to\infty}|n(e^{-\gamma_j\cdot r_n\mu}-1)+\gamma_j\cdot R_1\mu|\\
&=\lim_{n\to\infty}\left|n(-\gamma_j\cdot r_n\mu+\tfrac{(\gamma_j\cdot r_n)^2}{2!}\mu^2+\ldots)+\gamma_j\cdot R_1\mu\right.\\
&\left.-\gamma_j\cdot nr_n\mu+\gamma_j\cdot nr_n\mu\right|\\
&\leq\lim_{n\to\infty}n\sum_{k=2}^{\infty}\tfrac{n^k(\gamma_j\cdot r_n)^k}{n^k k!}+\lim_{n\to\infty}|\gamma_j\cdot nr_n-\gamma_j\cdot R_1|\\
&\leq\lim_{n\to\infty}\tfrac{1}{n}\sum_{k=0}^{\infty}\tfrac{(\gamma_j\cdot nr_n)^k}{k!}+\lim_{n\to\infty}|\gamma_j\cdot nr_n-\gamma_j\cdot R_1|\\
&\leq\lim_{n\to\infty}\tfrac{1}{n}e^{\gamma_j\cdot nr_n}+\lim_{n\to\infty}|\gamma_j\cdot nr_n-\gamma_j\cdot R_1|\\
&=0.
\end{aligned}$$

Concluding we proved that the sequence of functions $\bar{H}_n(\mu)$ converges unformly on $|\mu| \leq 1$ to $\bar{H}(\mu) = K\mu e^{-\nu_{\max}\cdot\mu}$ which has a zero in $\mu = 0$. When applying lemma A.1, there must be an $N \in \mathbb{N}$ such that for all $n \geq N$, $\exists g_n + ih_n$ such that $\bar{H}_n(\mu = g_n + ih_n) = 0$ and $\lim_{n\to\infty} g_n + ih_n = 0$. But this implies that for $n \geq N$, $H((c_n+g_n)+i(c_n+h_n), r_n, s) \triangleq H(\lambda_n, r_n, s) = 0$ which, after renumbering the indices, completes the proof.

Proof of $\Uparrow$ Consider the sequences

$$\{e_n r_n\}_{n\geq 1},\ \{f_n r_n\}_{n\geq 1}.$$

Define the index set $I \subseteq \{1, M\}$ whereby for all $i \in I$ not all $\gamma_{j,i}$, $\ j \in J_{\max}$, are zero and the subset of row indices S_1 such that

$$\{(f_n r_{n,k}) \bmod 2\pi\}_{n\geq 1, n\in S_1}$$

converges for each $k \in I$,

$$\lim_{n\to\infty} e_n r_{n,k}$$

exists for all $k \in I$ and

$$\lim_{n\to\infty, n\in S_1} \frac{e_n r_{n,k}}{e_n r_{n,l}}$$

exists for each value of k and $l \in I$. This is always possible because firstly when a sequence is unbounded there exists a subsequence with limit infinity and secondly when a sequence is bounded there exists a converging subsequence (Weierstrass-Bolzano).

Starting from

$$e^{\nu_{\max}\cdot s e_n}\left(1 - \sum_{i\in I} a_i e^{-(e_n+if_n)\gamma_i\cdot r_n} - \sum_{j\in J_{\max}} b_j e^{-(e_n+if_n)(\gamma_j\cdot r_n+\nu_j\cdot s)} - \sum_{j\in J\setminus J_{\max}} b_j e^{-(e_n+if_n)(\gamma_j\cdot r_n+\nu_j\cdot s)}\right) = 0$$

and because

$$\lim_{n\to\infty} e^{\nu_{\max}\cdot s e_n}\left(1 - \sum_{i\in I} a_i e^{-(e_n+if_n)\gamma_i\cdot r_n} - \sum_{j\in J\setminus J_{\max}} b_j e^{-(e_n+if_n)(\gamma_j\cdot r_n+\nu_j\cdot s)}\right) = 0$$

we end up with

$$\lim_{n\to\infty} \sum_{j\in J_{\max}} b_j e^{-e_n\gamma_j\cdot r_n} e^{-i\gamma_j\cdot f_n r_n} = 0 \tag{41}$$

Make a partition of index set $I =$ into I_1 and I_2 such that for each $i, j \in I_1$ and $k \in I_2$:

$$\lim_{n\to\infty, n\in S_1} \frac{e_n r_{n,i}}{e_n r_{n,j}} = K_{i,j} > \lim_{n\to\infty, n\in S_1} \frac{e_n r_{n,k}}{e_n r_{n,j}} = 0$$

and

$$\lim_{n\to\infty, n\in S_1} e_n r_{n,i} = \infty.$$

with $0 < K_{i,j} < \infty$. Suppose at this point that such a (non empty) set I_1 exists. Define R_1 such that for $i \in \{1, M\} \setminus I_1$, $R_{1,i} = 0$ and for $i, j \in I_1$,

$$\frac{R_{1,i}}{R_{1,j}} = K_{i,j}.$$

Now $\sum_{j \in J_{\max}} b_j e^{-e_n \gamma_j \cdot r_n} e^{-i\gamma_j \cdot f_n r_n}$ can be split in $\sum_{j \in J_1} b_j e^{-e_n \gamma_j \cdot r_n} e^{-i\gamma_j \cdot f_n r_n}$ and $\sum_{j \in J_{\max} \setminus J_1} b_j e^{-e_n \gamma_j \cdot r_n} e^{-i}$ whereby J_1 is dominant w.r.t. $J_{\max}$ with R_1.

Consider the sequences $\{\frac{e_n r_{n,i}}{R_{1,i}} - \frac{e_n r_{n,j}}{R_{1,j}}\}$ for $i, j \in I_1$ and define $S_2 \subseteq S_1$ such that

$$\lim_{n \to \infty, n \in S_2} \frac{e_n r_{n,i}}{R_{1,i}} - \frac{e_n r_{n,j}}{R_{1,j}}$$

exists for $i, j \in I_1$ and define a $\bar{k} \in I_1$ such that $\lim_{n \to \infty, n \in S_2} \left(\frac{e_n r_{n,\bar{k}}}{R_{1,\bar{k}}} - \frac{e_n r_{n,j}}{R_{1,j}} \right) \leq 0$ for all j $\in I_1$.

To remove the fastest growing component one can multiply equation (41) with

$$e^{\gamma_j \cdot \{e_n r_{n,\bar{k}} \frac{R_{1,i}}{R_{1,\bar{k}}}\}_{i=1}^{M}} = e^{\frac{e_n r_{n,\bar{k}}}{R_{1,\bar{k}}}(\gamma_j \cdot R_1)} \tag{42}$$

which is an expression *invariant* over each $j \in J_1$ because J_1 is dominant with R_1. Because (42) tends to zero as $n \to \infty$ this yields

$$\lim_{n \to \infty, n \in S_2} \sum_{j \in J_1} e^{\gamma_j \cdot \{e_n r_{n,i} - e_n r_{n,\bar{k}} \frac{R_{1,i}}{R_{1,\bar{k}}}\}_{i=1}^{M}} e^{-i\gamma_j \cdot f_n r_n} = 0 \tag{43}$$

Define now the sequences

$$\{q_{n,i}\}_{n \geq 1, n \in S_2} = \{e_n r_{n,i} - e_n r_{n,\bar{k}} \frac{R_{1,i}}{R_{1,\bar{k}}}\}_{n \geq 1, n \in S_2}, \quad i \in I'$$

whereby index set I' consists of all indices i such that not all $\gamma_{j,i}, \quad j \in J_1$ are zero. From the previous it follows that from a fixed number $N \in \mathbb{N}$ on the components of q_n are positive. Once again one can define a subset of S_2, S_3, such that for all $i, j \in I'$, $\lim_{n \to \infty, n \in S_3} q_{n,i}, \quad \lim_{n \to \infty, n \in S_3} \frac{q_{n,i}}{q_{n,j}}$ exist and define a partition of I' into I'_1 and I'_2 such that for $i, j \in I'_1$ and $k \in I'_2$,

$$\lim_{n \to \infty, n \in S_3} \frac{q_{n,i}}{q_{n,j}} = L_{i,j} > \lim_{n \to \infty, n \in S_3} \frac{q_{n,k}}{q_{n,j}} = 0$$

and

$$\lim_{n \to \infty, n \in S_3} q_{n,i} = \infty,$$

with $0 < L_{i,j} < \infty$.

If I'_1 is not empty one can define R_2 such that for $i \in \{1, M\} \setminus I'_1$, $R_{2,i} = 0$ and for $i, j \in I'_1$, $\frac{R_{2,i}}{R_{2,j}} = L_{i,j}$. From index set J_1 one can split off J'_1 which is dominant w.r.t. J_1 with R_2. Furthermore take a subset S_4 of S_3 such that $\lim_{n \to \infty, n \in S_4}(\frac{q_{n,i}}{R_{2,i}} - \frac{q_{n,j}}{R_{2,j}})$ converges, $\forall i, j \in I'_1$. Multiplying equation (43) with $e^{\frac{q_{n,\bar{l}}}{R_{2,\bar{l}}}(\gamma_j \cdot R_2)}$ whereby for each $j \in I'_1$, $\lim_{n \to \infty, n \in S_4}(\frac{q_{n,\bar{l}}}{R_{2,\bar{l}}} - \frac{q_{n,j}}{R_{2,j}}) \leq 0$ the following result is obtained:

$$\lim_{n \to \infty, n \in S_4} \sum_{j \in (J_1 \cap J'_1)} e^{\gamma_j \cdot \{q_{n,i} - q_{n,\bar{l}} \frac{R_{2,i}}{R_{2,\bar{l}}}\}_{i=1}^{M}} e^{-i\gamma_j \cdot f_n r_n} = 0.$$

By continuing the extraction of the fastest growing components, after a finite number of steps, we end up with

$$\lim_{n \to \infty, n \in S^*} \sum_{j \in (J^* = J_1 \cap J_1' \cap \ldots)} e^{\gamma_j \cdot (y_n)} e^{-i\gamma_j \cdot f_n r_n} = 0$$

whereby for all $i \in I^*$, the set of indices such that not all $\gamma_{j,i}$, $j \in J^*$ are zero, $\lim_{n \to \infty} y_n$ is a positive number. Finally we define

$$\begin{array}{ll} R_{,k} = \lim_{n \to \infty, n \in S^*} y_{n,k} & k \in I^* \\ R_{,k} = 0 & k \in I \setminus I^* \end{array}$$

and because the components of γ_j are integer numbers, $e^{-i\gamma_j \cdot f_n r_n} = e^{-i\gamma_j \cdot (f_n r_n) \mathrm{mod} 2\pi}$ and we set

$$\theta = \lim_{n \to \infty, n \in S^*} (f_n r_n) \mathrm{mod} 2\pi$$

By a continuity property we end up with

$$\sum_{j \in J^*} e^{\gamma_j \cdot R} e^{-i\gamma_j \cdot \theta} = 0.$$

which completes the proof. □