Decomposing the Secondary Cayley Polytope

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Abstract

The vertices of the secondary polytope of a point configuration correspond to its regular triangulations. The Cayley trick links triangulations of one point configuration, called the Cayley polytope, to the fine mixed subdivisions of a tuple of point configurations. In this paper we investigate the secondary polytope of this Cayley polytope. Its vertices correspond to all regular mixed subdivisions of a tuple of point configurations. We demonstrate that it equals the Minkowski sum of polytopes, which we call mixed secondary polytopes, whose vertices correspond to regular-cell configurations.

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DECOMPOSING THE SECONDARY CAYLEY POLYTOPE

TOM MICHIELS AND RONALD COOLS

Abstract. The vertices of the secondary polytope of a point configuration correspond to its regular triangulations. The Cayley trick links triangulations of one point configuration, called the Cayley polytope, to the fine mixed subdivisions of a tuple of point configurations. In this paper we investigate the secondary polytope of this Cayley polytope. Its vertices correspond to all regular mixed subdivisions of a tuple of point configurations. We demonstrate that it equals the Minkowski sum of polytopes, which we call mixed secondary polytopes, whose vertices correspond to regular-cell configurations.

1. Introduction.

Regular triangulations and mixed subdivisions play an important role in algebraic geometry (See [Stu94a],[Stu94b], [IR96],[LW98] and [Vir84]) and are used in homotopy continuation methods for solving polynomial systems (See [HS95] and [VVC94]). The vertices of the secondary polytope (See [GKZ94]) are in a one-to-one correspondence to regular triangulations of a point configuration. We show that for mixed subdivisions, this secondary polytope has a degenerate structure: it can be Minkowski decomposed.

In Sections 2 and 3 a brief introduction is given to regular triangulations and regular mixed subdivisions stressing the properties that are important to us. We define the I-mixed secondary polytopes in Section 4 and proof in Section 5 that the sum of these I-mixed secondary polytopes is the secondary polytope. The connection between bistellar flips and the edges of the I-mixed secondary polytopes is explained in Section 6. We conclude with an example in Section 7.

2. Preliminaries on Triangulations

A *triangulation* $\mathcal{T}$ of a finite point configuration $A \subset \mathbb{R}^d$ is a collection of cells $I \subset A$ of cardinality $d+1$ and $\dim(\mathrm{conv}(I)) = d$ such that $\bigcup_{I\in\mathcal{T}} \mathrm{conv}(I) = \mathrm{conv}(A)$ and $\forall I, J \in \mathcal{T}$: $\mathrm{conv}(I \cap J) = \mathrm{conv}(I) \cap \mathrm{conv}(J)$. A triangulation $\mathcal{T}$ is called *regular* (or *coherent*) if there exists a lifting vector $\omega \in \mathbb{R}^A$ such that $\{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in I\}$ with $I \in \mathcal{T}$ are the upper convex hull faces of $\{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in A\}$. (See [Lee91] for more on regular triangulations.)

A *circuit* is an affinely dependent point configuration Z with all proper subsets of Z affinely independent. Consequently there exists, up to real multiple, a unique affine relation between the points of the circuit: $\sum_{\mathbf{z}\in Z} \gamma_{\mathbf{z}}\mathbf{z} = 0; \sum_{\mathbf{z}\in Z} \gamma_{\mathbf{z}} = 1$. (See [Zie95].) In this paper we fix $|\gamma_{\mathbf{x}}| = \mathrm{vol}(Z \setminus \{\mathbf{x}\})$, where vol denotes a translation-invariant volume scaled to be 1 for the unit-simplex.

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A circuit has two triangulations $\mathcal{T}_+ = \{Z \setminus \{\mathbf{z}\} \mid \gamma_{\mathbf{z}} > 0\}$ and $\mathcal{T}_- = \{Z \setminus \{\mathbf{z}\} \mid \gamma_z < 0\}$. A triangulation $\mathcal{T}$ of $A \subset \mathbb{R}^d$ is called *flipable* over $Z \subset A$ if the simplices of $\mathcal{T}_{+(-)}$ are faces of simplices of $\mathcal{T}$ and if $\forall I \in \mathcal{T}, \forall J \in \mathcal{T}_{+(-)} : (J \subset I) \Rightarrow (\forall K \in \mathcal{T}_{+(-)} : K \cup (I \setminus J) \in \mathcal{T})$. This implies that $\exists F^{(1)}, F^{(2)}, \ldots, F^{(s)}$ such that $\forall I \in \mathcal{T}$: if $\exists I_+ \in \mathcal{T}_+$ with $I_+ \subset I$ then $\exists j : I = I_+ \cup F^{(j)}$ and $\forall j \in \{1, \ldots, s\}, \forall I_+ \in \mathcal{T}_+ : F^{(j)} \cup I_+ \in \mathcal{T}$.

Definition 2.1. *Given a circuit Z and a triangulation $\mathcal{T}$ flipable over Z then the* bistellar flip *of $\mathcal{T}$ over Z,* $\text{flip}_Z(\mathcal{T})$*, is the triangulation obtained by replacing all cells I of $\mathcal{T}$ having a cell $I_+ \in \mathcal{T}_+$ as a face of any dimension by $(I \setminus I_+) \cup I_-$ with $I_- \in \mathcal{T}_-$.*

A cell I of $\mathcal{T}$ is *involved* in a bistellar flip flip_Z if $I \notin \text{flip}_Z(\mathcal{T})$, i.e., if $\#(Z \setminus I) = 1$.

Theorem 2.2. *Given a circuit Z and two cells $I^{(1)} = Z \setminus \{\mathbf{z}^{(1)}\} \cup F$ and $I^{(2)} = Z \setminus \{\mathbf{z}^{(2)}\} \cup F$, not necessarily belonging to the same triangulation then*

$$\frac{\text{vol}(\text{conv}(I^{(1)}))}{\text{vol}(\text{conv}(I^{(2)}))} = \left| \frac{\gamma_{\mathbf{z}^{(1)}}}{\gamma_{\mathbf{z}^{(2)}}} \right|.$$

Proof. For notational convenience, a set of points is denoted by a $(d+1) \times (d+1)$-matrix whose rows are the affine coordinates of these points. Since the volume of a unit simplex is scaled to 1,

$$\text{vol}(\text{conv}(I^{(1)})) = |\det(I^{(1)})| = \left| \det \begin{pmatrix} F \\ Z^{(1)} \end{pmatrix} \right|.$$

Where $Z^{(1)}$ represent the points in $Z \setminus \{\mathbf{z}^{(1)}\}$. The points of $I^{(1)}$ are affinely independent and thus there exists an orthonormal transformation U such that

$$U. \begin{pmatrix} F \\ Z^{(1)} \end{pmatrix} = \begin{pmatrix} G & H \\ \mathbf{0} & Z'^{(1)} \end{pmatrix}$$

where $Z'^{(1)}$ is a square matrix with the same number of rows as $Z^{(1)}$. Then

$$\text{vol}(\text{conv}(I^{(1)})) = |\det(G)|.|\det(Z'^{(1)})|.$$

Observe that $\left|Z'^{(1)}\right|$ is the volume of $Z^{(1)}$ in its own dimension, i.e. $|\gamma_{\mathbf{z}^{(1)}}|$. Since Z is a circuit and thus affinely dependent, applying the same U to $I^{(2)}$ gives

$$U. \begin{pmatrix} F \\ Z^{(2)} \end{pmatrix} = \begin{pmatrix} G & H \\ \mathbf{0} & Z'^{(2)} \end{pmatrix}.$$

Combining all this proves

$$\frac{\text{vol}(\text{conv}(I^{(1)}))}{\text{vol}(\text{conv}(I^{(2)}))} = \left| \frac{\det(I^{(1)})}{\det(I^{(2)})} \right| = \left| \frac{\det(Z'^{(1)})}{\det(Z'^{(2)})} \right| = \left| \frac{\gamma_{\mathbf{z}^{(1)}}}{\gamma_{\mathbf{z}^{(2)}}} \right|.$$

Note that if $F = \emptyset$ then it follows immediately that

$$\frac{\text{vol}(\text{conv}(I^{(1)}))}{\text{vol}(\text{conv}(I^{(2)}))} = \left| \frac{\det(Z^{(1)})}{\det(Z^{(2)})} \right| = \left| \frac{\gamma_{\mathbf{z}^{(1)}}}{\gamma_{\mathbf{z}^{(2)}}} \right|.$$

□

Secondary polytopes were introduced in [GKZ94]. The vertices of a secondary polytope correspond to the regular triangulations of a polytope. This property was used in [dL95, TI97, MIK96] to enumerate all regular triangulations of point configurations. Secondary polytopes are generalized by fiber polytopes (See [BS92] and [Zie95, Lecture 9]) and universal polytopes (See [dLHSS96]).

Definition 2.3. *The characteristic function of a triangulation $\mathcal{T}$ of a point configuration $A \subset \mathbb{R}^d$ is*[1]

$$\varphi_{\mathcal{T}} : A \to \mathbb{R} : \mathbf{a} \mapsto \sum_{I | \mathbf{a} \in I} \mathrm{vol}(\mathrm{conv}(I))$$

where vol *is a translation-invariant volume scaled to be* 1 *for the unit simplex.*

Note that characteristic functions can be regarded as $\#A$-dimensional vectors.

Definition 2.4. *The* secondary polytope *of a point configuration $A \subset \mathbb{R}^d$ is*

$$\Sigma(A) := \mathrm{conv}\left(\{\varphi_{\mathcal{T}} \mid \mathcal{T} \text{ a triangulation of } A\}\right).$$

Theorem 2.5 (Theorem 1.7, page 221 and Theorem 2.11, page 233 [GKZ94])**.** *Let $A \subset \mathbb{R}^d$ be a point configuration then:*

1. *the vertices of $\Sigma(A)$ correspond to regular triangulations of A, i.e., the normal cones of $\Sigma(A)$ are exactly the cones of lifting vectors inducing the regular triangulations;*
2. *the edges of $\Sigma(A)$ correspond to the bistellar flips between regular triangulations.*

3. Preliminaries on Mixed Subdivisions

A *(regular) mixed subdivision $\mathcal{S}$* of $(A_1, A_2, \ldots, A_n)$ with $A_i \subset \mathbb{R}^d$ is a collection of cells $C = (C_1, C_2, \ldots, C_n)$ with $C_i \subset A_i$ such that the $\bigcup C_i \times \{\mathbf{e}^{(i-1)}\}$ make a (regular) triangulation of $\bigcup A_i \times \{\mathbf{e}^{(i-1)}\}$. This definition is equivalent with the usual definitions [Stu94b, GKZ94] of fine mixed subdivisions due to the Cayley trick [Stu94a, Lemma 5.2], [GKZ94, Proposition 1.7, page 274],[VGC96, Proposition 3.9]. All properties of (regular) triangulations can be formulated for (regular) mixed subdivisions.

A tuple $Z = (Z_1, Z_2, \ldots, Z_n)$ is a *mixed circuit* if $\bigcup Z_i \times \{\mathbf{e}^{(i-1)}\}$ is a circuit. We denote the affine coefficients of this (mixed) circuit by $\gamma_{i,\mathbf{z}}$ for $\mathbf{z} \in Z_i$. Note that since $\forall i : \sum_{\mathbf{z} \in Z_i} \gamma_{i,\mathbf{z}} = 0$, $\forall i : \#Z_i \neq 1$. A mixed subdivision $\mathcal{S}$ is flipable over Z if its corresponding triangulation is flipable over $\bigcup Z_i \times \{\mathbf{e}^{(i-1)}\}$. Two mixed subdivisions are connected by a bistellar flip flip_Z if their corresponding triangulations are connected by a bistellar flip over $\bigcup Z_i \times \{\mathbf{e}^{(i-1)}\}$.

A cell C of $\mathcal{S}$ is *involved* in a bistellar flip flip_Z if $C \notin \mathrm{flip}_Z(\mathcal{S})$, i.e., if $\sum_{i=1}^{n} \#(Z_i \setminus C_i) = 1$.

Definition 3.1. *For a mixed subdivision $\mathcal{S}$ the* characteristic function *is*

$$\varphi_{\mathcal{S}} = (\varphi_{\mathcal{S},1}, \varphi_{\mathcal{S},2}, \ldots, \varphi_{\mathcal{S},n})$$

with

$$\varphi_{\mathcal{S},i} : A_i \to \mathbb{R} : \mathbf{a} \mapsto \sum_{C | \mathbf{a} \in C_i} \mathrm{vol}\left(\mathrm{conv}\left(\bigcup C_j \times \{\mathbf{e}^{(j-1)}\}\right)\right).$$

For notational convenience from now on we will denote the volume of the convex hull of the simplex corresponding to a cell briefly with $\mathrm{vol}(C)$. Clearly $\varphi_{\mathcal{S}} \sim \varphi_{\mathcal{T}}$ for a triangulation $\mathcal{T}$ corresponding to a mixed subdivision $\mathcal{S}$.

[1] With $\sum_{X|Y}$ we mean *the summation over all X for which Y holds.*

Definition 3.2. *The* secondary polytope *of a tuple of point configurations* $(A_1, A_2, \ldots, A_n)$ *with* $A_i \subset \mathbb{R}^d$ *is*

$$\Sigma(A_1, A_2, \ldots, A_n) := \text{conv}\left(\{\varphi_{\mathcal{S}} \mid \mathcal{S} \textit{ a mixed subdivision of } (A_1, A_2, \ldots, A_n)\}\right).$$

As a consequence of Theorem 2.5 we have:

Theorem 3.3. *For a tuple of point configurations* $(A_1, A_2, \ldots, A_n)$*:*

1. *the vertices of* $\Sigma(A_1, A_2, \ldots, A_n)$ *correspond to regular mixed subdivisions;*
2. *the edges of* $\Sigma(A_1, A_2, \ldots, A_n)$ *correspond to bistellar flips between regular mixed subdivisions.*

4. Decomposing the Secondary Cayley Polytope

In this section we define I-mixed secondary polytopes and present our main theorem, stating that their Minkowski sum equals the secondary polytope.

Definition 4.1. *The type* $B(X)$ *of a tuple of point configurations (cell or mixed circuit)* $X = (X_1, X_2, \ldots, X_n)$ *is* $B(X) = \{i \in \{1, 2, \ldots, n\} \mid \#X_i > 1\}$.

Definition 4.2. *For* $I \subset \{1, 2, \ldots, n\}$ *the I-mixed characteristic function is*

$$\varphi^I_{\mathcal{S},i} : A_i \to \mathbb{R} : \mathbf{x} \mapsto \sum_{C \in S \mid B(C) \cup \{i\} = I, \mathbf{x} \in C_i} \text{vol}(C).$$

Theorem 4.3. $\varphi_{\mathcal{S},i} = \sum_{I \subset \{1,2,\ldots,n\}} \varphi^I_{\mathcal{S},i}$

Proof.

$$\begin{aligned}
\varphi_{\mathcal{S},i}(\mathbf{x}) &= \sum_{C \in \mathcal{S} \mid \mathbf{x} \in C_i} \text{vol}(C) \\
&= \sum_{I \subset \{1,2,\ldots,n\}} \sum_{C \in \mathcal{S} \mid \mathbf{x} \in C_i \wedge B(C) = I} \text{vol}(C) \\
&= \sum_{I \subset \{1,2,\ldots,n\} \mid i \in I} \sum_{C \in \mathcal{S} \mid \mathbf{x} \in C_i \wedge B(C) \cup \{i\} = I} \text{vol}(C) \\
&= \sum_{I \subset \{1,2,\ldots,n\} \mid i \in I} \varphi^I_{\mathcal{S},i}(\mathbf{x}) \\
&= \sum_{I \subset \{1,2,\ldots,n\}} \varphi^I_{\mathcal{S},i}(\mathbf{x})
\end{aligned}$$

□

Definition 4.4. *The* I-mixed secondary polytope *of a tuple* $(A_1, A_2, \ldots, A_n)$ *of point configurations with* $A_i \subset \mathbb{R}^d$ *is*

$$\Sigma^I(A_1, A_2, \ldots, A_n) := \text{conv}\left(\{\varphi^I_{\mathcal{S}} \mid \mathcal{S} \textit{ is a regular mixed subdivision of } (A_1, A_2, \ldots A_n)\}\right).$$

The following theorem follows directly from Definition 4.2:

Theorem 4.5. $\Sigma^{\{i\}}(A_1, A_2, \ldots, A_n) = \{0\}^{k_1+k_2+\ldots+k_{i-1}} \times \Sigma(A_i) \times \{0\}^{k_{i+1}+k_{i+2}+\ldots+k_n}$ *where* $k_j = \#A_j$.

Theorem 4.6 (Main Theorem). *The secondary polytope of a tuple of point configurations* $(A_1, A_2, \ldots, A_n)$ *can be decomposed in I-mixed secondary polytopes*

$$\Sigma(A_1, A_2, \ldots, A_n) = \sum_{I \subset \{1,2,\ldots,n\}} \Sigma^I(A_1, A_2, \ldots, A_n). \tag{1}$$

Note that the right inclusion $\subset$ in (1) follows from Theorem 4.3. We will postpone the proof of the left inclusion to the next section.

Theorem 4.7. *If* $I \subset \{1, 2, \ldots, n\}$, $\#I > d + 1$ *then* $\Sigma^I(A_1, A_2, \ldots, A_n) = \{\mathbf{0}\}$.

Proof. Since a cell C of a mixed subdivision $\mathcal{S}$ corresponds to a $(n + d - 1)$-dimensional simplex, we have $\sum \#C_i = n + d$ and $\forall i : \#C_i \geq 1$, thus $\#B(C) \leq d$. If $\#I > d + 1$ then there is no cell $C \in \mathcal{S}$ such that $B(C) \cup \{i\} = I$. □

5. Mixed Secondary Polytopes versus Bistellar Flips.

In this section we will focus on the difference between the characteristic functions of two neighbouring mixed subdivisions $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$. From what we know of a bistellar flip, there exist tuples of point configurations $F^{(1)}, F^{(2)}, \ldots, F^{(s)}$ with $\forall k, j : F_k^{(j)} \cap Z_k = \emptyset$ such that de cells of $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ involved in the bistellar flip flip_Z can be written as

$$C^{(j,k,\mathbf{z})} = (Z_1 \cup F_1^{(j)}, Z_2 \cup F_2^{(j)}, \ldots, Z_k \setminus \{\mathbf{z}\} \cup F_k^{(j)}, \ldots, Z_n \cup F_n^{(j)}) \tag{2}$$

with $\mathbf{z} \in Z_k$ where $\gamma_{k,\mathbf{z}} > 0$ for cells of $\mathcal{S}^{(1)}$ and $\gamma_{k,\mathbf{z}} < 0$ for $\mathcal{S}^{(2)}$. Using Theorem 2.2 we can write the volume of (2) as

$$\text{vol}(C^{(j,k,\mathbf{z})}) = f^{(j)}.|\gamma_{k,\mathbf{z}}|$$

where $f^{(j)}$ is a constant independent of $\mathbf{z}$. Using these notations we formulate the following theorem:

Theorem 5.1. *Given a mixed subdivision* $\mathcal{S}^{(1)}$ *and a neighbour* $\mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$ *then*

$$\varphi_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}(\mathbf{x}) = -\gamma_{i,\mathbf{x}}. \sum_{j=1}^{s} f^{(j)}. \tag{3}$$

Proof. We only need the volumes of cells involved in the bistellar flip to express the difference between the characteristic function:

$$\begin{aligned}
\varphi_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}(\mathbf{x}) &= \sum_{j=1}^{s}\sum_{k=1}^{n} \sum_{\mathbf{z}\in Z_k|\mathbf{x}\in C_i^{(j,k,\mathbf{z})}} \operatorname{sign}(\gamma_{k,\mathbf{z}}).\operatorname{vol}(C^{(j,i,\mathbf{z})}) \\
&= \sum_{j=1}^{s}\Big(\sum_{k=1|i\neq k}^{n} \sum_{\mathbf{z}\in Z_k|\mathbf{x}\in F_i^{(j)}\cup Z_i} f^{(j)}.\gamma_{k,\mathbf{z}} + \sum_{\mathbf{z}\in Z_i|\mathbf{x}\in F_i^{(j)}\cup Z_i\setminus\{\mathbf{z}\}} f^{(j)}.\gamma_{i,\mathbf{z}}\Big) \\
&= \sum_{j=1}^{s} f^{(j)}.\Big(\sum_{k=1|i\neq k}^{n} \underbrace{\sum_{\mathbf{z}\in Z_k|\mathbf{x}\in F_i^{(j)}\cup Z_i} \gamma_{k,\mathbf{z}}}_{=0} + \sum_{\mathbf{z}\in Z_i|\mathbf{x}\in F_i^{(j)}\cup Z_i\setminus\{\mathbf{z}\}} \gamma_{i,\mathbf{z}}\Big) \qquad (4)\\
&= \sum_{j=1}^{s} f^{(j)}.\Big(\underbrace{\sum_{\mathbf{z}\in Z_i|\mathbf{x}\in F_i^{(j)}} \gamma_{i,\mathbf{z}}}_{=0} + \sum_{\mathbf{z}\in Z_i|\mathbf{x}\in Z_i\setminus\{\mathbf{z}\}} \gamma_{i,\mathbf{z}}\Big) \qquad (5)\\
&= \sum_{j=1}^{s} f^{(j)}. \sum_{\mathbf{z}\in Z_i|\mathbf{z}\neq\mathbf{x}\in Z_i} \gamma_{i,\mathbf{z}} \\
&= -\gamma_{i,\mathbf{x}}. \sum_{j=1}^{s} f^{(j)} \quad \text{where } \gamma_{i,\mathbf{x}} = 0 \text{ for } \mathbf{x}\notin Z_i.
\end{aligned}$$

The simplifications of (4) and (5) are based on the observation that $\sum_{\mathbf{z}\in Z_k}\gamma_{k,\mathbf{z}} = 0$. □

Observe that the factor $\sum_{j=1}^{s} f^{(j)}$ in (3) does not depend on $\mathbf{x}$ and thus, is a constant, scaling the vector γ.

Theorem 5.2. *Given a mixed subdivision $\mathcal{S}^{(1)}$ and a neighbour $\mathcal{S}^{(2)} = \operatorname{flip}_Z(\mathcal{S}^{(1)})$ then*

$$\varphi^I_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi^I_{\mathcal{S}^{(2)},i}(\mathbf{x}) = \Big(\sum_{j|I=B(F^{(j)})\cup B(Z)} f^{(j)}\Big).(-\gamma_{i,\mathbf{x}}).$$

Proof. The only cells that have influence on $\varphi^I_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi^I_{\mathcal{S}^{(2)},i}(\mathbf{x})$ are those that are involved in the flip, and obey the restrictions of Definition 4.2, i.e., $B(C^{(j,k,\mathbf{z})}) \cup \{i\} = I$. Note that $B(C^{(j,k,\mathbf{z})})$ is independent of the choice of $\mathbf{z}$. Hence we can make the same simplifications as done in Equations (4) and (5) of the proof of Theorem 5.1 :

$$\varphi^I_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi^I_{\mathcal{S}^{(2)},i}(\mathbf{x}) = \Big(\sum_{j\in\{1,\dots,s\}|B(C^{(j,i,z)})\cup\{i\}=I} f^{(j)}\Big).(-\gamma_{i,\mathbf{x}}).$$

Since $\forall k : \#Z_k \neq 1$, from (2) follows $B(C^{(j,i,z)}) \cup \{i\} = B(Z) \cup B(F^{(j)})$, and this completes the proof. □

Corollary 5.3. *Given a mixed subdivision $\mathcal{S}^{(1)}$ and a neighbour $\mathcal{S}^{(2)} = \operatorname{flip}_Z(\mathcal{S}^{(1)})$ then for all $I \subset \{1, 2, \dots, n\}$ there exists a $c_I \in [0, 1]$ such that*

$$\varphi^I_{\mathcal{S}^{(1)}} - \varphi^I_{\mathcal{S}^{(2)}} = c_I \left(\varphi_{\mathcal{S}^{(1)}} - \varphi_{\mathcal{S}^{(2)}}\right).$$

Furthermore

$$\sum_{I\subset\{1,\dots,n\}} c_I = 1.$$

Proof. This follows directly form Theorem 4.3, Theorem 5.1 and Theorem 5.2. □

We can now prove Theorem 4.6.

Proof. We only need to prove the right inclusion $\supset$. We will show that every vertex $\mathbf{f} = \sum \varphi^I_{\mathcal{S}(I)}$ of $\sum \Sigma^I(A_1, A_2, \dots, A_n)$ belongs to $\Sigma(A_1, A_2, \dots, A_n)$.

$\mathbf{f}$ is a vertex of $\sum \Sigma^I(A_1, A_2, \dots, A_n)$ maximizing the inproduct $< \cdot\, , \mathbf{v} >$ for some vector $\mathbf{v}$ on $\sum \Sigma^I(A_1, A_2, \dots, A_n)$. Let $\varphi_{\mathcal{S}^*}$ be the vertex maximizing this inproduct $< \cdot\, , \mathbf{v} >$ on $\Sigma(A_1, A_2, \dots, A_n)$. Using linear programming, for all I one can build a path of neighbouring regular mixed subdivisions

$$\mathcal{S}^{(I)} = \mathcal{S}^{(I,1)}, \mathcal{S}^{(I,2)}, \dots, \mathcal{S}^{(I,s_I)} = \mathcal{S}^*$$

such that

$$\left\langle \varphi_{\mathcal{S}^{(I,j+1)}} - \varphi_{\mathcal{S}^{(I,j)}}, \mathbf{v}\right\rangle > 0 \quad \text{for } j = 1, 2, \dots, s_I.$$

Applying Theorem 5.3 gives

$$\left\langle \varphi^I_{\mathcal{S}^{(I,j+1)}} - \varphi^I_{\mathcal{S}^{(I,j)}}, \mathbf{v}\right\rangle \geq 0 \quad \text{for } j = 1, 2, \dots, s_I$$

and thus

$$\left\langle \varphi^I_{\mathcal{S}^*}, \mathbf{v}\right\rangle \geq \left\langle \varphi^I_{\mathcal{S}^{(I)}}, \mathbf{v}\right\rangle. \tag{6}$$

Summing (6) over all I's gives

$$\left\langle \varphi_{\mathcal{S}^*}, \mathbf{v}\right\rangle \geq \left\langle \sum \varphi^I_{\mathcal{S}^{(I)}}, \mathbf{v}\right\rangle = \left\langle \mathbf{f}, \mathbf{v}\right\rangle$$

which proves that $\mathbf{f} \in \Sigma(A_1, A_2, \dots, A_n)$. □

6. Mixed Secondary Polytopes versus Regular Mixed-Cell Configurations.

In this section we will see that the vertices and edges of a I-mixed secondary polytope play a similar role as for the secondary polytope.

Definition 6.1. *A cell $C = (C_1, C_2, \dots, C_n)$ is called I*-mixed *if $I = B(C)$. A (regular) I*-mixed-cell configuration *is the set*

$$\mathcal{S}^I := \{C_I = (C_{k_1}, C_{k_2}, \dots, C_{k_l}) \mid C \in \mathcal{S} \text{ and } B(C) = I = \{k_1, k_2, \dots, k_l\}\}$$

of a (regular) mixed subdivision $\mathcal{S}$.

This definition generalises the definition of [MV].

Theorem 6.2. *Given a circuit Z, two neighbouring regular mixed subdivisions $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)} = \mathrm{flip}_Z(\mathcal{S}^{(1)})$ and $I \subset \{1, \dots, n\}$ with $\#I \leq d$, then*

$$\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I} \iff \varphi^I_{\mathcal{S}^{(1)}} = \varphi^I_{\mathcal{S}^{(2)}}. \tag{7}$$

Proof. We denote the cells involved in the bistellar flip flip_Z between $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ by $C^{(j,i,\mathbf{z})}$ as in (2).

$$\begin{aligned}\mathcal{S}^{(1)I} \neq \mathcal{S}^{(2)I} &\overset{(1)}{\Leftrightarrow} \exists i,j,\mathbf{z} \in Z_i : \gamma_{i,\mathbf{z}} > 0, B(C^{(j,i,\mathbf{z})}) = I \text{ and } C_I^{(j,i,\mathbf{z})} \notin \mathcal{S}^{(2)I} \\ &\overset{(2)}{\Leftrightarrow} \exists i,j,\mathbf{z} \in Z_i : \gamma_{i,\mathbf{z}} > 0, B(C^{(j,i,\mathbf{z})}) = I \text{ and } i \in I \\ &\overset{(3)}{\Leftrightarrow} \exists j : B(F^{(j)}) \cup B(Z) = I \text{ and } \exists i : \#(Z_i \cup F_i^{(j)}) > 2 \\ &\overset{(4)}{\Leftrightarrow} \exists j : B(F^{(j)}) \cup B(Z) = I \\ &\overset{(5)}{\Leftrightarrow} \varphi^I_{\mathcal{S}^{(1)}} \neq \varphi^I_{\mathcal{S}^{(2)}}\end{aligned}$$

1. A difference between $\mathcal{S}^{(1)I}$ and $\mathcal{S}^{(2)I}$ can only be caused by a cell $C^{(j,i,\mathbf{z})}$ involved in the bistellar flip between $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$.
2. Since a bistellar flip only affects $C_i^{(j,i,\mathbf{z})}$ we have $C_I^{(j,i,\mathbf{z})} \notin \mathcal{S}^{(2)I} \Leftrightarrow i \in I$.
3. This follows from $\forall i : \#Z_i \neq 1$.
4. If $B(F^{(j)}) \cup B(Z) = I$ then there is always an i such that $\#(Z_i \cup F_i^{(j)}) > 2$ because $\sum \#(Z_i \cup F_i^{(j)}) = n + d + 1$, $\forall i : \#(Z_i \cup F_i^{(j)}) \geq 1$ and the number of i's for which $\#(Z_i \cup F_i^{(j)}) > 1$ is smaller than $\#I (\leq d)$. □
5. This follows from Theorem 5.2.

Theorem 6.3. *The set of regular mixed subdivisions* $\{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \dots, \mathcal{S}^{(r)}\}$ *whose I-mixed-cell configurations are equal for a given* $I \subset \{1, 2, \dots, n\}$, *i.e.* $\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I} = \cdots = \mathcal{S}^{(r)I}$ *is interconnected by bistellar flips.*

Proof. Consider the union of normal cones on the vertices $\varphi_{\mathcal{S}^{(1)}}, \varphi_{\mathcal{S}^{(2)}}, \dots, \varphi_{\mathcal{S}^{(r)}}$. This union is the set of lifting vectors inducing one I-mixed-cell configuration. One can in an *ad-hoc*-way describe this set as the solution of a system of homogeneous linear inequalities. Consequently this union of normal cones is convex, and thus all normal cones are interconnected by facets. These facets correspond to the bistellar flips between the regular mixed subdivisions. □

Theorem 6.4. *Given a set of all regular mixed subdivisions* $\{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \dots, \mathcal{S}^{(r)}\}$ *whose I-mixed-characteristic functions are equal for a given* $I \subset \{1, 2, \dots, n\}$, *i.e.* $\varphi^I_{\mathcal{S}^{(1)}} = \varphi^I_{\mathcal{S}^{(2)}} = \cdots = \varphi^I_{\mathcal{S}^{(r)}}$, *then the vertices* $\varphi_{\mathcal{S}^{(1)}}, \varphi_{\mathcal{S}^{(2)}}, \dots, \varphi_{\mathcal{S}^{(r)}}$ *are interconnected by edges.*

Proof. This follows directly from Theorem 4.6 and basic properties of Minkowski sums. □

Theorem 6.3 and 6.4 allows us to generalise Theorem 6.2 for non-neighbouring regular mixed-cell configurations.

Theorem 6.5. *Given two regular mixed subdivisions* $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$ *and* $I \subset \{1, \dots, n\}$ *with* $\#I \leq d$, *then*

$$\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I} \Longleftrightarrow \varphi^I_{\mathcal{S}^{(1)}} = \varphi^I_{\mathcal{S}^{(2)}}. \tag{8}$$

Proof. If $\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I}$ then Theorem 6.3 ensures that one can construct a path of regular mixed-cell configurations from $\mathcal{S}^{(1)}$ to $\mathcal{S}^{(2)}$ all sharing the same I-mixed-cell configuration. Using Theorem 6.2 we know that they all have the same characteristic function. At the other hand, if $\varphi^I_{\mathcal{S}^{(1)}} = \varphi^I_{\mathcal{S}^{(2)}}$ then Theorem 6.4 ensures that one can construct a path of regular mixed-cell configurations from $\mathcal{S}^{(1)}$ to $\mathcal{S}^{(2)}$ all sharing the same I-characteristic function. Using Theorem 6.2 we know that they all have the same I-mixed-cell configuration. □

Theorem 6.6. *For a $I \subset \{1, \ldots, n\}$ with $\#I \leq d$:*

1. *the vertices of $\Sigma^I(A_1, A_2, \ldots, A_n)$ correspond to the I-mixed-cell configurations of $(A_1, A_2, \ldots, A_n)$;*
2. *the edges of $\Sigma^I(A_1, A_2, \ldots, A_n)$ correspond to bistellar flips involving I-mixed cells.*

Proof. This follows from Theorem 6.5. □

7. An Example

Consider the following point configurations $A_1 = A_2 = \{(0,0), (1,0), (1,1)\}$ and $A_3 = \{(0,0), (1,0)\}$. Figure 1 shows the secondary polytope of (A_1, A_2, A_3) with for each vertex, a regular mixed subdivision. Figures 2, 3, 4, 5, 6, 7 and 8 denote the mixed secondary polytopes. Note that $\Sigma^{\{1\}}, \Sigma^{\{2\}}, \Sigma^{\{3\}}$ (Figures 2, 3 and 4) are singletons corresponding to the only triangulation of A_1, A_2 and A_3. The vertices of $\Sigma^{\{1,2\}}, \Sigma^{\{1,3\}}, \Sigma^{\{2,3\}}$ (Figures 5, 6 and 7) correspond to mixed-cell configurations $\mathcal{S}^{\{1,2\}}$, $\mathcal{S}^{\{1,3\}}$ and $\mathcal{S}^{\{2,3\}}$. The mixed-cell configurations are depicted by drawing them as $\sum_{i \in I} C_i$.

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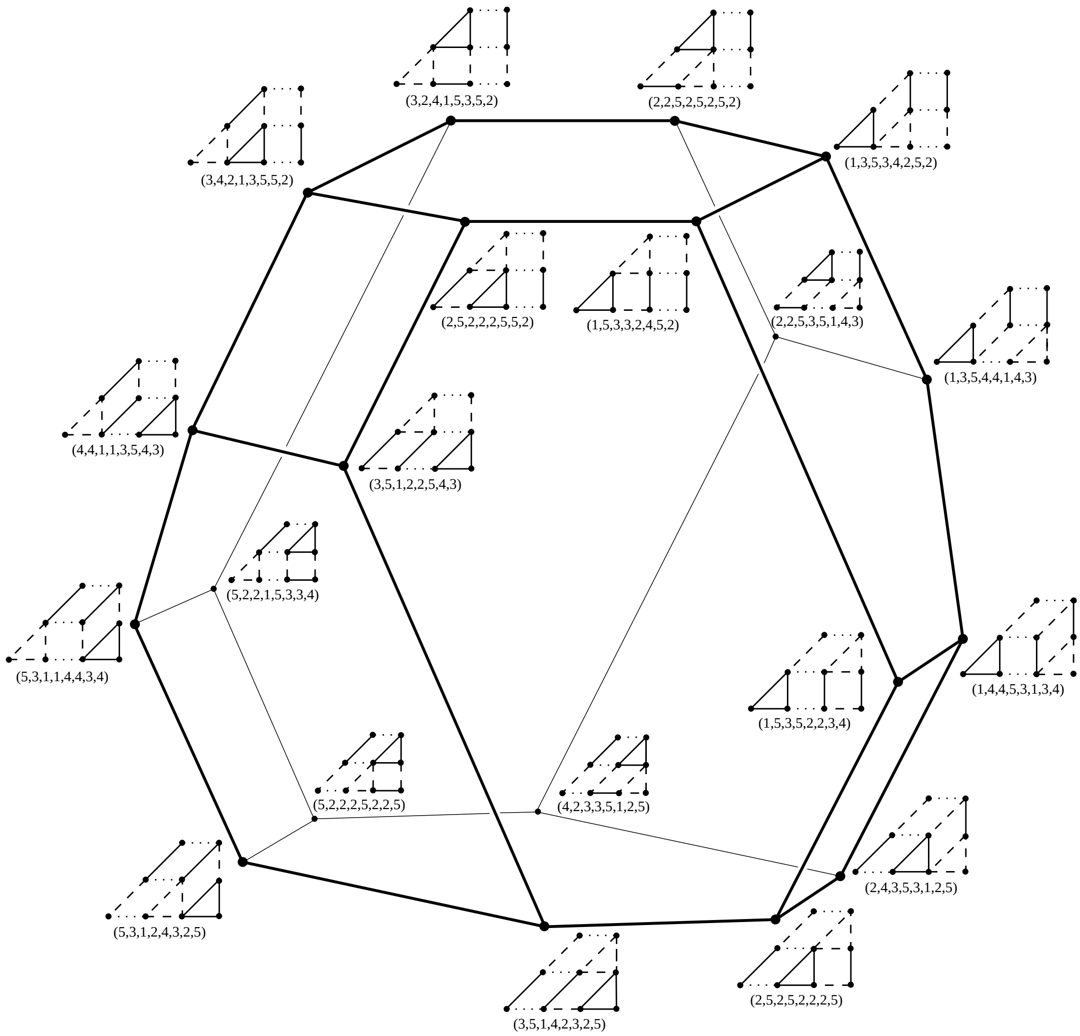

FIGURE 1. $\Sigma(A_1, A_2, A_3)$

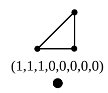

FIGURE 2. $\Sigma^{\{1\}}(A_1, A_2, A_3)$

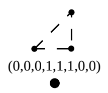

FIGURE 3. $\Sigma^{\{2\}}(A_1, A_2, A_3)$

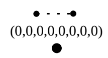

FIGURE 4. $\Sigma^{\{3\}}(A_1, A_2, A_3)$

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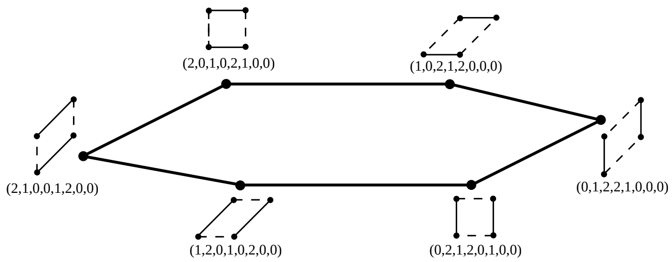

FIGURE 5. $\Sigma^{\{1,2\}}(A_1, A_2, A_3)$

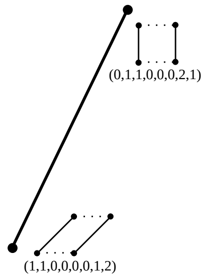

FIGURE 6. $\Sigma^{\{1,3\}}(A_1, A_2, A_3)$

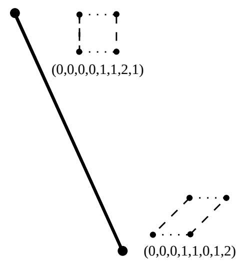

FIGURE 7. $\Sigma^{\{2,3\}}(A_1, A_2, A_3)$

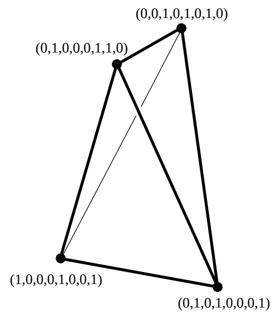

FIGURE 8. $\Sigma^{\{1,2,3\}}(A_1, A_2, A_3)$

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