Taylor-Waadeland modifications of continued fractions

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Report TW 284, September 1998

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Abstract

In this paper we discuss a method for accelerating the convergence of continued fractions based on Taylor series expansions. This method is due to Haakon Waadeland who used the first three terms of the Taylor series expansion of the tails of a continued fraction to define a (convergence accelerating) modification of this continued fraction.

This paper was written for Haakon Waadeland's seventieth birthday.

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Some 10 years ago, at a conference in Antwerp organised by Annie Cuyt, Haakon Waadeland gave a talk entitled "Some recent results in the analytic theory of continued fractions" [10] (see also [6], [7], [8], [9]). In this talk Waadeland discussed a method of convergence acceleration for continued fractions based on Taylor series expansions. The idea is the following: we consider a continued fraction of the form

$$\operatorname*{K}_{n=1}^{\infty} \left(\frac{z_n}{1}\right) \tag{1}$$

with $z_i \in \mathbb{C}$. Such an expression defines a function of infinitely many variables: to see this let

$$\overline{z} = (z_1, z_2, \ldots, z_N, z_{N+1}, \ldots)$$

and then define

$$H^{(0)} \; : \; A \subset \mathbb{C}^{\infty} \to \; \mathbb{C} : \; \overline{z} \mapsto H^{(0)}(\overline{z}) = \operatorname*{K}_{n=1}^{\infty} \left(\frac{z_n}{1}\right)$$

where we assume that the continued fraction converges to a finite value for all $\overline{z} \in A$, an open subset of $\mathbb{C}^{\infty}$. Let us furthermore assume that

$$\overline{\delta} = (\delta_1, \delta_2, \ldots, \delta_N, \delta_{N+1}, \ldots)$$

The function $H^{(0)}$ has a Taylor series expansion which we can write in the following form:

$$H^{(0)}(\overline{z} + \overline{\delta}) = H^{(0)}(\overline{z}) + \sum_{i=1}^{\infty} \frac{\partial H^{(0)}}{\partial z_i}(\overline{z})\delta_i + \frac{1}{2!} \sum_{i,j=1}^{\infty} \frac{\partial^2 H^{(0)}}{\partial z_i \partial z_j}(\overline{z})\delta_i\delta_j + \ldots \tag{2}$$

If we use the notation:

$$H^{(N)}(\overline{z}) = \operatorname*{K}_{n=N+1}^{\infty} \left(\frac{z_n}{1}\right)$$

for the tails of the continued fraction (1), we can write down similar series expansions for these tails. In the rest of this paper we assume that all these tails converge to a finite value.

The classical method of obtaining an approximation to the value of the continued fraction (1) is by replacing the N-th tail $H^{(N)}$ by zero. Here the choice of the value zero is a rather arbitrary one: if an approximation to the value of the N-th tail is known, then it is better to use this approximation. In some cases this will accelerate the convergence of the continued fraction (1).

The partial sums of the Taylor series expansion of this N-th tail give us approximations for the value of the N-th tail. The only problem is to calculate these partial sums. To do this we have to calculate the partial derivatives in (2). In [10] this was done for the derivatives up to the second order. Waadeland used

$$H^{(0)}(\overline{z}) + \sum_{i=1}^{\infty} \frac{\partial H^{(0)}}{\partial z_i}(\overline{z})\delta_i$$

and

$$H^{(0)}(\overline{z}) + \sum_{i=1}^{\infty} \frac{\partial H^{(0)}}{\partial z_i}(\overline{z})\delta_i + \frac{1}{2!}\sum_{i,j=1}^{\infty} \frac{\partial^2 H^{(0)}}{\partial z_i \partial z_j}(\overline{z})\delta_i\delta_j$$

We will get expressions for the partial derivatives of the third order. For the notations used we refer to [10].
Starting from

$$H^{(0)} = \frac{A_n + A_{n-1}H^{(n)}}{B_n + B_{n-1}H^{(n)}}$$

and using

$$H^{(n)} = \frac{z_{n+1}}{1 + H^{(n+1)}}$$

$$A_{n-1}B_n - A_nB_{n-1} = (-1)^n \prod_{k=1}^{n} z_k$$

$$B_n + B_{n-1}H^{(n)} = \prod_{k=1}^{n}(1 + H^{(k)})$$

$$B_{n-1} = \sum_{k=1}^{n}(-1)^{n-k}\left(\prod_{\nu=k}^{n-1} H^{(\nu)}\right)\left(\prod_{\nu=1}^{k-1}(1 + H^{(\nu)})\right)$$

(see [10]) and

$$\frac{\partial B_{n-1}}{\partial z_{m+1}} = B_{m-1}B^{(m+1)}_{n-m-2} \quad (n-1 \geq m)$$

with $B^{(m)}_{-1} = 0, B^{(m)}_{0} = 1$, $B^{(m)}_{n+1} = B^{(m)}_{n} + z_{n+m+1}B^{(m)}_{n-1}$ (this is not difficult to prove using induction), we've recalculated some of the partial derivatives of $H^{(0)}$. Here they are, with

$$t_\nu = -\frac{H^{(\nu)}}{1 + H^{(\nu)}}$$

First order

$$\begin{aligned}\frac{\partial H^{(0)}}{\partial z_{n+1}} &= \frac{B_nA_{n-1} - A_nB_{n-1}}{(1 + H^{(n+1)})(B_n + B_{n-1}H^{(n)})^2} \\ &= \frac{H^{(0)}}{H^{(n)}(1 + H^{(n+1)})}\prod_{k=1}^{n} t_k\end{aligned}$$

Second order

$$\begin{aligned}\frac{\partial^2 H^{(0)}}{\partial z^2_{n+1}} &= (-2)\frac{B_nA_{n-1} - A_nB_{n-1}}{(1 + H^{(n+1)})^2(B_n + B_{n-1}H^{(n)})^3}B_{n-1} \\ &= \frac{-2\,H^{(0)}}{H^{(n)}(1 + H^{(n+1)})^2}\prod_{k=1}^{n}\left(-\frac{H^{(k)}}{(1 + H^{(k)})^2}\right)\cdot B_{n-1} \\ &= \frac{\partial H^{(0)}}{\partial z_{n+1}}\cdot\frac{2}{H^{(n)}(1 + H^{(n+1)})}\cdot\sum_{k=1}^{n}\prod_{\nu=k}^{n} t_\nu\end{aligned}$$

and for $n > m$

$$\begin{aligned}
\frac{\partial^2 H^{(0)}}{\partial z_{m+1}\partial z_{n+1}} &= \frac{(-1)^n z_n \cdot \ldots \cdot z_{m+2} \cdot z_m \cdot \ldots \cdot z_1}{(1+H^{(n+1)})[(1+H^{(n)}) \cdot \ldots \cdot (1+H^{(m+1)})]^2} \frac{B_m - B_{m-1}H^{(m)}}{(B_m + B_{m-1}H^{(m)})^3} \\
&= \frac{\partial H^{(0)}}{\partial z_{n+1}} \cdot \frac{1}{H^{(m)}(1+H^{(m+1)})} \prod_{k=1}^{m} \left(\frac{1}{1+H^{(k)}}\right) \cdot (B_m - B_{m-1}H^{(m)}) \\
&= \frac{\partial H^{(0)}}{\partial z_{n+1}} \cdot \frac{1}{H^{(m)}(1+H^{(m+1)})} \cdot \left[1 + 2\sum_{k=1}^{m}\prod_{\nu=k}^{m} t_\nu\right]
\end{aligned}$$

Third order

$$\begin{aligned}
\frac{\partial^3 H^{(0)}}{\partial z_{n+1}^3} &= 6\frac{B_n A_{n-1} - A_n B_{n-1}}{(1+H^{(n+1)})^3(B_n + B_{n-1}H^{(n)})^4} B_{n-1}^2 \\
&= \frac{\partial^2 H^{(0)}}{\partial z_{n+1}^2} \cdot \frac{3}{H^{(n)}(1+H^{(n+1)})} \cdot \sum_{k=1}^{n}\prod_{\nu=k}^{n} t_\nu
\end{aligned}$$

and for $n > m$

$$\begin{aligned}
\frac{\partial^3 H^{(0)}}{\partial z_{m+1}\partial z_{n+1}^2} &= (-2)\frac{(-1)^n z_n \cdot \ldots \cdot z_{m+2} \cdot z_m \cdot \ldots \cdot z_1}{(1+H^{(n+1)})^2[(1+H^{(n)}) \cdot \ldots \cdot (1+H^{(m+1)})]^3} \cdot \\
&\quad \left[\frac{B_m - 2B_{m-1}H^{(m)}}{(B_m + B_{m-1}H^{(m)})^4} \cdot B_{n-1} + B_{m-1}B_{n-m-2}^{(m+1)} \frac{z_{m+1}}{(B_m + B_{m-1}H^{(m)})^3}\right] \\
&= \frac{\partial H^{(0)}}{\partial z_{n+1}} \cdot \frac{2}{H^{(n)}(1+H^{(n+1)})} \cdot \frac{1}{H^{(m)}(1+H^{(m+1)})} \cdot \\
&\quad \left[\left\{1 + 3\sum_{k=1}^{m}\prod_{\nu=k}^{m} t_\nu\right\} \cdot \sum_{k=1}^{n}\prod_{\nu=k}^{n} t_\nu - \sum_{k=1}^{m}\prod_{\nu=k}^{m} t_\nu \cdot \sum_{k=m+2}^{n}\prod_{\nu=k}^{n} t_\nu\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 H^{(0)}}{\partial z_{m+1}^2\partial z_{n+1}} &= \frac{(-1)^n z_n \cdot \ldots \cdot z_{m+2} \cdot z_m \cdot \ldots \cdot z_1}{(1+H^{(n+1)})[(1+H^{(n)}) \cdot \ldots \cdot (1+H^{(m+1)})]^2(1+H^{(m+1)})} \cdot \\
&\quad \frac{-4B_m + 2B_{m-1}H^{(m)}}{(B_m + B_{m-1}H^{(m)})^4} \cdot B_{m-1} \\
&= \frac{\partial H^{(0)}}{\partial z_{n+1}} \cdot \frac{2}{(H^{(m)})^2(1+H^{(m+1)})^2} \cdot \left[2 + 3\sum_{k=1}^{m}\prod_{\nu=k}^{m} t_\nu\right] \cdot \sum_{k=1}^{m}\prod_{\nu=k}^{m} t_\nu
\end{aligned}$$

and for $n > m > p$

$$\begin{aligned}
\frac{\partial^3 H^{(0)}}{\partial z_{p+1}\partial z_{m+1}\partial z_{n+1}} &= \text{... too complicated} \\
&= \frac{\partial H^{(0)}}{\partial z_{n+1}} \cdot \frac{1}{H^{(m)}H^{(p)}(1+H^{(m+1)})(1+H^{(p+1)})} \cdot \\
&\quad \left[\left\{1 + 2\sum_{k=1}^{m}\prod_{\nu=k}^{m} t_\nu\right\} \cdot \left\{1 + 2\sum_{k=1}^{p}\prod_{\nu=k}^{p} t_\nu\right\} + \right. \\
&\qquad\qquad \left. 2\sum_{k=1}^{p}\prod_{\nu=k}^{p} t_\nu \cdot \sum_{k=1}^{p+1}\prod_{\nu=k}^{m} t_\nu\right]
\end{aligned}$$

With these expressions we can now try to accelerate the convergence of a continued fraction of the form

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{z_n + \delta_n}{1}\right) \tag{3}$$

if we know the values of all the tails of the 'nearby' continued fraction

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{z_n}{1}\right)$$

As an example Waadeland uses

$$z_i = a \ , \qquad a \in \mathbb{C} \backslash] -\infty, -1/4]$$

In this case we have:

$$H^{(n)}(\overline{z}) = \Gamma \ \ \text{with} \ \ \Gamma = \frac{-1 + \sqrt{1+4a}}{2}$$

Setting

$$G^{(N)}(z) = \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{a + \delta_{n+N}\, z}{1}\right) \qquad z \in \mathbb{C}$$

we can simplify things by using the Maclaurin series expansion of this function of the variable z at the point $z = 1$:

$$G^{(N)}(1) = G^{(N)}(0) + \sum_{i=1}^{k} \frac{1}{i!} \cdot \left(\frac{\mathrm{d}^i G^{(N)}}{\mathrm{d}z^i}\right)(0)$$

The first three derivatives at the righthand side may be calculated from the expressions for the partial derivatives given above. We denote the partial sums of this series by $w_{k,N}$:

$$w_{k,N} = \Gamma + \sum_{i=1}^{k} \frac{1}{i!} \cdot \left(\frac{\mathrm{d}^i G^{(N)}}{\mathrm{d}z^i}\right)(0)$$

For $k = 1$ we find in [10] :

$$w_{1,N} = \Gamma + \frac{1}{1+\Gamma} \sum_{n=0}^{\infty} \left(-\frac{\Gamma}{1+\Gamma}\right)^n \delta_{n+N+1}$$

and for $k = 2$:

$$\begin{aligned} w_{2,N} \ &= \ \Gamma + \frac{1}{1+\Gamma} \sum_{n=0}^{\infty} \left(-\frac{\Gamma}{1+\Gamma}\right)^n \delta_{n+N+1} \\ &\quad - \frac{1}{(1+2\Gamma)(1+\Gamma)^2} \sum_{n=0}^{\infty} \left(-\frac{\Gamma}{1+\Gamma}\right)^n \left[1 - \left(-\frac{\Gamma}{1+\Gamma}\right)^n\right] \delta_{n+N+1}^2 \\ &\quad + \frac{1}{\Gamma(1+2\Gamma)(1+\Gamma)^2} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \left(-\frac{\Gamma}{1+\Gamma}\right)^n \left[1 + 2\left(-\frac{\Gamma}{1+\Gamma}\right)^m\right] \delta_{m+N+1}\delta_{n+N+1} \end{aligned}$$

If we use the notation

$$G_{n,n,n} = \left(\frac{\partial^3 H^{(0)}}{\partial z_{n+1}^3}\right)_0 \qquad G_{m,n,n} = \left(\frac{\partial^3 H^{(0)}}{\partial z_{m+1}\partial z_{n+1}^2}\right)_0$$
$$G_{m,m,n} = \left(\frac{\partial^3 H^{(0)}}{\partial z_{m+1}^2\partial z_{n+1}}\right)_0 \qquad G_{p,m,n} = \left(\frac{\partial^3 H^{(0)}}{\partial z_{p+1}\partial z_{m+1}\partial z_{n+1}}\right)_0$$

where $()_0$ denotes evaluation at $z_1 = z_2 = \ldots = z_i = \ldots = a$, then we find for $k = 3$

$$\begin{aligned} w_{3,N} &= w_{2,N} + \frac{1}{6}\sum_{n=0}^{\infty} G_{n,n,n}\delta_{n+N+1}^3 + \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty} G_{m,n,n}\delta_{m+N+1}\delta_{n+N+1}^2 \\ &+ \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty} G_{m,m,n}\delta_{m+N+1}^2\delta_{n+N+1} + \sum_{p=0}^{\infty}\sum_{m=p+1}^{\infty}\sum_{n=m+1}^{\infty} G_{p,m,n}\delta_{p+N+1}\delta_{m+N+1}\delta_{n+N+1} \end{aligned}$$

We used MAPLEV to calculate the extra terms in this expression, for the special case

$$\delta_k = C \cdot T^k$$

Here is a transcript of the MAPLE session:

```
> t1:=t^n*(1-t^n)^2*T^(3*n);
```

$$t1 := t^n\,(\,1 - t^n\,)^2\,T^{(\,3\,n\,)}$$

```
> t1:=expand(t1);
```

$$t1 := t^n\,(\,T^n\,)^3 - 2\,(\,t^n\,)^2\,(\,T^n\,)^3 + (\,t^n\,)^3\,(\,T^n\,)^3$$

```
> t1:=sum(t1,n=0..infinity);
```

$$t1 := -\frac{1}{t\,T^3 - 1} + 2\,\frac{1}{t^2\,T^3 - 1} - \frac{1}{t^3\,T^3 - 1}$$

```
> t1:=subs(t=(-Gamma/(Gamma+1)),t1):
> t1:=simplify(t1);
```

$$\begin{aligned} t1 := {} & (\,\Gamma + 1\,)\,\Gamma\,T^3\,(\,1 + 6\,\Gamma + 13\,\Gamma^2 + 4\,\Gamma^4\,T^3 + 4\,\Gamma^4 + 12\,\Gamma^3 + 4\,\Gamma^3\,T^3 + \Gamma^2\,T^3\,)\Big/\big(\\ & (\,\Gamma\,T^3 + \Gamma + 1\,)\,(\,\Gamma^2\,T^3 - \Gamma^2 - 2\,\Gamma - 1\,)\,(\,\Gamma^3\,T^3 + \Gamma^3 + 3\,\Gamma^2 + 3\,\Gamma + 1\,)\,\big) \end{aligned}$$

```
> term1:=1/(Gamma+1)^3*1/(1+2*Gamma)^2*t1:
> term1:=simplify(term1);
```

$$\begin{aligned} term1 := {} & (\,\Gamma^2\,T^3 + \Gamma^2 + 2\,\Gamma + 1\,)\,\Gamma\,T^3\Big/\Big((\,\Gamma^3\,T^3 + \Gamma^3 + 3\,\Gamma^2 + 3\,\Gamma + 1\,)\,(\,\Gamma + 1\,)^2 \\ & (\,\Gamma\,T^3 + \Gamma + 1\,)\,(\,\Gamma^2\,T^3 - \Gamma^2 - 2\,\Gamma - 1\,)\Big) \end{aligned}$$

```
> t2:=t^n*(1-2*t^n+(1+Gamma)*t^(n-m)+2*Gamma*t^m-3*Gamma*t^(n+m))*T^(2*n+m);
```

$$t2 := t^n\,(\,1 - 2\,t^n + (\,\Gamma + 1\,)\,t^{(\,n - m\,)} + 2\,\Gamma\,t^m - 3\,\Gamma\,t^{(\,n + m\,)}\,)\,T^{(\,2\,n + m\,)}$$

```
> t2:=expand(t2);
```

$$\mathit{t2} := t^n\,(T^n)^2\,T^m - 2\,(t^n)^2\,(T^n)^2\,T^m + \frac{(t^n)^2\,(T^n)^2\,T^m\,\Gamma}{t^m} + \frac{(t^n)^2\,(T^n)^2\,T^m}{t^m} + 2\,t^n\,(T^n)^2\,T^m\,\Gamma\,t^m - 3\,(t^n)^2\,(T^n)^2\,T^m\,\Gamma\,t^m$$

```
> t2:=sum(t2,n=m+1..infinity):
> t2:=sum(t2,m=0..infinity);
```

$$\mathit{t2} := \frac{t\,T^2}{(t\,T^2-1)(t\,T^3-1)} - 2\,\frac{t^2\,T^2}{(t^2\,T^2-1)(t^2\,T^3-1)} + \frac{t^2\,T^2\,\Gamma}{(t^2\,T^2-1)(t\,T^3-1)} + \frac{t^2\,T^2}{(t^2\,T^2-1)(t\,T^3-1)} + 2\,\frac{t\,T^2\,\Gamma}{(t\,T^2-1)(t^2\,T^3-1)} - 3\,\frac{t^2\,T^2\,\Gamma}{(t^2\,T^2-1)(t^3\,T^3-1)}$$

```
> t2:=subs(t=(-Gamma/(Gamma+1)),t2):
> t2:=simplify(t2);
```

$$\mathit{t2} := (-1 - 10\,\Gamma - 104\,\Gamma^3 - 3\,\Gamma^2\,T^3 - 22\,\Gamma^3\,T^3 - 43\,\Gamma^2 - 64\,\Gamma^4\,T^3 - 155\,\Gamma^4 + 4\,\Gamma^7\,T^8 + 4\,\Gamma^7\,T^5 - 11\,\Gamma^6\,T^5 - 20\,\Gamma^5\,T^5 - 28\,\Gamma^7\,T^3 - 73\,\Gamma^6\,T^3 - 94\,\Gamma^5\,T^3 - 2\,\Gamma^3\,T^5 + \Gamma^6\,T^8 - 11\,\Gamma^4\,T^5 - 28\,\Gamma^7 - 85\,\Gamma^6 - 146\,\Gamma^5 + 4\,\Gamma^8\,T^8 - 4\,\Gamma^8 + 4\,\Gamma^8\,T^5 - 4\,\Gamma^8\,T^3)(\Gamma+1)\,\Gamma\,T^2 \Big/ ((\Gamma^3\,T^3 + \Gamma^3 + 3\,\Gamma^2 + 3\,\Gamma + 1)(\Gamma^2\,T^3 - \Gamma^2 - 2\,\Gamma - 1)(\Gamma^2\,T^2 - \Gamma^2 - 2\,\Gamma - 1)(\Gamma\,T^3 + \Gamma + 1)(\Gamma\,T^2 + \Gamma + 1))$$

```
> term2:=-1/(Gamma*(1+Gamma)^3*(1+2*Gamma)^2)*t2:
> term2:=simplify(term2);
```

$$\mathit{term2} := -T^2(\Gamma^6\,T^8 - \Gamma^6 + \Gamma^6\,T^5 - \Gamma^6\,T^3 - 6\,\Gamma^5 - 6\,\Gamma^5\,T^3 - 12\,\Gamma^4\,T^3 - 3\,\Gamma^4\,T^5 - 15\,\Gamma^4 - 20\,\Gamma^3 - 10\,\Gamma^3\,T^3 - 2\,\Gamma^3\,T^5 - 3\,\Gamma^2\,T^3 - 15\,\Gamma^2 - 6\,\Gamma - 1) \Big/ \Big((\Gamma+1)^2 (\Gamma\,T^2 + \Gamma + 1)(\Gamma\,T^3 + \Gamma + 1)(\Gamma^2\,T^2 - \Gamma^2 - 2\,\Gamma - 1)(\Gamma^2\,T^3 - \Gamma^2 - 2\,\Gamma - 1)(\Gamma^3\,T^3 + \Gamma^3 + 3\,\Gamma^2 + 3\,\Gamma + 1)\Big)$$

```
> t3:=t^n*(1-t^m)*(1+2/Gamma+3*t^m)*T^(n+2*m);
```

$$\mathit{t3} := t^n\,(1 - t^m)\left(1 + 2\,\frac{1}{\Gamma} + 3\,t^m\right) T^{(n+2\,m)}$$

```
> t3:=expand(t3);
```

$$\mathit{t3} := t^n\,T^n\,(T^m)^2 + 2\,\frac{t^n\,T^n\,(T^m)^2}{\Gamma} + 2\,t^n\,T^n\,(T^m)^2\,t^m - 2\,\frac{t^n\,T^n\,(T^m)^2\,t^m}{\Gamma} - 3\,t^n\,T^n\,(T^m)^2\,(t^m)^2$$

```
> t3:=sum(t3,n=m+1..infinity):
> t3:=sum(t3,m=0..infinity);
```

$$\mathit{t3} := \frac{t\,T}{(t\,T-1)(t\,T^3-1)} + 2\,\frac{t\,T}{\Gamma\,(t\,T-1)(t\,T^3-1)} + 2\,\frac{t\,T}{(t\,T-1)(t^2\,T^3-1)}$$

$$-\,2\,\frac{t\,T}{\Gamma\,(\,t\,T-1\,)\,(\,t^2\,T^3-1\,)}-3\,\frac{t\,T}{(\,t\,T-1\,)\,(\,t^3\,T^3-1\,)}$$

```
> t3:=subs(t=(-Gamma/(Gamma+1)),t3):
> t3:=simplify(t3);
```

$$\mathit{t3} := (\,\Gamma+1\,)\,(\,-2-11\,\Gamma-11\,\Gamma^3+\Gamma^3\,T^3-20\,\Gamma^2+4\,\Gamma^4\,T^3+4\,\Gamma^4+4\,\Gamma^5\,T^3+4\,\Gamma^5\,)\,T^4\,\Gamma\Big/(\,(\,\Gamma^3\,T^3+\Gamma^3+3\,\Gamma^2+3\,\Gamma+1\,)\,(\,\Gamma^2\,T^3-\Gamma^2-2\,\Gamma-1\,)\,(\,\Gamma\,T^3+\Gamma+1\,)\,(\,T\,\Gamma+\Gamma+1\,)\,)$$

```
> term3:=-1/((1+Gamma)^3*(1+2*Gamma)^2)*t3:
> term3:=simplify(term3);
```

$$\mathit{term3} := -\Gamma\,T^4\,(\,\Gamma^3+\Gamma^3\,T^3-3\,\Gamma-2\,)\Big/\Big((\,T\,\Gamma+\Gamma+1\,)\,(\,\Gamma\,T^3+\Gamma+1\,)\,(\,\Gamma^2\,T^3-\Gamma^2-2\,\Gamma-1\,)\,(\,\Gamma^3\,T^3+\Gamma^3+3\,\Gamma^2+3\,\Gamma+1\,)\,(\,\Gamma+1\,)^2\Big)$$

```
> t4:=t^n*(1+4*Gamma*t^m+2*Gamma^2*t^(m-p-1)+2*Gamma*t^p+6*Gamma^2*t^(m+p))*
> T^(p+m+n);
```

$$\mathit{t4} := t^n\,(\,1+4\,\Gamma\,t^m+2\,\Gamma^2\,t^{(\,m-p-1\,)}+2\,\Gamma\,t^p+6\,\Gamma^2\,t^{(\,m+p\,)}\,)\,T^{(\,p+m+n\,)}$$

```
> t4:=expand(t4);
```

$$\mathit{t4} := t^n\,T^p\,T^m\,T^n+4\,t^n\,T^p\,T^m\,T^n\,\Gamma\,t^m+2\,\frac{t^n\,T^p\,T^m\,T^n\,\Gamma^2\,t^m}{t^p\,t}+2\,t^n\,T^p\,T^m\,T^n\,\Gamma\,t^p+6\,t^n\,T^p\,T^m\,T^n\,\Gamma^2\,t^m\,t^p$$

```
> t4:=sum(t4,n=m+1..infinity):
> t4:=sum(t4,m=p+1..infinity):
> t4:=sum(t4,p=0..infinity);
```

$$\mathit{t4} := -\frac{t^2\,T^3}{(\,t\,T-1\,)\,(\,t\,T^2-1\,)\,(\,t\,T^3-1\,)}-4\,\frac{t^3\,T^3\,\Gamma}{(\,t\,T-1\,)\,(\,t^2\,T^2-1\,)\,(\,t^2\,T^3-1\,)}-2\,\frac{t^2\,T^3\,\Gamma^2}{(\,t\,T-1\,)\,(\,t^2\,T^2-1\,)\,(\,t\,T^3-1\,)}-2\,\frac{t^2\,T^3\,\Gamma}{(\,t\,T-1\,)\,(\,t\,T^2-1\,)\,(\,t^2\,T^3-1\,)}-6\,\frac{t^3\,T^3\,\Gamma^2}{(\,t\,T-1\,)\,(\,t^2\,T^2-1\,)\,(\,t^3\,T^3-1\,)}$$

```
> t4:=subs(t=(-Gamma/(Gamma+1)),t4):
> t4:=simplify(t4);
```

$$\mathit{t4} := (1+9\,\Gamma+61\,\Gamma^3+13\,\Gamma^7\,T^6+6\,\Gamma^6\,T^6+\Gamma^5\,T^6-61\,\Gamma^7\,T^2-85\,\Gamma^6\,T^2-70\,\Gamma^5\,T^2-34\,\Gamma^4\,T^2-9\,\Gamma^3\,T^2-4\,\Gamma^9+4\,\Gamma^9\,T^8+4\,\Gamma^9\,T^6-4\,\Gamma^9\,T^2+12\,\Gamma^8\,T^6-24\,\Gamma^8\,T^2+\Gamma^2\,T^3+4\,\Gamma^3\,T^3+33\,\Gamma^2-2\,\Gamma^4\,T^3+51\,\Gamma^4-\Gamma^2\,T^2+\Gamma^7\,T^8-60\,\Gamma^7\,T^5-65\,\Gamma^6\,T^5-30\,\Gamma^5\,T^5-44\,\Gamma^7\,T^3-59\,\Gamma^6\,T^3-32\,\Gamma^5\,T^3-5\,\Gamma^4\,T^5-57\,\Gamma^7-61\,\Gamma^6-9\,\Gamma^5+4\,\Gamma^8\,T^8-24\,\Gamma^8-20\,\Gamma^8\,T^5-12\,\Gamma^8\,T^3)(\,\Gamma+1\,)\,\Gamma^2\,T^3\Big/(\,(\,\Gamma^3\,T^3+\Gamma^3+3\,\Gamma^2+3\,\Gamma+1\,)\,(\,\Gamma^2\,T^3-\Gamma^2-2\,\Gamma-1\,)\,(\,\Gamma^2\,T^2-\Gamma^2-2\,\Gamma-1\,)$$

$$(\Gamma T^3+\Gamma+1)(\Gamma T^2+\Gamma+1)(T\Gamma+\Gamma+1))$$

```
> term4:=1/(Gamma^2*(1+Gamma)^3*(1+2*Gamma)^2)*t4:
> term4:=simplify(term4);
```

$$\begin{aligned}\mathit{term4} := {} & T^3(1+5\Gamma+5\Gamma^3+\Gamma^7T^6+2\Gamma^6T^6+\Gamma^5T^6-\Gamma^7T^2-5\Gamma^6T^2-10\Gamma^5T^2\\ & -10\Gamma^4T^2-5\Gamma^3T^2+\Gamma^2T^3+9\Gamma^2-6\Gamma^4T^3-5\Gamma^4-\Gamma^2T^2+\Gamma^7T^8-5\Gamma^6T^5\\ & -10\Gamma^5T^5-3\Gamma^6T^3-8\Gamma^5T^3-5\Gamma^4T^5-\Gamma^7-5\Gamma^6-9\Gamma^5)\Big/\Big((T\Gamma+\Gamma+1)\\ & (\Gamma T^2+\Gamma+1)(\Gamma T^3+\Gamma+1)(\Gamma^2T^2-\Gamma^2-2\Gamma-1)(\Gamma^2T^3-\Gamma^2-2\Gamma-1)\\ & (\Gamma^3T^3+\Gamma^3+3\Gamma^2+3\Gamma+1)(\Gamma+1)^2\Big)\end{aligned}$$

```
> Total:=term1+term2+term3+term4;
```

$$\begin{aligned}\mathit{total} := {} & \frac{(\Gamma^2T^3+\Gamma^2+2\Gamma+1)\Gamma T^3}{\%1(\Gamma+1)^2(\Gamma T^3+\Gamma+1)(\Gamma^2T^3-\Gamma^2-2\Gamma-1)}-T^2(\Gamma^6T^8-\Gamma^6+\Gamma^6T^5\\ & -\Gamma^6T^3-6\Gamma^5-6\Gamma^5T^3-12\Gamma^4T^3-3\Gamma^4T^5-15\Gamma^4-20\Gamma^3-10\Gamma^3T^3-2\Gamma^3T^5\\ & -3\Gamma^2T^3-15\Gamma^2-6\Gamma-1)\Big/\Big((\Gamma+1)^2(\Gamma T^2+\Gamma+1)(\Gamma T^3+\Gamma+1)\\ & (\Gamma^2T^2-\Gamma^2-2\Gamma-1)(\Gamma^2T^3-\Gamma^2-2\Gamma-1)\%1\Big)\\ & -\frac{\Gamma T^4(\Gamma^3+\Gamma^3T^3-3\Gamma-2)}{(T\Gamma+\Gamma+1)(\Gamma T^3+\Gamma+1)(\Gamma^2T^3-\Gamma^2-2\Gamma-1)\%1(\Gamma+1)^2}+T^3(1\\ & +5\Gamma+5\Gamma^3+\Gamma^7T^6+2\Gamma^6T^6+\Gamma^5T^6-\Gamma^7T^2-5\Gamma^6T^2-10\Gamma^5T^2-10\Gamma^4T^2\\ & -5\Gamma^3T^2+\Gamma^2T^3+9\Gamma^2-6\Gamma^4T^3-5\Gamma^4-\Gamma^2T^2+\Gamma^7T^8-5\Gamma^6T^5-10\Gamma^5T^5\\ & -3\Gamma^6T^3-8\Gamma^5T^3-5\Gamma^4T^5-\Gamma^7-5\Gamma^6-9\Gamma^5)\Big/\Big((T\Gamma+\Gamma+1)(\Gamma T^2+\Gamma+1)\\ & (\Gamma T^3+\Gamma+1)(\Gamma^2T^2-\Gamma^2-2\Gamma-1)(\Gamma^2T^3-\Gamma^2-2\Gamma-1)\%1(\Gamma+1)^2\Big)\\ & \%1 := \Gamma^3T^3+\Gamma^3+3\Gamma^2+3\Gamma+1\end{aligned}$$

```
> simplify(total);
```

$$\frac{T^2(T+1)}{(T\Gamma+\Gamma+1)^3(\Gamma T^3+\Gamma+1)(\Gamma T^2+\Gamma+1)}$$

The last expression, multiplied by $C^3\cdot T^{3N+3}$, gives us

$$w_{3,N}-w_{2,N}$$

for this special case. So, combining this with the results from the paper [10], we finally get

$$\begin{aligned}w_{3,N} &= \Gamma+\frac{CT^{N+1}}{T\Gamma+\Gamma+1}-\frac{C^2T^{2N+3}}{(T\Gamma+\Gamma+1)^2(T^2\Gamma+\Gamma+1)}\\ &\qquad +\frac{C^3(T+1)T^{3N+5}}{(T\Gamma+\Gamma+1)^3(T^2\Gamma+\Gamma+1)(T^3\Gamma+\Gamma+1)}\end{aligned}$$

There is of course another method to calculate these Taylor-Waadeland modifications. For instance for $k=4$. We start with the recurrence relation for the tails:

$$z_{n+1}-H^{(n)}(1+H^{(n+1)})=0$$

and we assume that:

$$w_{4,n} = w_{3,n} + K \cdot T^{4n}$$

We put this into the equation for the tails, we expand the result and we choose K in such a way that all the terms of the form $c \cdot T^{4n}$ cancel out.
Here is a transcript of the MAPLE session:

```
> w:=n->Gamma+C*T^(n+1)/(1+Gamma+Gamma*T)-C^2*T^(2*n+3)/((1+Gamma+Gamma*T)^
> 2*(1+Gamma+Gamma*T^2))+C^3*T^(3*n+5)*(1+T)/((1+Gamma+Gamma*T)^3*(1+Gamma+Gamma
> *T^2)*(1+Gamma+Gamma*T^3))+K*T^(4*n):
> w(n)*w(n+1)+w(n)-Gamma-Gamma*Gamma-C*T^(n+1):
> expand("):
> normal("):
> expand(numer(")/T^(4*n)):
> subs(T^n=0,"):
> solve("=0,K):
> factor(");
```

$$-\frac{(T^5\Gamma + 2\Gamma T^4 + 2\Gamma T^3 + T^3 + 2\Gamma T^2 + T^2 + 2\Gamma T + 2T + 1 + \Gamma)C^4 T^7}{(1+\Gamma+\Gamma T^3)(\Gamma T^4 + 1 + \Gamma)(1+\Gamma+\Gamma T^2)^2(1+\Gamma+\Gamma T)^4}$$

Hence:

$$w_{4,N} = w_{3,N} - \frac{(T^5\Gamma + 2\Gamma T^4 + 2\Gamma T^3 + T^3 + 2\Gamma T^2 + T^2 + 2\Gamma T + 2T + 1 + \Gamma)C^4 T^{4N+7}}{(1+\Gamma+\Gamma T^3)(\Gamma T^4 + 1 + \Gamma)(1+\Gamma+\Gamma T^2)^2(1+\Gamma+\Gamma T)^4}$$

EXAMPLE: Here are the results if we use these modifications in an example:

```
N=15  T=0.999  C=1  A=30  GAMMA=5
    exact     5.0901212003055
    k=0       5.0961963614170
    k=1       5.0900725866929
    k=2       5.0901219882156
    k=3       5.0901211843188
    k=4       5.0901212006691
```

NOTE: There is an easier way to get $w_{k,N}$ in this special case. For instance for $k = 4$, we replace $H^{(n)}$ in the lefthand side of the recurrence relation for the tails by $w_{3,n}$, we expand and take the coefficient of T^{4n} in the result. This expression divided by $1+\Gamma+\Gamma T^4$ gives us $w_{4,n}$:

```
> w:=n->Gamma+C*T^(n+1)/(1+Gamma+Gamma*T)-C^2*T^(2*n+3)/((1+Gamma+Gamma*T)^2
> *(1+Gamma+Gamma*T^2))+C^3*T^(3*n+5)*(1+T)/((1+Gamma+Gamma*T)^3*(1+Gamma+Gamma*T
> ^2)*(1+Gamma+Gamma*T^3)):
> u:=Gamma*(1+Gamma)+C*T^(n+1)-w(n)*(1+w(n+1)):
> u:=expand(u):
```

```
> u:=normal(u):
> u:=u/(T^(n))^4:
> u:=subs(T^n=0,u):
> u:=simplify(u):
> u:=u/(1+Gamma+Gamma*T^4);
```

$$u := -\frac{C^4\,T^7\,(\,T^5\,\Gamma + 2\,T^4\,\Gamma + 2\,\Gamma\,T^3 + 2\,\Gamma\,T^2 + 2\,\Gamma\,T + \Gamma + T^3 + 1 + 2\,T + T^2\,)}{(\,1 + \Gamma + \Gamma\,T^3\,)\,(\,1 + \Gamma + \Gamma\,T\,)^4\,(\,1 + \Gamma + \Gamma\,T^2\,)^2\,(\,1 + \Gamma + T^4\,\Gamma\,)}$$

This method is related to the one described in [1, 2]. (See also [4, 5])

To understand why the numerator in $w_{4,N}$ is such a complicated expression, we have to make a link with the method described in [3].
We go back to the general case and we consider two continued fractions:

$$(1)\ \operatorname*{K}_{n=1}^{\infty}\left(\frac{z_n}{1}\right) \qquad\qquad (3)\ \operatorname*{K}_{n=1}^{\infty}\left(\frac{z_n+\delta_n}{1}\right)$$

We assume that we know the exact values of all the tails $H^{(n)}$ (we omit the argument $\overline{z}$) of the continued fraction (1), and that the second is in some sense near to the first one. We want to use this information to accelerate the convergence of (3). To do this the standard procedure is to replace the n-th tail of (3) by $H^{(n)}$:

$$W_{0,n} = H^{(n)}$$

We try to get a better approximation for the tails of (3), and therefore we assume that the n-th tail $h^{(n)}(= H^{(n)}(\overline{z}+\overline{\delta})$ is of the form

$$h^{(n)} = H^{(n)} + \epsilon_n^{(1)}$$

(an idea developed in [9]), with $\epsilon_n^{(1)}$ in some sense small compared with $H^{(n)}$. Now the tails of (3) satisfy the equation

$$z_{n+1} + \delta_{n+1} - h^{(n)}(1 + h^{(n+1)}) = 0$$

Substituting gives us

$$z_{n+1} + \delta_{n+1} - (H^{(n)} + \epsilon_n^{(1)})(1 + H^{(n+1)} + \epsilon_{n+1}^{(1)}) = 0$$

or, using the fact that $H^{(n)}$ is the n-th tail of (1):

$$\delta_{n+1} - H^{(n)}\epsilon_{n+1}^{(1)} - (1 + H^{(n+1)})\epsilon_n^{(1)} - \epsilon_n^{(1)}\epsilon_{n+1}^{(1)} = 0$$

We leave out the last term on the left, assuming it will be small compared with the others, and rewrite the equation:

$$H^{(n)}\hat{\epsilon}_{n+1}^{(1)} + (1 + H^{(n+1)})\hat{\epsilon}_n^{(1)} = \delta_{n+1}$$

with $\hat{\epsilon}_n^{(1)}$ an approximation to $\epsilon_n^{(1)}$.
Hence $\hat{\epsilon}_n^{(1)}$ is a solution of the recurrence relation:

$$H^{(n)}y_{n+1} + (1 + H^{(n+1)})y_n = \delta_{n+1}$$

We assume that $\hat{\epsilon}_n^{(1)}$ is the minimal solution of this recurrence relation, and we calculate it using backward recursion starting from

$$y_{N+1} = 0$$

We get

$$y_N = \frac{\delta_{N+1}}{1 + H^{(N+1)}}$$

and

$$y_{N-1} = \frac{\delta_N}{1 + H^{(N)}} - \frac{H^{(N-1)}\delta_{N+1}}{(1 + H^{(N)})(1 + H^{(N+1)})}$$

Continuing in this way we find:

$$y_i = \sum_{n=0}^{N-i} \frac{H^{(i)}}{H^{(n+i)}(1 + H^{(n+i+1)})} \prod_{m=1}^{n} \left(- \frac{H^{(m+i)}}{1 + H^{(m+i)}} \right) \delta_{n+i+1}$$

We now let $N \to \infty$ to find $\hat{\epsilon}_i^{(1)}$:

$$\hat{\epsilon}_i^{(1)} = H^{(i)} \sum_{n=0}^{\infty} \frac{1}{H^{(n+i)}(1 + H^{(n+i+1)})} \prod_{m=1}^{n} \left(- \frac{H^{(m+i)}}{1 + H^{(m+i)}} \right) \delta_{n+i+1}$$

For $i = 0$ this gives us:

$$\hat{\epsilon}_0^{(1)} = H^{(0)} \sum_{n=0}^{\infty} \frac{1}{H^{(n)}(1 + H^{(n+1)})} \prod_{m=1}^{n} \left(- \frac{H^{(m)}}{1 + H^{(m)}} \right) \delta_{n+1}$$

Hence our new modification is given by:

$$W_{1,n} = H^{(n)} + \hat{\epsilon}_n^{(1)}$$

This is exactly the result given in [9]. It reduces to $w_{1,n}$ if $z_1 = z_2 = \ldots = z_i = \ldots = a$. Using an argument similar to that in [2, 3] it is possible to prove convergence acceleration in this case.

NOTE 1: If the series above converges to a finite value, then $\hat{\epsilon}_n^{(1)}$ is the minimal solution of the first order recurrence relation, in the sense that $\hat{\epsilon}_n^{(1)}/y_n \to 0$ for $n \to \infty$ for every other solution y_n of the recurrence. This is easy to prove.

NOTE 2: The generalization to the case of a more general continued fraction of the form $\mathrm{K}(a_n/b_n)$ is straightforward.

We now try to improve our approximation for the tails. Let us assume that the n-th tail of the continued fraction (3) is of the form

$$h^{(n)} = H^{(n)} + \hat{\epsilon}_n^{(1)} + \epsilon_n^{(2)}$$

with $\epsilon_n^{(2)}$ small compared with $\hat{\epsilon}_n^{(1)}$. Substituting into the equation for the tails gives us:

$$z_{n+1} + \delta_{n+1} - (H^{(n)} + \hat{\epsilon}_n^{(1)} + \epsilon_n^{(2)})(1 + H^{(n+1)} + \hat{\epsilon}_{n+1}^{(1)} + \epsilon_{n+1}^{(2)}) = 0$$

or, using the recurrence relation for $\hat{\epsilon}_n^{(1)}$:

$$-H^{(n)}\epsilon_{n+1}^{(2)} - (1 + H^{(n+1)})\epsilon_n^{(2)} - \hat{\epsilon}_n^{(1)}\hat{\epsilon}_{n+1}^{(1)} - \hat{\epsilon}_n^{(1)}\epsilon_{n+1}^{(2)} - \epsilon_n^{(2)}\hat{\epsilon}_{n+1}^{(1)} - \epsilon_n^{(2)}\epsilon_{n+1}^{(2)} = 0$$

We leave out the three last terms on the left, assuming they will be small compared with the others, and rewrite the equation:

$$H^{(n)}\hat{\epsilon}_{n+1}^{(2)} + (1 + H^{(n+1)})\hat{\epsilon}_n^{(2)} = -\hat{\epsilon}_n^{(1)}\hat{\epsilon}_{n+1}^{(1)}$$

with $\hat{\epsilon}_n^{(2)}$ an approximation to $\epsilon_n^{(2)}$.
Continuing in the same way as above, we find:

$$\hat{\epsilon}_i^{(2)} = -H^{(i)} \sum_{n=0}^{\infty} \frac{1}{H^{(n+i)}(1 + H^{(n+i+1)})} \prod_{m=1}^{n} \left(-\frac{H^{(m+i)}}{1 + H^{(m+i)}} \right) \hat{\epsilon}_{n+i}^{(1)}\hat{\epsilon}_{n+i+1}^{(1)}$$

Our new modification is given by:

$$W_{2,n} = H^{(n)} + \hat{\epsilon}_n^{(1)} + \hat{\epsilon}_n^{(2)}$$

It is possible to prove that $W_{2,n} = w_{2,n}$ in the special case $z_1 = z_2 = \ldots = z_i = \ldots = a$.

EXAMPLE: Let us look at the following example from [9]:

$$z_n = n(n+2) \ , \quad \delta_n = (n+1)y$$

it is easy to see that

$$H^{(n)} = n + 1, \qquad W_{1,n} = n + 1 + \frac{y}{2}$$

We find for $\hat{\epsilon}_i^{(2)}$:

$$\hat{\epsilon}_i^{(2)} = -(i+1) \sum_{n=0}^{\infty} \frac{1}{(n+i+1)(n+i+3)} (-1)^n \frac{i+2}{n+i+2} \frac{y^2}{4}$$

which can be simplified (using partial fractions) to:

$$\hat{\epsilon}_i^{(2)} = \frac{i}{4} + \frac{5}{8} - \frac{(i+1)(i+2)}{2}(-1)^i \left(\ln 2 - \sum_{k=1}^{i} (-1)^{k-1} \frac{1}{k} \right) \cdot y^2$$

If we use the corresponding modifications for the case $y = 1$, we find

```
N=25
    exact     1.444713868
    k=0       1.447170660
    k=1       1.444691157
    k=2       1.444714294
```

It is clear that we can iterate the process described above: in each step we get a first order recurrence relation of the form

$$H^{(n)}y_{n+1} + (1 + H^{(n+1)})y_n = \text{r.h.s.}$$

In the next step the r.h.s. will be equal to

$$-\hat{\epsilon}_n^{(1)}\hat{\epsilon}_{n+1}^{(2)} - \hat{\epsilon}_n^{(2)}\hat{\epsilon}_{n+1}^{(1)}$$

These are some of the terms that we neglected earlier on. Note that so far each term of $\hat{\epsilon}_n^{(k)}$ contains a product of k δ's (Compare this with the Taylor-Waadeland modifications!). If we take the expression above as the r.h.s. in the recurrence relation, then the solution $\hat{\epsilon}_n^{(3)}$ will be an infinite sum and its terms contain 3 (=1+2=2+1) factors δ. The r.h.s. in the next step will be

$$-\hat{\epsilon}_n^{(2)}\hat{\epsilon}_{n+1}^{(2)} - \hat{\epsilon}_n^{(1)}\hat{\epsilon}_{n+1}^{(3)} - \hat{\epsilon}_n^{(3)}\hat{\epsilon}_{n+1}^{(1)}$$

since 2+2=1+3=3+1=4.
So the r.h.s.'s get more complicated, and so do the $\hat{\epsilon}_n^{(k)}$.

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