A unifying approach for studying the behaviour and control of perturbations in variable structure systems

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Abstract

The behaviour of perturbations around trajectories of variable structure systems is studied and a sensitivity formula is derived. This formula is used to adapt classical shooting algorithms for computing periodic solutions to piecewise smooth systems. The influence of switching boundaries on the stability of periodic solutions is studied within a control theory approach.

1 Introduction

In this paper we discuss some properties of variable structure systems. We use this general term to denote systems which switch between several configurations, each of them modelled by an (non) autonomous ODE. The switching of the structure can occur when a trajectory of the system crosses a (possibly time-varying) boundary (figure 1-a) or when the value of a discrete state changes. This usually occurs when the continuous state satisfies a spatial condition. In the latter case the systems are usually denoted by *hybrid* systems due to the mixture of discrete and continuous states. The difference between the two mentioned cases is rather subtile: hybrid systems can have trajectories which return after switching to the same side of the boundary (see figure 1-b). The theory derived in the next sections can also easily be adapted to systems where once a boundary is hit, an additional mapping in the state space occurs (see figure 1-c). The unifying property of all the described system types is the geometrical aspect of the switching from one structure into another.

Variable structure systems are not only interesting from a theoretical point of view, but they also play an important role in many engineering applications varying from static power convertors, the steady state-behaviour of which can be considered as switched limit cycles, over electrical diode network to mechanical systems with forbidden regions. The latter type of systems covers the well studied class of impact oscillators.

In this paper we calculate the sensitivity of points on a switched trajectory to changes of the starting point. The resulting sensitivity formula can be seen as an extension of the classical sensitivity formula. This is an interesting result on its own, and is of importance for several applications. Some of these applications are discussed in this paper.

First we deal with the calculation of periodic solutions of variable structure systems using a shooting method. This technique is based on time integration and allows to deal relatively easy with changes in the structure of the system. It will be shown that *classical* shooting algorithms, i.e. developed for smooth systems, may not give rise to a (quadratically converging) Newton process, but can easily be adapted using the sensitivity formula derived in this paper. A related issue is the calculation of Lyapunov exponents.

Secondly, we start from a periodic solution which is switched at a time-varying boundary and take a *control theory* point of view: we investigate whether the position of the boundary can be used to control perturbations around the limit cycle. This covers in fact some major aspects of the linear

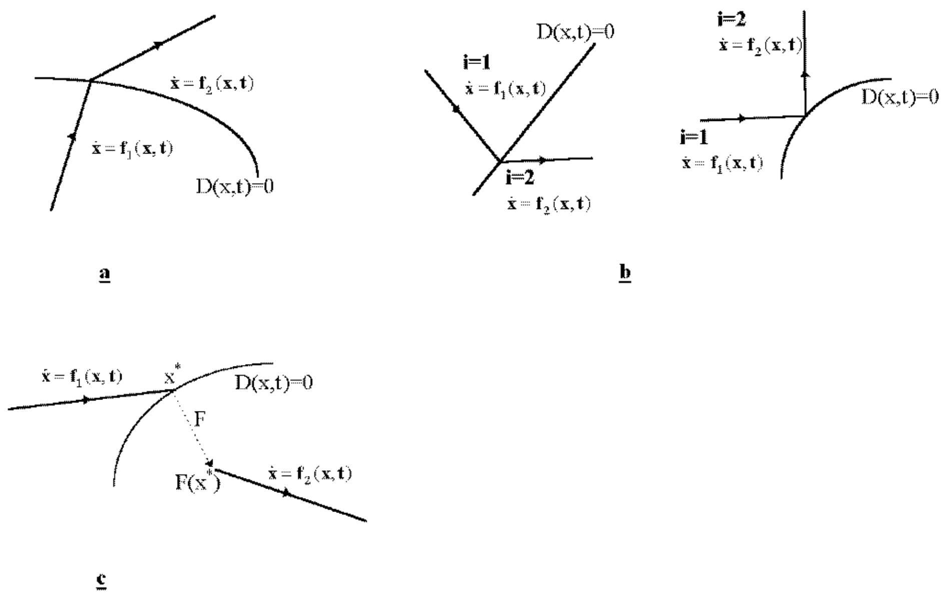

Figure 1: Variable structure systems. (a) Switching occurs when a trajectory crosses a boundary. (b) Hybrid system: the discrete state i changes on the hypersurface $D(x,t) = 0$ where x is the continuous state. (c) On the boundary an additional mapping of the state occurs.

control theory used in many practical applications such as static power convertors, but now it is described from a more abstract mathematical point of view, reasoning in terms of limit cycles, monodromy matrices, switching surfaces, etc., instead of starting from the specific properties of an application dependent model. Consequently, this way of discussing the problem unifies concepts and methods used in different engineering disciplines.

2 Sensitivity formula

2.1 Derivation

In this section we discuss the sensitivity of a point on a switched trajectory to changes of the initial point. We consider a hybrid system. The analysis is exactly the same for the other types of variable structure systems, except when an additional mapping on the boundary is performed. The latter case is discussed in the appendix. During the derivation a distinction is made between driven and non-driven systems but both cases lead to the same sensitivity formula.

We consider a non-autonomous hybrid system of the form

$$\left\{ \begin{array}{l} \dot{x} = f_i(x,t) \\ \left. \begin{array}{l} i^+ = 2 \quad \text{if} \quad i^- = 1 \\ i^+ = 1 \quad \text{if} \quad i^- = 2 \end{array} \right\} \text{ when } D(x,t) = 0, \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the continuous state, $i \in \{1,2\}$ the discrete state and $f_i(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are smooth mappings. $D(x,t) = 0$ represents the time-varying boundary on which the discrete state i and consequently the system dynamics change. By i^- resp. i^+ we denote the value of the discrete state just before resp. after the switching.

Consider a part of a trajectory starting in x_0 at time t_0 integrated over a fixed time T reaching point y_0 at time $t_1 = t_0 + T$ with $D(x_0, t_0) \neq 0$ and $D(x_1, t_1) \neq 0$. This is depicted in figure 2. Suppose this trajectory hits the boundary at time $\tilde{t}_0$ in $\tilde{x}_0$ in a non-tangential way relative to the (moving) plane D. This can be expressed mathematically as

$$\left. \left(\text{grad} D^T f_1 + \frac{\partial D}{\partial t} \right) \right|_{(\tilde{x}_0, \tilde{t}_0)} \neq 0. \tag{2}$$

Now the sensitivity of the end point y_0 to changes of the initial point x_0 will be investigated under the following conditions:

- the total integration time T remains constant,
- the starting time t_0 is constant. For reasons of simplicity of the formulae derived next, this time is set to zero, without loosing generality.

We define functions

$$y = \varphi_i(x, \tau, t) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n \tag{3}$$

which map x into y by integrating the differential equation $\dot{x} = f_i(x,t)$ starting in x at time t over a time interval τ. Obviously

$$\left\{ \begin{array}{l} \tilde{x}_0 = \varphi_1(x_0, \tilde{t}_0, 0), \\ y_0 = \varphi_2(\tilde{x}_0, T - \tilde{t}_0, \tilde{t}_0). \end{array} \right. \tag{4}$$

Now assign to *each* x in an open ε-ball $B(x_0, \varepsilon)$ around x_0, $\varepsilon > 0$, a vector $\tilde{x}$ and a time $\tilde{t}$ such that

$$\left\{ \begin{array}{l} \tilde{x} = \varphi_1(x, \tilde{t}, 0), \\ D(\tilde{x}, \tilde{t}) = 0. \end{array} \right. \tag{5}$$

Such an ε always exists due to (2) and the smoothness assumption on f_1 and D. Integration of the differential equations from arbitrary $x \in B(x_0, \varepsilon)$ at time $t = 0$ over a fixed time T delivers:

$$y(x) = \varphi_2(\tilde{x}(x), T - \tilde{t}(x), \tilde{t}(x)). \tag{6}$$

At this point the desired sensitivity of y w.r.t. x at x_0 under the mentioned conditions can be calculated by differentiating (6) using the chain rule and evaluating the result at the values corresponding to the central trajectory: $(x_0, \tilde{x}_0, \tilde{t}_0)$. Applying this to the i-th component of y, y_i, gives:

$$\left.\frac{\partial y_i}{\partial x_j}\right|_{x_0} = \sum_{k=1}^{n} \left.\frac{\partial \varphi_{2,i}}{\partial x_k}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \left.\frac{\partial \tilde{x}_k}{\partial x_j}\right|_{x_0} + \left.\left(\frac{\partial \varphi_{2,i}}{\partial t} - \frac{\partial \varphi_{2,i}}{\partial \tau}\right)\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \left.\frac{\partial \tilde{t}}{\partial x_j}\right|_{x_0}. \tag{7}$$

The partial derivatives $\frac{\partial \tilde{x}_k}{\partial x_j}$ and $\frac{\partial \tilde{t}}{\partial x_j}$ follow from differentiating the defining equations for $\tilde{x}$ and $\tilde{t}$ (5):

$$\begin{cases} \sum_{k=1}^{n} \left.\frac{\partial D}{\partial x_k}\right|_{(\tilde{x}_0, \tilde{t}_0)} \left.\frac{\partial \tilde{x}_k}{\partial x_j}\right|_{x_0} + \left.\frac{\partial D}{\partial t}\right|_{(\tilde{x}_0, \tilde{t}_0)} \left.\frac{\partial \tilde{t}}{\partial x_j}\right|_{x_0} = 0 \\ \left.\frac{\partial \tilde{x}_k}{\partial x_j}\right|_{x_0} = \left.\frac{\partial \varphi_{1,k}}{\partial x_j}\right|_{(x_0, \tilde{t}_0, 0)} + \left.\frac{\partial \varphi_{1,k}}{\partial \tau}\right|_{(x_0, \tilde{t}_0, 0)} \left.\frac{\partial \tilde{t}}{\partial x_j}\right|_{x_0} \end{cases} \tag{8}$$

Combining (7) and (8), for all i and j, the following result is obtained (in a compact notation):

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x_0} = {} & \left.\frac{\partial \varphi_2}{\partial x}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \left.\frac{\partial \varphi_1}{\partial x}\right|_{(x_0, \tilde{t}_0, 0)} + \\ & \frac{\left(\left.\frac{\partial \varphi_2}{\partial \tau}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} - \left.\frac{\partial \varphi_2}{\partial t}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} - \left.\frac{\partial \varphi_2}{\partial x}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \left.\frac{\partial \varphi_1}{\partial \tau}\right|_{(x_0, \tilde{t}_0, 0)} \right)}{\left.\mathrm{grad} D^T\right|_{(\tilde{x}_0, \tilde{t}_0)} \left.\frac{\partial \varphi_1}{\partial \tau}\right|_{(x_0, \tilde{t}_0, 0)} + \left.\frac{\partial D}{\partial t}\right|_{(\tilde{x}_0, \tilde{t}_0)}} \\ & \left.\mathrm{grad} D^T\right|_{(\tilde{x}_0, \tilde{t}_0)} \left.\frac{\partial \varphi_1}{\partial x}\right|_{(\tilde{x}_0, \tilde{t}_0, 0)} \end{aligned} \tag{9}$$

where

$$\begin{aligned} M_1 &= \left.\frac{\partial \varphi_1}{\partial x}\right|_{(x_0, \tilde{t}_0, 0)} \in \mathbb{R}^{n \times n} & M_2 &= \left.\frac{\partial \varphi_2}{\partial x}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \in \mathbb{R}^{n \times n} \\ R_1 &= \left.\frac{\partial \varphi_1}{\partial \tau}\right|_{(x_0, \tilde{t}_0, 0)} \in \mathbb{R}^{n \times 1} & R_2 &= \left.\frac{\partial \varphi_2}{\partial \tau}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \in \mathbb{R}^{n \times 1} \end{aligned} \tag{10}$$

From the definition of φ_1 and φ_2 it follows that M_1 would be the sensitivity of $\tilde{x}$ w.r.t. changes of x when spatial perturbations were integrated with righthand side f_1 over a fixed time $\tilde{t_0}$. In other words, M_1 is the result of integration of the variational equations corresponding to $\dot{x} = f_1(x,t)$, namely $\dot{x} = \frac{\partial f_1}{\partial x}\Big|_{x_r(t)} x$ with $x_r(t) = \varphi_1(x_0, t, 0)$ the solution of (1), the unity matrix as initial condition, over fixed time $\tilde{t_0}$ starting in x_0 at time $t = 0$. Similary, M_2 is the result of integration of the variational equations around $\dot{x} = f_2(x,t)$ starting in $\tilde{x}_0$ at $\tilde{t_0}$ over a time interval $T - \tilde{t_0}$. R_1 and R_2 are the velocity vectors of the central trajectory at the end of the first and the second segment respectively. In other words

$$R_1 = f_1(\tilde{x}_0, \tilde{t_0}), \quad R_2 = f_2(y_0, T). \tag{11}$$

Equation (9) now becomes (for simplicity evaluation points are omitted):

$$\frac{dy}{dx}\Big|_{x_0} = M_2 M_1 + \frac{(R_2 - \frac{\partial \varphi_2}{\partial t}\Big|_{(\tilde{x}_0, T - \tilde{t_0}, \tilde{t_0})} - M_2 R_1)\mathrm{grad} D^T}{\mathrm{grad} D^T R_1 + \frac{\partial D}{\partial t}} M_1 \tag{12}$$

At this moment we will distinguish between two cases: driven and non-driven systems. By driven (non-driven) systems we denote systems with piecewise dynamics described by equations in the form $\dot{x} = f_i(x,t)$ ($\dot{x} = f_i(x)$). The analysis will show that both cases will lead to the same sensitivity formula (18). In section 4 we will use the term autonomous to denote systems with trajectories which are invariant w.r.t. time-shifts i.e. non-driven systems switched at *fixed* boundaries.

Case 1: non-driven systems

The second term between brackets in the numerator of (12) vanishes and because of the invariance of the trajectories w.r.t. time-shifts in the 'free' areas.

$$R_2 = M_2 \acute{R}_1 \tag{13}$$

where $\acute{R}_1$ is the velocity vector at the beginning of the second segment, $f_2(\tilde{x}_0, \tilde{t_0})$. Therefore (12) can be simplified to (18).

Case 2: driven systems

In this case, (13) is not valid because a perturbation in the direction of the trajectory (parallel to $\acute{R}_1$) would disturb the phase relation between source and state.

However when one couples spatial perturbations of the starting point $\tilde{x}_0$ on the original trajectory with perturbations in the initial time (i.e. source-phase) $\tilde{t}_0$ in such a way that the new starting point is consistent (in time and space) with the original trajectory, the perturbed trajectory wil exactly track the original, a property analogous to non-driven systems. Mathematically, according to figure 3 one can first assign to each $t \in [\tilde{t}_0, T]$ a corresponding vector

$$z(t) = \varphi_2(\tilde{x}_0, t - \tilde{t}_0, \tilde{t}_0) \tag{14}$$

Thus $z(t)$ is the point on the original trajectory corresponding to source-phase t. Next one can look at the end point of a trajectory starting at time t in $z(t)$ integrated over $T - \tilde{t}_0$:

$$w(t) = \varphi_2(z(t), T - \tilde{t}_0, t) \tag{15}$$

Since then $z(t)$ and $w(t)$ exactly track the original trajectory, the derivative of w with respect to t in $\tilde{t}_0$ is the velocity vector R_2. Differentiating (15) delivers

$$\left.\frac{dw}{dt}\right|_{\tilde{t}_0} = \left.\frac{\partial \varphi_2}{\partial x}\right|_{(z(\tilde{t}_0)=\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \left.\frac{dz}{dt}\right|_{\tilde{t}_0} + \left.\frac{\partial \varphi_2}{\partial t}\right|_{(z(\tilde{t}_0)=\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)}. \tag{16}$$

Because $\left.\frac{dz}{dt}\right|_{\tilde{t}_0} = \left.\frac{\partial \varphi_2}{\partial \tau}\right|_{(\tilde{x}_0, 0, \tilde{t}_0)} = f_2(\tilde{x}_0, \tilde{t}_0) = \acute{R}_1$, this equation can be rewritten as:

$$R_2 = M_2 \acute{R}_1 + \left.\frac{\partial \varphi_2}{\partial t}\right|_{(\tilde{x}_0, T-\tilde{t}_0, \tilde{t}_0)} \tag{17}$$

Combining (17) with (12) yields a formula which is valid for driven as well as for non-driven systems:

$$\left.\frac{dy}{dx}\right|_{x_0} = M_2 \left[I + \frac{(\acute{R}_1 - R_1)\mathrm{grad}D^T}{\mathrm{grad}D^T R_1 + \frac{\partial D}{\partial t}} \right] M_1 = M_2 \Pi M_1 \tag{18}$$

2.2 Properties of the sensitivity formula

The sensitivity $\left.\frac{dy}{dx}\right|_{x_0}$ consists of a product of three factors M_1, Π and M_2, where the M_i can be calculated by integrating variational equations over

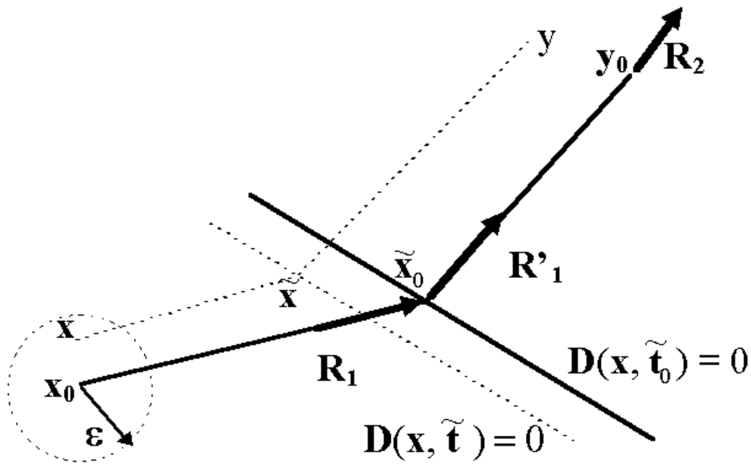

Figure 2: Switched trajectory

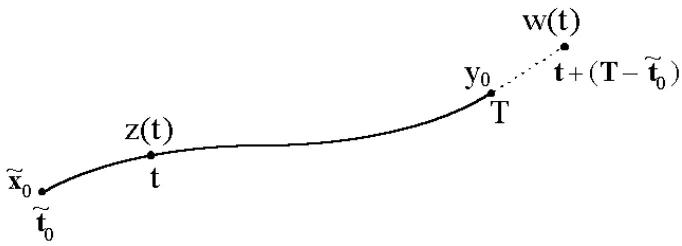

Figure 3: Driven systems. When spatial perturbations are coupled with perturbations in the starting time the original trajectory is tracked.

fixed times and where

$$\Pi = I + \frac{(\acute{R}_1 - R_1)\text{grad}D^T}{\text{grad}D^T R_1 + \frac{\partial D}{\partial t}} \tag{19}$$

is a rank-1 matrix which can be seen as a correction for the non-smoothness of the trajectories at the boundaries. The matrix Π has following properties:

- Only in the case of 'hard' discontinuities, where $f(x,t)$ is not continuous at the boundaries, Π differs from unity. In the rest of the paper this kind of discontinuities is assumed. For systems where the righthand side of the equations is continuous (but possibly with discontinuous derivates) no adaption of the classical sensitivity formula is needed. The underlying fact is that, due to the integration process, the trajectories have continuous velocity vectors.

- Π is not defined when $\text{grad}D^T R + \frac{\partial D}{\partial t} = 0$ i.e. when the trajectory is tangent to the discontinuity surface. That condition is known in literature as the *grazing* condition (see for example [5]). The fact that the components of the correction matrix Π tend to infinity when a grazing point is approached reflects the locally infinite stretching of the vector field approaching a grazing point from inside (i.e. from the side at which the boundary is effectively hit). Obviously the nonlinear mapping $x \to \varphi(x,\tau,t)$ (following definition (3)) is discontinuous in each $\hat{x}$ where $\hat{x}$ is chosen so that the corresponding trajectory gives rise to the grazing phenomenon (i.e. will somewhere be tangent to a boundary).

3 Application to the computation of periodic solutions by a shooting method

3.1 Classical shooting method

The sensitivity formula derived in the previous section can be used in the context of solving arbitrary two-point boundary value problems by a shooting methed. A shooting method (see for example [3]) is based on time integration and allows to deal relatively easy with changes in the structure of the system. In thie section we focus on the computation of periodic solutions of dynamical systems described by ordinary differential equations

$$\dot{x} = F(x,(t)), \quad x \in \mathbb{R}^n. \tag{20}$$

When the system is non-autonomous the righthand side of the equation is assumed to be periodic with period T, i.e. $f(x,t) = f(x,t+T)$. The (single) shooting method can be summarized as follows: denote by $\varphi(x_0, \tau, t_0)$ the mapping consisting of integrating the differential equations from x_0 at t_0 over a time τ. Then a periodic solution with period T satisfies

$$x_0 - \varphi(x_0, T, t_0) = 0 \tag{21}$$

with unknown x_0 for forced systems and

$$\begin{cases} x_0 - \varphi(x_0, T) = 0 \\ p(x_0, T) = 0 \end{cases} \tag{22}$$

with unknowns x_0 and T for autonomous systems. The 'phase-condition' $p(x_0, T)$ has to be added for uniqueness of the solution due to the invariance of the trajectories with respect to time shifts. The shooting process consists of solving the nonlinear equations (21) and (22) using Newton's method. Therefore, in order to calculate the Jacobian matrix, in both cases, $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_0,(t_0),T)}$ is needed. In classical shooting algorithms developed for smooth system its elements are usually calculated using finite differences, or more accurately, by integrating the variational equations around a trajectory $x_r(t) = \varphi(x_0, t, t_0)$ passing in x_0 at time t_0:

$$\begin{cases} \dot{x}_r = F(x_r, t) \\ \delta \dot{x}_1 = \left.\frac{\partial F}{\partial x}\right|_{x_r(t)} \delta x_1 \\ \vdots \\ \delta \dot{x}_n = \left.\frac{\partial F}{\partial x}\right|_{x_r(t)} \delta x_n \end{cases} \tag{23}$$

over fixed time T starting from n independent perturbations $\delta x_i(t_0)$.

3.2 Shooting methods for variable structure systems

3.2.1 Problems with classical shooting software

When one uses shooting algorithms based on the integration of the variational equations (23) for the calculation of periodic solutions of variable structure systems, a failure may occur although the underlying integration process can easily deal with jumps in the righthand sides of the equations. This can be seen as follows: suppose the system $\dot{x} = F(x,t)$ has the structure of (1) and one integrates the variational equations from x_0 at time t_0, then the last equation of (23) can be written as

$$\begin{cases} \delta \dot{x}_n = \left.\frac{\partial f_1}{\partial x}\right|_{x_r(t)} \delta x_n, & t \leq \tilde{t}_0, \\ \delta \dot{x}_n = \left.\frac{\partial f_2}{\partial x}\right|_{x_r(t)} \delta x_n, & t > \tilde{t}_0. \end{cases} \tag{24}$$

because the derivates of F with respect to x are evaluated at $x_r(t)$ which switches at $\tilde{t}_0$. In fact equation (23) calculates $M_2 M_1$ (defined in (10)). But this implies that all perturbed trajectories would reach the switching surface $D(x,t)$ at the same time $\tilde{t}_0$ which contradicts with the geometrical aspect of the switching: the trajectories should switch when they 'hit' the boundary.

3.2.2 Adaptation of the algorithm

In order to calculate correctly the Jacobian matrix one should use formula (18) for $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_0,T,(t_0))}$. Because (18) is well separated into the two terms M_1 and M_2 (which can be calculated by integrating the variational equations over fixed times dictated by the central trajectory) and a correction Π at the boundary, the following algorithm can be used:

Algorithm 3.1 *Calculation of* $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_0,(t_0),T)}$*:*

step 1 $J = I$ *(unity matrix) and* $i = 1$

step 2 Integrate the variational equation corresponding to f_i*, matrix* J *as initial condition. Stop when the total integration time is* T *or when a boundary* D_i *is detected (i.e.* $D_i(x_r(t), t) = 0$*) The result is stored in* J*. When the accumulated integration time is* T *go to step 5.*

step 3 Calculate the correction mapping Π_i*; set* $J = \Pi_i J$

step 4 $i = i + 1$*. Go to step 2*

step 5 $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_0,(t_0),T)} = J.$

Consequently classical shooting algorithms are extremely simple to adapt to piecewise-smooth systems. One just has to build in a detector of the boundaries and the correction matrices Π_i can be calculated with hardly extra cost.

Remark:

When switching is *always* performed at fixed times, for example at a prescribed phase of a periodic reference signal, the correction matrices Π are unity and one doesn't have to interrupt the integration process if switching occurs. Mathematically, time switching can indeed be seen as switching on an infinitely fast moving switching plane: $\frac{\partial D}{\partial t} = \infty$. This result is logical because as shown in (24) applying shooting algorithms for smooth systems (which calculate $\prod_i M_i$) to variable structure systems implies switching at fixed times.

Because the correction matrix Π only depends on information about the central trajectory $x_r(t)$ and not on the whole perturbation behaviour in the neighbourhood of the switching plane, calculating the evolution of a perturbation along a trajectory passing a boundary just requires one extra time integration.

3.3 Eigenvectors and eigenvalues of the monodromy matrix

3.3.1 Properties

When the Newton process of the shooting method converges, $\left.\frac{\partial \varphi}{\partial x}\right|_{(x_0,(t_0),T)}$ tends to the monodromy matrix the eigenvalues of which determine the stability of the solution. We will show some interesting properties of the monodromy matrices of variable structure systems.

According to figure 4 we define the 'sensitivity' matrices

$$M_i = \frac{\partial x_i}{\partial x_{i-1}} \tag{25}$$

for integration over time intervals $t_{i+1} - t_i$ starting in x_i at t_i, without taking switching into account. Now we establish relations between the eigenvectors of the monodromy matrices in x_0 and x_2, lying on different segments of the cycle. The obtained results are generally valid but the terminology of figure 4 is used for simplicity.

Denote by E_1 the eigenvector with eigenvalue (Floquet multiplier) λ of the monodromy matrix $M^{(1)}$ of the original trajectory starting in x_0, in other words

$$M^{(1)} E_1 = \lambda E_1, \tag{26}$$

or using (25) and (18)

$$(M_4 \Pi_2 M_3 M_2 \Pi_1 M_1) E_1 = \lambda E_1 \tag{27}$$

for the situation shown in figure 4. In this equation Π_1 and Π_2 are the correction mappings defined in the previous section in order to obtain correct sensitivity formulae. Then it follows after multiplication of both side of (27) with $M_2\Pi_1 M_1$ that

$$M^{(2)}E_2 = \lambda E_2 \tag{28}$$

where $E_2 = M_2\Pi_1 M_1 E_1$ and $M^{(2)} = M_2\Pi_1 M_1 M_4 \Pi_2 M_3$ the monodromy matrix calculated in x_2 with starting time t_2. This means that, as in the case of an overall smooth vector field, the Floquet multipliers are independent of the point in which the monodromy matrix is calculated. Furthermore, the eigenvectors corresponding to the same Floquet multiplier are mapped into each other by the sensitivity matrices, with corrections on the boundaries. However, the manifold of those eigenvectors is *not smooth* when passing the boundary. In fact the eigenvectors can only be defined in each point not on the boundary and the limits of these eigenvectors on each side of the boundary are mapped into each other by the correction matrices Π_i. An illustration of this property are the velocity vectors (eigenvectors corresponding to the trivial Floquet multiplier) which indeed change discontinuously.

When the switching boundaries are *not* time-varying (i.e. $\frac{\partial D}{\partial t} = 0$), using

$$\acute{E} = \left(I + \frac{(\acute{R} - R)\text{grad}D^T}{\text{grad}D^T R} \right) E \tag{29}$$

one can show that

$$\underbrace{\left(I - \frac{R\text{grad}D^T}{\text{grad}D^T R} \right) E}_{\text{projection on grad}D^T x=0, \text{ direction } R} = \underbrace{\left(I - \frac{\acute{R}\text{grad}D^T}{\text{grad}D^T R} \right) \acute{E}}_{\text{projection on grad}D^T x=0, \text{ direction } \acute{R}} \equiv E_p \tag{30}$$

which states that the projection of E on the plane tangent to the boundary in the direction of the velocity vector R equals the projection of $\acute{E}$ on the same plane, in the direction $\acute{R}$. Hence we have proven the following result.

Theorem 3.1 *Denote by $M(x)$ the monodromy matrix calculated in a point x of a limit cycle $x(t)$ switching in $\tilde{x}$ at $\tilde{t}$ on the fixed boundary $D(x) = 0$. Let E be an eigenvector of $M^- = \lim_{t\to\tilde{t}-} M(x(t))$. Then*
$\acute{E} = \Pi E$ is an eigenvector of $M^+ = \lim_{t\to\tilde{t}+} M(x(t))$, with $I + \frac{(\acute{R}-R)\text{grad}D^T}{\text{grad}D^T R}$. Furthermore $P_D^R E = P_D^{\acute{R}} \acute{E} \equiv E_p$ where P_D^R respectively $P_D^{\acute{R}}$ are the projection matrices on $\text{grad}D^T\big|_{\tilde{x}}\, x = 0$ in the direction of R respectively $\acute{R}$.

Suppose that the system is autonomous and instead of working with monodromy matrices, the $(n-1)$-dimensional surface $D(x) = 0$ is used as

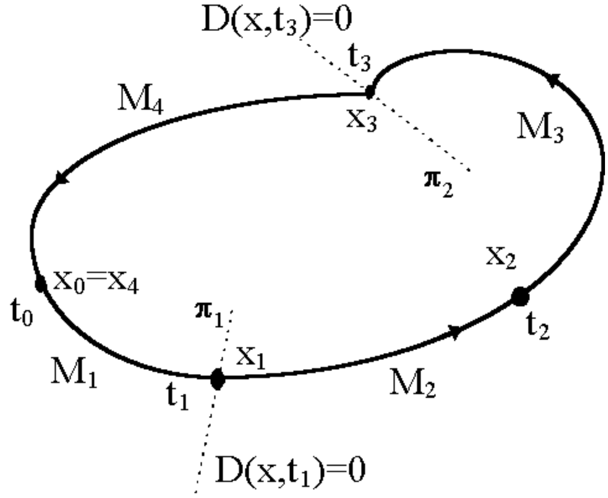

Figure 4: Switched limit cycle of a variable structure system. Π_1 and Π_2 are the correction mappings at the swithching boundary. $D(x_1, t_1) = 0$ and $D(x_3, t_3) = 0$.

Poincaré-section to determine stability, i.e., one considers the eigenstructure of the linearized map mapping successive intersection points of the trajectories with the surface into each other. Then E_p would be an eigenvector of this linearized Poincaré-map with section $\text{grad}D^T x = 0$ except for the velocity vectors (=eigenvectors corresponding to the trivial eigenvalue 1 of the monodromy matrix) which are projected into zero. This is shown in figure 5. This relation between the eigenvectors of the monodromy matrix and those of the linearized Poincaré-map is a natural extension of the results for smooth systems: in that case $\acute{R} = R$, $\acute{E} = E$ and obviously $E_p = P_D^R E$.

3.3.2 Consequences for stability analysis

- It is possible that the spectral radius of $M^{(1)}$ is greater than one although the product $M_4 M_3 M_2 M_1$ indicates stability, in other words that the geometrical component of the switching, expressed mathematically by the mappings Π_i, destabilizes the limit-cycle. A practical example can be found in [4]. It concerns a static DC-to-DC buck-convertor where switching between two possible configurations occurs when a feedback signal concerning information about voltages and cur-

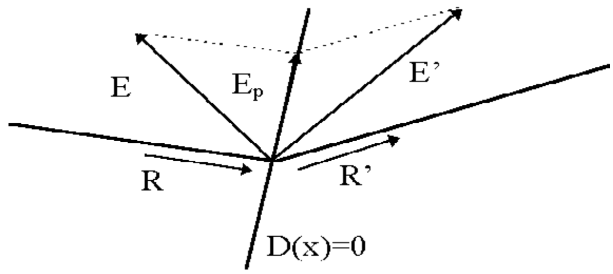

Figure 5: If the boundary $D(x) = 0$ is fixed, the projections of the eigenvectors coincide. E_p is an eigenvector of the linearized Poincaré map with section $D = 0$.

rents (state variables) equals an externally generated sawtooth time signal. In a particular situation chaos was detected in the output signal (normally a switched periodic signal) but the underlying periodic orbit could be stabilized by replacing the feedback control by pure time-switching. Mathematically the elimination of the geometric component is expressed by the fact that the corrections Π_i become unity.

- Also the opposite is true: sometimes an unstable switched cycle can be stabilized by manipulating the boundaries. This is extensively explained in the next section but the underlying idea is the following: when the angle between the normal vector of the surface and the outgoing trajectory is close to $\pi/2$, the mapping Π performs strong contraction on the vector field which can possibly compensate the diverging behaviour between switching instants. This phenomenon is mentioned in [7].

3.4 Calculating Lyapunov exponents of variable structure systems

Lyapunov exponents (see for example [2]) play an important role in the characterization of the long term behaviour of dynamical systems. They give useful information about the converging/diverging behaviour of perturbations around a trajectory and can be used to classify attractors of the dynamical system (periodic, quasiperiodic, chaotic, hyperchaotic,...). For variable structure systems they are even more important because chaotic behaviour is often observed.

The problem of the calculating the Lyapunov exponents is strongly related to the previous one because the long term evolution of perturbations around a trajectory (involved with the calculation of Lyapunov exponents) can be calculated by integrating the variational equations and when the structure of the system changes, the algorithm has to be adapted. In [10] this is applied to an impact oscillator.

4 Control theory point of view

4.1 Introduction

In this section we investigate the influence of the geometric position of the switching planes on the eigenvalues of the monodromy matrix and consequently on the stability of the periodic solution. Thereby links with classical linear control are made.

In linear control theory one considers systems of the form

$$\dot{x} = Ax + Bu \tag{31}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ the input. Controllability of the system is expressed mathematically by the non-singularity of the controllability matrix $T = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$. Using a suitable state feedback $u = K^T x$ with $K \in \mathbb{R}^{n\times 1}$, the poles (i.e. eigenvalues) of the closed loop system $\dot{x} = (A + BK^T)x$ can be chosen arbitrary.

The starting point of this discussion is a switched limit cycle of an (non)autonomous system $\dot{x} = f(x,t)$. The influence of the position of the plane $D = 0$ is investigated. As depicted in figure 6, we denote by P the point on the limit cycle just before switching occurs. The monodromy matrix in P can thus be calculated as $M\Pi$ with $\Pi = I + \frac{(\acute{R}-R)\mathrm{grad}D^T}{\mathrm{grad}^T R + \frac{\partial D}{\partial t}}$. Hereby we consider a fixed matrix M, i.e. we investigate the influence of the geometric postion of the switching plane near the intersection point P while the geometric structure in the other switching points remains unchanged.

Define δx^k as the mapping of a n-dimensional perturbation δx^0 around P after a time kT with T the period of the limit cycle. Then the following holds:

$$\delta x^{k+1} = M\Pi\delta x^k, \tag{32}$$

or

$$\delta x^{k+1} = M\delta x^k + M(\acute{R} - R)\frac{\mathrm{grad}D^T}{\mathrm{grad}D^T R + \frac{\partial D}{\partial t}}\delta x^k, \tag{33}$$

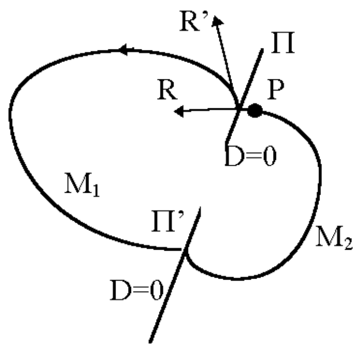

Figure 6: Switched limit cycle. The influence of the intersection near P is investigated, $M = M_2\acute{\Pi}M_1$.

which can be seen as a standard linear control system

$$\delta x^{k+1} = Ax^k + Bu, \quad A = M, \quad B = M(\acute{R} - R), \tag{34}$$

with a static state feedback law $u = K^T \delta x^k = \frac{\mathrm{grad} D^T}{\mathrm{grad}^T R + \frac{\partial D}{\partial t}} \delta x^k$.

Now we suppose that the position and the 'velocity' of the switching plane at the intersection point are free parameters and we investigate how they can be determined to result in a desired behaviour of perturbations around the limit cycle, i.e. in a desired position of the Floquet multipliers.

From (34) it is clear that the possibility of choosing all the Floquet multipliers is expressed by the controllability of the pair $\Big(M, M(\acute{R} - R)\Big)$ but the set of all possible feedback matrices K satisfies a parametrization constraint:

$$K \in W = \left\{ K : \exists D(x,t) \ni K = \frac{\mathrm{grad} D}{\mathrm{grad} D^T R + \frac{\partial D}{\partial t}} \right\} \tag{35}$$

- When the plane is fixed (i.e. $\frac{\partial D}{\partial t} = 0$) the set W is a hyperplane:

$$W = \left\{ K \ni K^T R = 1 \right\} \tag{36}$$

- When the plane is allowed to vary in time ($\frac{\partial D}{\partial t} \neq 0$) we have $W = \mathbb{R}^n$.

4.2 Case 1: time-varying boundary (i.e. $\frac{\partial D}{\partial t} \neq 0$)

The following theorem holds:

Theorem 4.1 *When the boundary is time-varying and $(M, M(\acute{R} - R))$ is controllable, all the Floquet multipliers can be placed arbitrary.*

Proof 4.1 *The set W of all admissible controls is the whole space. Combining with the controllability property proves the statement of the theorem.* □

Generally, i.e. in the non-degenerate case, $(M, M(\acute{R} - R))$ is controllable, also in the case when the piecewise dynamics are described by equations of the form $\dot{x} = f_i(x)$. Indeed, due to the time-varying boundary the system is non-autonomous and therefore there is no trivial Floquet-multiplier and thus no constraint on the position of the Floquet multipliers.

The following algorithm, based on Ackermann's formula [1] can be used to solve the pole-placement problem. Thereby the simplest case is considered where all desired poles are separated.

Algorithm 4.1 *Pole-placement method*

- *calculate*
$$K^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} T^{-1} \alpha_c(M)$$
where T is the controllability matrix $\begin{bmatrix} M(\acute{R} - R) & M^2(\acute{R} - R) & \cdots & M^n(\acute{R} - R) \end{bmatrix}$ and $\alpha_c(M) = M^n + \alpha_1 M^{n-1} + \alpha_2 M^{n-2} + \cdots + \alpha_n I$ the polynomial with the desired Floquet multipliers as roots.

- *calculate*
$$\text{grad} D^T = cK^T$$
$$\frac{\partial D}{\partial t} = c(1 - K^T R)$$
for some $c \neq 0$

4.3 Case 2: fixed boundary (i.e. $\frac{\partial D}{\partial t} = 0$)

In non-degenerate cases the system $\left(M, M(\acute{R} - R)\right)$ is controllable, but there is a parametrization constraint on all admissible feedback gains K,

$$K^T R = 1$$

which prohibits the free choice of all the Floquet multipliers. This can be considered as a new form of uncontrollability.

However when the system is *autonomous* the parametrization constraint is natural and corresponds to the fact that each 'control' should create the trivial Floquet multiplier and its corresponding eigenvector (the velocity vector), as stated in the following theorem.

Remark

When the system is autonomous, this implies that

- the piecewise dynamics are governed by equations in the form $\dot{x} = f_i(x)$ (non-driven systems),
- all the boundaries are fixed.

Using the notation of figure 6 this can be expressed mathematically by the following invariance property:

$$M\acute{R} = R.$$

Theorem 4.2 *When the system is autonomous, the parametrization constraint $K^T R = 1$ is equivalent with: R is the eigenvector of the monodromy matrix corresponding to the trivial Floquet multiplier* 1.

Proof 4.2 *Because $\acute{R} \neq R$ and consequently $MR \neq R$:*

$$\begin{gathered} K^T R = 1 \\ \Updownarrow \\ MR(1 - K^T R) + R(K^T R - 1) = 0 \\ \Updownarrow \\ M\left(I + (\acute{R} - R)K^T\right) R = R \\ \Updownarrow \\ M\Pi R = R \end{gathered}$$

□

From theorem 4.2 it follows that each control which satisfies the parametrization constraint creates the trivial Floquet multiplier. Now we prove the opposite: each control which creates an eigenvalue 1 satisfies the parametrization constraint i.e. the control can be realized by a suitable geometric position of the switching plane.

Theorem 4.3 *All feedback gains K (taken the control theory point of view) creating an eigenvalue 1 satisfy the parametrization constraint (36).*

Proof 4.3 *Take an arbitrary fixed position of the switching plane (expressed by a choice of Π). This implies that R is an eigenvector of $M\Pi$ with eigenvalue* 1*. Then because of the controllability the system*

$$\delta x^{k+1} = M\Pi\delta x^k = M\delta x^k + M(\acute{R} - R)K^T\delta x^k$$

can be transformated using the (linearized) state transformation

$$\delta z^k = L^{-1}\delta x^k$$

with $L^{-1} = \begin{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} T^{-1}M^{n-1} \\ \vdots \\ \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} T^{-1}M \\ \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} T^{-1} \end{bmatrix}$
and T the controllability matrix $\begin{bmatrix} M(\acute{R}-R) & M^2(\acute{R}-R) & \cdots & M^n(\acute{R}-R) \end{bmatrix}$ *into the* control canonical form*:*

$$\delta z^{k+1} = \begin{bmatrix} c_1 & c_2 & \cdots & & c_n \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix} \delta z^k + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} K^T L \delta z^k \left(= L^{-1}M\Pi L\delta z^k \right)$$

$\lambda = 1$ is also an eigenvalue of $L^{-1}M\Pi L$ and the corresponding eigenvector in the z-space is given by

$$\begin{bmatrix} \lambda^{n-1} & \lambda^{n-2} & \cdots & \lambda & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix}^T$$

Transforming this eigenvector to the x-space yields:

$$R = \alpha L \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

for some α.

Because the transformation matrix L only depends on M, R and $\acute{R}$ (not on K!) and the eigenvector corresponding to the eigenvalue 1 *is always* $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$ *in the z-space, each feedback gain K creating an eigenvalue*

1 *automatically creates as corresponding eigenvector the velocity vector* R. *But following theorem 4.2, this implies that* K *satisfies the parametrization constraint.* $\square$

Now we can prove the key result of this section:

Theorem 4.4 *If the system is autonomous and* $\left(M, M(\acute{R} - R)\right)$ *is controllable, then* $(n-1)$ *Floquet multipliers can be chosen arbitrary.*

Proof 4.4 *Because of the controllability of* $\left(M, M(\acute{R} - R)\right)$ *there exists a feedback* K *for each desired position of the Floquet multipliers. When one Floquet multiplier is set to one and the other are chosen arbitrary, due to theorem 4.3, the corresponding feedback* K *automatically satisfies the parametrization constraint and is realizable by a fixed position of the switching plane.* $\square$

In the autonomous case algorithm 4.1 can be modified as follows: choose 1 as one root of α_c and grad$D$ is a multiple of K. Because K has to satisfy the parametrization constraint, one can use $K^T R = 1$ as a test.

4.4 Applications

As already mentioned in the introduction we have given a rather mathematical point of view reasoning in terms of limit cycles, Floquet multipliers, etc. There are many engineering disciplines where one implicitly uses the theory developed in this paper. Let us focus on the domain of static DC-to-DC convertors. Such systems switch between several configurations, each of them described by a (mostly linear) ODE. Their steady state solutions can be seen as switched limit cycles.

As we have seen in the previous sections, for the system's (small gain) behaviour to be completely determined by interaction with the boundary (i.e. free choice of all Floquet multipliers), a necessary condition is that the (equivalent) switching boundary is time-varying. This condition holds for most practical applications which enables the application of optimal control strategies [6], pole placement techniques, sliding mode control [8], etc. to control perturbations around the limitcycle.

Example: PWM-techniques

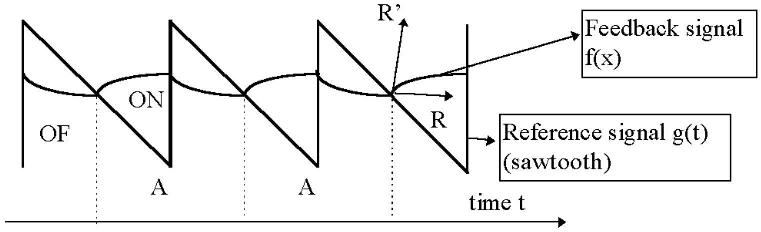

Figure 7: Principle of PWM: the state of the system is determined by the sign of $f(x) - g(t)$

In PWM (pulse width modulation) controlled convertors switching commands are given when an externally generated reference time signal (usually a sawtooth) equals a feedback control signal containing information about the state of the system. This is schematically shown in figure 7. In such systems the solution is in fact a limit cycle switched in the phase plane on the time-varying boundary $D(x,t) = f(x) - g(t)$. Two different types of switching occur, namely one pure time switching (point A) and a switching with a geometrical component (point B). Because the slope of the reference signal, i.e. $\frac{\partial D}{\partial t}$, is nonzero, the dynamics of the system can generally be controlled by a (designed) value of $\left.\frac{\partial f}{\partial x}\right|_{\tilde{x}}$ with $\tilde{x}$ the intersection point B. Usually $f(x)$ is linear. Thus f can be written as $f(x) = \alpha^T x$ and α can be chosen according a designed pole position or an optimal control strategy.

Another frequently used control design technique for switched DC-to-DC convertors is based on state space averaging [9] of the linearized power stage model leading to a transfer function from the duty-rate (ON-time/switching period) to the filtered (i.e. averaged) output voltage. Next the duty rate is used to control the output via (averaged) output feedback. When the averaged system's dynamics are sufficiently slow compared to the period of the reference signal, one actuallly changes the position of the 'switching plane' in order to control the output.

5 Conclusion

In this paper we have studied properties and algorithmic issues of variable structure systems. A general formula has been derived, which describes the evolution of perturbations around a solution of a variable structure system. This formula plays a crucial role in the adaptation of many algorithms based on simulation and developed for smooth systems to the piecewise smooth case. This is illustrated for the shooting method for computing periodic solutions. The clear structure of this formula allows an investigation of the geometric aspect of switching with respect to the stability of periodic solutions. We have taken a mathematical viewpoint by reasoning in terms of the theory of dynamical systems and a theoretical framework is developed in which many practical applications are unified, as illustrated for switched power convertors.

Further work will include the investigation of superstable limit cycles: when the switching plane contains the outgoing velocity vector, one Floquet multiplier is set to zero and consequently some perturbations can disappear in a finite time which implies superstability.

6 Acknowledgements

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A Mapping in the state space

In the article one basically has taken into account the geometric aspect of switching. A slight extension of the problem is formed by systems where, once a boundary is hit, a mapping in the state space is performed. Examples are mechanical systems where moving masses can interact with obstacles (hard constraints). For such systems the mapping in the state space at the boundary is determined by the kind of collisions involved (for example perfectly elastic collisions give rise to a reversal of the (relative) velocity).

Generally consider the following system where x^+ denotes the mapped state:

$$\left\{\begin{array}{lll} \dot{x} = f_i(x,t) & & \\ \quad i^+ = 2 & \text{if} \quad i^- = 1 & \\ \quad i^+ = 1 & \text{if} \quad i^- = 2 & \\ \quad x^+ = F(x^-) & & \end{array}\right. \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{when } D(x,t) = 0 \tag{37}$$

The last equation describes the smooth mapping of the state vector to be performed at the boundary $D(x,t) = 0$. Consider a trajectory starting in x_0 ($i = 1$) at $t = 0$ which reaches the boundary in a non-tangential way at $\tilde{t}_0$ in $\tilde{x}_0$ where swichting occurs ($i = 2$) and reaching y_0 in a time $T > \tilde{t}_0$. For each x in a sufficiently small neighbourhood of x_0 define point y as the point reached by integrating the differential equations (37) from x over fixed time T with the same starting time $t = 0$. Then using the definition of φ_i (3), y satisfies:

$$y = \varphi_2(F(\tilde{x}), T - \tilde{t}, \tilde{t}) \tag{38}$$

where $\tilde{t}$ and $\tilde{x}$ satisfy:

$$\begin{cases} D(\tilde{x}, \tilde{t}) = 0 \\ \tilde{x} = \varphi_1(x, \tilde{t}, 0) \end{cases} \tag{39}$$

Then analoguously as done in section 2 the sensitivity of y to changes of x at x_0 can be calculated as:

$$\left.\frac{dy}{dx}\right|_{x_0} = M_2 \left[J + \frac{\acute{R}_1 \text{grad} D^T}{\text{grad} D^T R_1 + \frac{\partial D}{\partial t}} - J \frac{R_1 \text{grad} D^T}{\text{grad} D^T R_1 + \frac{\partial D}{\partial t}} \right] M_1 \tag{40}$$

where M_1, M_2, R_1 are defined in (10), $\acute{R}_1 = \left.\frac{\partial \varphi_2}{\partial \tau}\right|_{(F(\tilde{x}_0), 0, \tilde{t}_0)}$ and J is the jacobian matrix of the map $F(x)$ evaluated at $\tilde{t}_0$, $\left.\frac{dF(x)}{dx}\right|_{\tilde{x}_0}$.

Formula (40) would equal $M_2 J M_1$ when all the trajectories would reach the surface D in the same time. The other terms thus reflect once again the influence of the true geometrical position of the boundary.