Wavelets by orthogonal rational kernels

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Abstract

Suppose $\{\phi_k\}_{k=0}^n$ is an orthonornal basis for the function space $\mathcal{L}_n$ of polynomials or rational functions of degree n with prescribed poles. Suppose $n = 2^s$ and set $\mathcal{V}_s = \mathcal{L}_n$. Then $k_n(z,w) = \sum_{k=0}^n \phi_k(z)\overline{\phi_k(w)}$, is a reproducing kernel for $\mathcal{V}_s$. For fixed w, such reproducing kernels are known to be functions localized in the neighborhood of $z = w$. Moreover, by an appropriate choice of the parameters $\{\xi_{nk}\}_{k=0}^n$, the functions $\{\varphi_{n,k}(z) = k_n(z,\xi_{nk})\}_{k=0}^n$ will be an orthogonal basis for $\mathcal{V}_s$. The orthogonal complement $\mathcal{W}_s = \mathcal{V}_{s+1} \ominus \mathcal{V}_s$ is spanned by the functions $\{\psi_{n,k}(z) = l_n(z,\eta_{nk})\}_{k=0}^{n-1}$ for an appropriate choice of the parameters $\{\eta_{nk}\}_{k=0}^{n-1}$ where $l_n = k_{n+1} - k_n$ is the reproducing kernel for $\mathcal{W}_s$. These observations form the basic ingredients for a wavelet type of analysis for orthogonal rational functions on the unit circle or the real line with respect to an arbitrary probability measure.

Keywords : orthogonal rational functions, wavelets, reproducing kernel.
AMS(MOS) Classification : Primary : 42C05, Secondary : 42C05, 46E22, 42A38.

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Abstract

Suppose $\{\phi_k\}_{k=0}^n$ is an orthonornal basis for the function space $\mathcal{L}_n$ of polynomials or rational functions of degree n with prescribed poles. Suppose $n = 2^s$ and set $\mathcal{V}_s = \mathcal{L}_n$. Then $k_n(z,w) = \sum_{k=0}^n \phi_k(z)\overline{\phi_k(w)}$, is a reproducing kernel for $\mathcal{V}_s$. For fixed w, such reproducing kernels are known to be functions localized in the neighborhood of $z = w$. Moreover, by an appropriate choice of the parameters $\{\xi_{nk}\}_{k=0}^n$, the functions $\{\varphi_{n,k}(z) = k_n(z,\xi_{nk})\}_{k=0}^n$ will be an orthogonal basis for $\mathcal{V}_s$. The orthogonal complement $\mathcal{W}_s = \mathcal{V}_{s+1} \ominus \mathcal{V}_s$ is spanned by the functions $\{\psi_{n,k}(z) = l_n(z,\eta_{nk})\}_{k=0}^{n-1}$ for an appropriate choice of the parameters $\{\eta_{nk}\}_{k=0}^{n-1}$ where $l_n = k_{n+1} - k_n$ is the reproducing kernel for $\mathcal{W}_s$. These observations form the basic ingredients for a wavelet type of analysis for orthogonal rational functions on the unit circle or the real line with respect to an arbitrary probability measure.

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1 Introduction

Consider a Hilbert space of complex functions analytic in $X \subset \mathbb{C}$, equipped with an inner product $\langle\cdot,\cdot\rangle$, then it is a reproducing kernel Hilbert space if there exists a function k_w such that $k_w \in H$ for all $w \in X$ and $\langle f, k_w\rangle = f(w)$ for all $w \in X$ and for all $f \in H$. This $k_w(z) \equiv k(z,w)$, which is in fact unique, is called the reproducing kernel for H.

In such a reproducing kernel Hilbert space, it is well known that the solution of the problem

$$\inf_{f\in H}\{\|f\| : f(w) = 1\}$$

for some $w \in X$ is given by $f(z) = k(z,w)/k(w,w)$.

This property is the key to this paper, because it characterizes the reproducing kernel as a function which is localized near $z = w$. Indeed, without the constraint $f(w) = 1$, the solution would be $f \equiv 0$. With the constraint, the function is forced to take the value 1 at $z = w$ but to minimize the norm, it should be as close to zero as possible outside $z = w$.

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So what really happens is that the solution will be an approximation of this impulse by a function from H. If we solve this problem for subspaces of H with increasing dimension, then the reproducing kernels for these subspaces will approximate the Dirac impulse better and better, hence will be "narrower" near the peak at $z = w$ and will oscillate more by a Gibbs-like phenomenon. Thus the better we localize the function in the z-domain, the worse it will be localized in the frequency domain. This is a manifestation of the Heisenberg uncertainty principle.

This shows that two main ingredients of a wavelet analysis are present: localized functions (and it will turn out that they can indeed be used to generate basis functions) and a multiresolution idea where a function can be represented at increasing resolution levels.

This is the idea which will be elaborated in this paper. It is inspired by the paper of Fischer and Prestin [20] where a similar construction was used for orthogonal polynomials on the real line.

2 The function spaces

In this paper, we consider spaces of rational functions analytic in $\mathbb{O}$ where $\mathbb{O}$ is either the unit disk or the upper half plane. The inner product is given by an integral over the boundary $\partial\mathbb{O}$ which is the unit circle or the real line. The poles of the rational functions are fixed in advance. There are two possibilities: either the poles are all inside $\mathbb{O}$ or they are all located on the boundary $\partial\mathbb{O}$.

In order not to complicate the notation too much we will consider only two cases explicitly. In the case of the circle the poles are among the points $\mathfrak{P} = \{1/\overline{\alpha}_k : k = 1, 2, \ldots\}$ with all $\alpha_k \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the inner product is given by

$$\langle f, g\rangle = \int_{\mathbb{T}} f(t)\overline{g(t)}\mathrm{d}\mu(t), \quad t = e^{i\theta},$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle and $\int \mathrm{d}\mu(t) = 1$. Thus if Π_n denotes the polynomials of degree at most n, and if we set $\pi_0 = 1$ and $\pi_n(z) = \prod_{k=1}^n (1 - \overline{\alpha}_k z)$ for $n = 1, 2, \ldots$, then we consider spaces of rational functions

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \Pi_n \right\}.$$

In the case of the real line, we assume that the poles are among the points $\mathfrak{P} = \{1/\alpha_k : k = 1, 2, \ldots\}$ with all $\alpha_k \in \mathbb{R}$. In this case, we assume that the functions are real valued on $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and the inner product is given by

$$\langle f, g\rangle = \int_{\hat{\mathbb{R}}} f(x)g(x)\mathrm{d}\mathring{\mu}(x),$$

with $\mathrm{d}\mathring{\mu}(t) = (1 + x^2)^{-1}\mathrm{d}\mu(x)$ and $\int_{\hat{\mathbb{R}}} \mathrm{d}\mathring{\mu}(t) = 1$. The factor $(1 + x^2)$ originates from the fact that by mapping the unit circle $\mathbb{T}$ to the extended real line $\hat{\mathbb{R}}$ via a Cayley transform τ, a measure $\mathrm{d}\mu(t)$, $t \in \mathbb{T}$ is mapped onto the measure $\mathrm{d}\mathring{\mu}(x)$, $x \in \hat{\mathbb{R}}$ with $x = \tau(t)$.

Thus, again using Π_n to denote the space of polynomials of degree at most n and setting $\pi_0 = 1$ and $\pi_n(z) = \prod_{k=1}^n (1 - \alpha_k z)$ for $n = 1, 2, \ldots$, the spaces we consider are of the form

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \Pi_n \right\}.$$

Note that unlike the polynomial case, a function in $\mathcal{L}_n$ can be finite at ∞, so that μ can have a mass point at ∞.

We isolate these two cases because if we set all $\alpha_k = 0$, then $\mathcal{L}_n = \Pi_n$, and the polynomial case is recovered.

The orthonormal rational functions on the circle and on the line, i.e., the functions $\{\phi_0, \phi_1, \ldots\}$ such that $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $\phi_n \perp \mathcal{L}_{n-1}$ and $\|\phi_n\| = 1$ were studied by M.M. Djrbashian [14, 13, 15, 16, 17, 18], (see also [24]), later by Bultheel, K. Pan, Xin Li, and in a long list of papers by the authors and they are the subject of a forthcoming monograph [9]. They are related to polynomials orthogonal with respect to varying measures as studied in several papers by López-Lagomasino.

3 Reproducing kernels and orthogonal rational functions

Assume that, whatever the case, i.e., the circle or the real line, we have available the orthonormal rational functions $\{\phi_0, \phi_1, \ldots\}$ as introduced above. The following properties concerning the reproducing kernels are well known (see for example [2, 23, 19])

Theorem 3.1 *We consider an m-dimensional subspace $\mathcal{K}$ of $\mathcal{L}_n$. Assume that its reproducing kernel is given by $k(z, w)$. Then*

1. *For any orthonormal basis $\{e_1, e_2, \ldots, e_m\}$ of $\mathcal{K}$, the reproducing kernel is given by*

$$k(z, w) = \sum_{k=1}^{m} e_k(z)\overline{e_k(w)}.$$

2. *For any set of distinct points $\{w_1, w_2, \ldots, w_m\}$ among the points of analyticity for $\mathcal{K}$,*

$$\langle k(z, w_j), k(z, w_i)\rangle = k(w_i, w_j)$$

and the matrix

$$M = [k(w_i, w_j)]_{i,j=1}^{m}$$

is positive semi-definite.

3. *The orthogonal projection of $f \in \mathcal{L}_n$ onto $\mathcal{K}$ is given by*

$$P_{\mathcal{K}} f = \langle f(z), k(z, w)\rangle .$$

4. *For any point w where it makes sense,*

$$\inf_{f \in \mathcal{K}} \{\|f\| : f(w) = 1\} = [k(w, w)]^{-1}$$

and the minimizer is $f(z) = k(z, w)/k(w, w)$.

Also the following lemma is easy to verify.

Lemma 3.2 *Let $\{\phi_0, \ldots \phi_n\}$ be the orthonormal basis for $\mathcal{L}_n$ as introduced above, then for any set of distinct points $\mathbf{x} = \{x_0, x_1, \ldots, x_n\}$ which are points of analyticity for functions in $\mathcal{L}_n$, the matrix*

$$\Phi_n(\mathbf{x}) = \begin{bmatrix} \phi_0(x_0) & \cdots & \phi_0(x_n) \\ \vdots & & \vdots \\ \phi_n(x_0) & \cdots & \phi_n(x_n) \end{bmatrix} \tag{3.1}$$

is regular.

Proof. If it were not regular, then there would exist a nonzero vector $\mathbf{c} = [c_0, \ldots, c_n]$ such that $\mathbf{c}\Phi_n(\mathbf{x}) = 0$. In other words, the function $\phi(z) = \sum_{k=0}^{n} c_k\phi_k(z) \in \mathcal{L}_n$ would vanish at $n+1$ distinct points $\{x_k\}_{k=0}^{n}$. Because ϕ is rational of degree at most n, it would be identically zero and this would imply that the functions $\{\phi_k\}_{k=0}^{n}$ were linearly dependent, which is impossible. □

This lemma shows the unisolvence (Haar condition) of the system $\{\phi_k\}$. See Davis [12, chap. II, Sect. 2.4]. This lemma entails immediately

Corollary 3.3 *If $k_n(z, w)$ is the reproducing kernel for $\mathcal{L}_n$ and the $\mathbf{x} = \{x_k\}_{k=0}^{n}$ are $n+1$ distinct points of analyticity for $\mathcal{L}_n$, then the functions $\{\varphi_{nj}(z) = k_n(z, x_j)\}_{j=0}^{n}$ form a basis for $\mathcal{L}_n$.*

Proof. Since, with the matrix $\Phi_n(\mathbf{x})$ as in the previous lemma:

$$\begin{bmatrix} k_n(z, x_0) \\ \vdots \\ k_n(z, x_n) \end{bmatrix} = \Phi_n^H(\mathbf{x}) \begin{bmatrix} \phi_0(z) \\ \vdots \\ \phi_n(z) \end{bmatrix}, \tag{3.2}$$

the statement follows because $\{\phi_k\}_{k=0}^{n}$ is a basis and $\Phi_n(\mathbf{x})$ is regular. □

4 Multiresolution

Now we want a multiscale representation of a function. If we use the orthonormal basis ϕ_n and consider the Fourier expansion

$$f(z) \sim c_0\phi_0 + c_1\phi_1(z) + c_2\phi_2(z) + \cdots, \quad c_k = \langle f, \phi_k \rangle,$$

then as we add more and more terms, we shall add more and more details to the low resolution approximation. We shall consider the spaces $\mathcal{L}_{2^s}$ for $s = 0, 1, \ldots$ and set $\mathcal{V}_s = \mathcal{L}_{2^s}$. The orthogonal complement of $\mathcal{L}_{2^s}$ in $\mathcal{L}_{2^{s+1}}$ is denoted by $\mathcal{W}_s$. Thus if we assume from now on that n always has the meaning of 2^s, then we can define the nested spaces $\mathcal{V}_s$ and the orthogonal complements $\mathcal{W}_s$ in $\mathcal{V}_{s+1}$ as

$$\mathcal{V}_{-1} = \mathcal{L}_0, \quad \mathcal{W}_{-1} = \mathcal{V}_0 \ominus \mathcal{V}_{-1}, \quad \mathcal{V}_s = \mathcal{L}_n \quad \text{and} \quad \mathcal{W}_s = \mathcal{V}_{s+1} \ominus \mathcal{V}_s, \quad s = 0, 1, \ldots$$

We call the orthogonal projection of a function onto $\mathcal{V}_s$ the representation of that function at scale or resolution level s. For $s = -1$, for example, this is just a constant c_0, which is the weighted (by μ) average of the function.

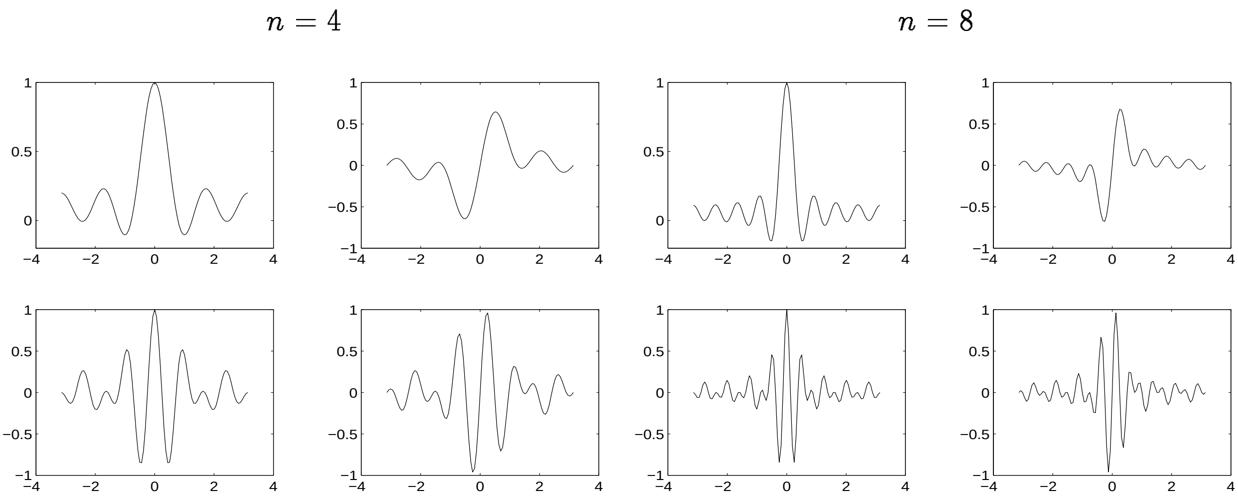
Obviously, $\mathcal{V}_s = \text{span}\{\phi_k\}_{k=0}^{n}$ and $\mathcal{W}_s = \text{span}\{\phi_k\}_{k=n+1}^{2n}$ can both be generated by our orthonormal basis functions. However, these basis functions are in general not localized both in the z and in the frequency domain.

To give a trivial example, consider the case of the unit circle with all $\alpha_k = 0$ and with μ the normalized Lebesgue measure on $\mathbb{T}$. Then the rational functions are just polynomials and the orthogonal polynomials are just the powers z^k. They form the basis used in Fourier analysis. Each basis function contains only one frequency, but $|z^k| = 1$ on $\mathbb{T}$. The kernels $k_{2n}(z, w) = \sum_{k=0}^{2n} z^k \overline{w}^k$ for $w \in \mathbb{T}$ are

$$k_{2n}(z, w) = \left(\frac{z}{w}\right)^n \sum_{k=-n}^{n} \left(\frac{z}{w}\right)^k = e^{in(\theta-\omega)} \frac{\sin(n + \frac{1}{2})(\theta - \omega)}{\sin \frac{1}{2}(\theta - \omega)}, \quad z = e^{i\theta}, \quad w = e^{i\omega}.$$

Hence these kernels are complex exponentials modulated by the Dirichlet kernel, which is localized around $\theta = \omega$. This can be verified in the top row of Figure 1. For the spaces

Figure 1: Circular case. The real and imaginary part of the functions $k_n(z, w) = \sum_{k=0}^{n} (z/w)^k$ (top) and $l_n(z, w) = \sum_{k=n+1}^{2n} (z/w)^k$ (bottom) for $z = e^{i\theta}$ and $w = 1$, plotted as functions of θ. The figures on the left are for $n = 4$, the figures on the right for $n = 8$. Note that the real parts are even functions, while the imaginary parts are odd functions.



$\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$, the reproducing kernels are given by $l_n(z, w) = k_{2n}(z, w) - k_n(z, w) = \sum_{k=n+1}^{2n} \phi_k(z)\overline{\phi_k(w)}$. In our example this is just $(z\overline{w})^{n+1} \sum_{k=0}^{n-1} (z\overline{w})^k$. The plots for this trivial example are given in the bottom row of Figure 1. It is immediately observed that these oscillate more but otherwise have properties similar to the kernels for $\mathcal{L}_n$.

For further details about the orthogonal rational functions and the wavelet-like analysis, at several places we have to discuss the circle case and the real line case separately, in order not to complicate the notation too much. However, if we can easily avoid a separate treatment, we will give a uniform treatment.

5 The ORF basis

5.1 The circle case

We introduce the artificial point $\alpha_0 = 0$, the Blaschke factors for $k = 1, 2, \ldots$

$$\zeta_k(z) = z_k \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \quad \text{with } z_k = 1 \text{ if } \alpha_k = 0 \text{ and } z_k = -\frac{\overline{\alpha}_k}{|\alpha_k|} \text{ otherwise,}$$

and the Blaschke products

$$B_0 = 1, \quad B_n = \zeta_1 \cdots \zeta_n, \quad n = 1, 2, \ldots$$

For any function f we set $f_*(z) = \overline{f(1/\overline{z})}$ and if $f_n \in \mathcal{L}_n$ (it should be clear from the context what space $\mathcal{L}_n$ is involved) we define $f_n^*(z) = B_n(z) f_*(z)$.

We can now give the Christoffel-Darboux relation which is a summation formula, expressing the reproducing kernel as follows.

Theorem 5.1 (Christoffel-Darboux) *Let $\{\phi_k\}_{k=0}^n$ with $\phi_k \in \mathcal{L}_k \backslash \mathcal{L}_{k-1}$ be an orthonormal basis for $\mathcal{L}_n$, then the reproducing kernels $k_n(z,w)$ for $\mathcal{L}_n$ satisfy*

$$k_n(z,w) = \frac{\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1 - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}.$$

The zeros of the orthogonal rational functions (ORF) are known to be in the open unit disk $\mathbb{D}$, so they are not suitable for the construction of the quadrature formulas representing integrals over the unit circle $\mathbb{T}$. We should rather have knots which are located on $\mathbb{T}$. Such knots are provided by the zeros of para-orthogonal functions. The para-orthogonal functions are defined as

$$Q_n(z, \tau_n) = \phi_n(z) + \tau_n \phi_n^*(z)$$

for $\tau_n \in \mathbb{T}$, $n = 0, 1, \ldots$ and the zeros of these are simple and on $\mathbb{T}$. We have the following rational Szegő quadrature formula.

Theorem 5.2 (Rational Szegő quadrature) *The zeros of the para-orthogonal rational function $Q_{n+1}(z, \tau_{n+1})$ are all simple and on $\mathbb{T}$. Let us denote them by $\{\xi_{n,k}\}_{k=0}^n$. Moreover, defining $\lambda_{nk} = [k_n(\xi_{nk}, \xi_{nk})]^{-1}$, with $k_n(z,w)$ the reproducing kernel for $\mathcal{L}_n$, then equality holds in*

$$\langle f, g \rangle = \sum_{k=0}^n \lambda_{nk} f(\xi_{nk}) g_*(\xi_{nk}), \quad \forall f, g \in \mathcal{L}_n.$$

Conversely, if the above equality holds, then the ξ_{nk} should be the zeros of the para-orthogonal function $Q_{n+1}(z, \tau_{n+1})$ for some $\tau_{n+1} \in \mathbb{T}$ and $\lambda_{nk} = [k_n(\xi_{nk}, \xi_{nk})]^{-1}$.

The $n+1$ zeros $\{\xi_{nk}\}_{k=0}^n$ can also be characterized as follows.

Theorem 5.3 *Let $k_n(z,w)$ be the reproducing kernel for $\mathcal{L}_n$. Define $\xi_{n0} = w$ with w arbitrary on $\mathbb{T}$ and $\{\xi_{nk}\}_{k=1}^n$ the n zeros of $k_n(z,w)$. Then the numbers $\{\xi_{nk}\}_{k=0}^n$ are the zeros of the para-orthogonal function $Q_{n+1}(z, \tau_{n+1})$ with*

$$\tau_{n+1} = -\frac{\phi_{n+1}(w)}{\phi_{n+1}^*(w)}.$$

Conversely, if $\{\xi_{nk}\}_{k=0}^{n}$ *are the zeros of some para-orthogonal function* $Q_{n+1}(z,\tau_{n+1})$*, then there exists a number* $w \in \mathbb{T}$*, such that* $\xi_{n0} = w$ *and* $\tau_{n+1} = -\frac{\phi_{n+1}(w)}{\phi_{n+1}^*(w)}$*, while* $\{\xi_{nk}\}_{k=1}^{n}$ *are the* n *zeros of* $k_n(z,w)$*.*

Proof. First note that neither ϕ_{n+1} nor ϕ_{n+1}^* can be zero on $\mathbb{T}$. Thus τ_{n+1} is well defined and is on $\mathbb{T}$ for all $w \in \mathbb{T}$ because $|\phi_{n+1}^*| = |B_{n+1}||\phi_{n+1}| = |\phi_{n+1}|$ on $\mathbb{T}$. Thus it is immediately seen that the zeros of $Q_{n+1}(z,\tau_{n+1})$ are the zeros of

$$\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}.$$

Obviously $z = w$ is one of the zeros. In the Christoffel-Darboux formula, this zero is canceled by the denominator and it is of course not a zero of $k_n(z,w)$. The n other zeros of $Q_{n+1}(z,\tau_{n+1})$ however are precisely the zeros of $k_n(z,w)$.

Conversely, recall that all the zeros of ϕ_{n+1} are in $\mathbb{D}$ and hence none of the zeros of ϕ_{n+1}^* are in $\mathbb{D} \cup \mathbb{T}$. Thus it follows that $-\phi_{n+1}/\phi_{n+1}^*$ has winding number $n+1$, so that there are $n+1$ values of $w \in \mathbb{T}$ such that $\tau_{n+1} = -\phi_{n+1}(w)/\phi_{n+1}^*(w)$. Each of these values of w has to be a zero of $Q_{n+1}(z,\tau_{n+1})$, thus we can pick the one which is ξ_{n0}. The other zeros of $Q_{n+1}(z,\tau_{n+1})$ have to be zeros of $k_n(z,w)$ as was explained above. □

5.2 The case of the real line

The development for the case of the real line is similar to the development given for the unit circle, except that we now require that the points α_k are all in $\mathbb{R}$, such that the poles $1/\alpha_k$ are all on $\hat{\mathbb{R}} \setminus \{0\}$.

Define as elementary factors

$$\zeta_k(z) = \frac{z}{1-\alpha_k z}, \quad k = 0, 1, \ldots$$

and set $B_0 = 1$ and $B_n = \zeta_1 \cdots \zeta_n$ for $n = 1, 2, \ldots$ and $\mathcal{L}_n = \text{span}\{B_0, B_1, \ldots, B_n\}$. The substar is defined by $f_*(z) = \overline{f(\overline{z})}$. Note that the basis functions B_k satisfy $B_{k*} = B_k$ so that $f \in \mathcal{L}_n \Rightarrow f_* \in \mathcal{L}_n$. Let the orthonormal functions ϕ_k be generated by orthonormalizing the basis $B_0, B_1, \ldots$, then it is easily checked that the coefficients of the ϕ_k with respect to the basis B_k are real. We remark that if all the $\alpha_k = 0$, then $B_k(z) = z^k$ and hence, $\mathcal{L}_n = \Pi_n$, the set of polynomials of degree at most n. We say that ϕ_n is regular (or that n is a regular index) if $p_n(1/\alpha_{n-1}) \neq 0$ where p_n is the numerator polynomial of ϕ_n, i.e., $\phi_n = p_n/\pi_n$ with $\pi_n(z) = \prod_{k=1}^{n}(1-\alpha_k z)$. We say that the system $\{\phi_k\}$ is regular if ϕ_k is regular for all $k = 0, 1, \ldots$. If the system is regular, then the following generalization of the three-term recurrence relation for orthogonal polynomials exists [9].

Theorem 5.4 *Suppose that the orthonormal system* $\{\phi_k\}$ *is regular, then there holds a recurrence relation of the following form*

$$\phi_n(z) = \left(A_n\zeta_n(z) + B_n\frac{\zeta_n(z)}{\zeta_{n-1}(z)}\right)\phi_{n-1}(z) + C_n\frac{\zeta_n(z)}{\zeta_{n-2}(z)}\phi_{n-2}(z), \quad n = 2, 3, \ldots$$

with initial conditions

$$\phi_0(z) = 1, \quad \phi_1(z) = (A_1\zeta_1(z) + B_1)\phi_0(z).$$

Moreover A_n and C_n are all real and nonzero, and $A_n = -C_n A_{n-1}$, $n = 2, 3, \ldots$

Conversely, if the functions ϕ_k are given by such a relation, then they will be orthonormal with respect to some positive measure $\mathring{\mu}$ on $\hat{\mathbb{R}}$.

This theorem can also be found in several pieces and in slightly varying forms for example in [6, 8] where analogs are given for the unit circle.

The relation $A_n = -C_n A_{n-1}$ monitors the norms. This means that ϕ_k, $k = 2, 3, \ldots$ will be normalized to norm 1 if we choose ϕ_0 and ϕ_1 to be normalized. Recall that $\phi_0 = 1$ is normalized because we assumed that $\int \mathrm{d}\mathring{\mu} = 1$. So, if all the numbers C_n are given, then all the A_n will be uniquely defined once that A_1 is fixed. This value of A_1 is related to the orthonormality of ϕ_1. From the viewpoint of the Favard theorem, given all the C_n and the B_n, we can choose A_1 nonzero and then all the remaining A_n will be fixed. Thus all the orthogonal functions are fixed, and therefore also the orthogonality measure is fixed to a large extend. In this sense, A_1 will impose a certain condition on the orthogonality measure. The meaning is as follows.

Lemma 5.5 *The value of A_1 in the above recurrence is related to the orthogonality measure by a generalized standard deviation, i.e.,*

$$A_1 = \frac{\pm 1}{s}, \quad \textit{with} \quad s^2 = m_2^2 - m_1^2, \quad m_2^2 = \int_{\hat{\mathbb{R}}} |\zeta_1(x)|^2 \mathrm{d}\mathring{\mu}(x), \quad m_1 = \int_{\hat{\mathbb{R}}} \zeta_1(x) \mathrm{d}\mathring{\mu}(x).$$

Proof. First assume that $B_1 = 0$, then $\langle \phi_0, \phi_1 \rangle = 0$ implies that the generalized mean $m_1 = \int \zeta_1(x)\mathrm{d}\mathring{\mu}(x) = 0$. The normality condition $\|\phi_1\|^2 = 1$ then gives $A_1^2 m_2^2 = 1$.

If $B_1 \neq 0$, then the condition $\langle \phi_0, \phi_1 \rangle = 0$ gives $A_1 = -B_1/m_1$, while $\|\phi_1\|^2 = 1$ leads to

$$B_1^2(m_2^2 - m_1^2) = m_1^2.$$

In both cases, this gives the desired formulas. □

Recall that in the polynomial case where all $\alpha_k = 0$, then $\zeta_1(x) = x$ and we then get the usual definition of standard deviation.

Thus, if $\mathrm{d}\mathring{\mu}(x) = w(x)\mathrm{d}x$, in some sense we can say that a larger $|A_1|$ will correspond to a smaller s which means that the graph of w will be wider, a small $|A_1|$ will correspond to a more peaked w.

A useful property we shall need later is that $A_n = 0$ iff ϕ_n is not regular (see Appendix A).

Like for the circular case, we also have a Christoffel-Darboux type formula, which can be formulated as follows.

Theorem 5.6 (Christoffel-Darboux) *Let $\{\phi_k\}_{k=0}^n$ with $\phi_k \in \mathcal{L}_k \backslash \mathcal{L}_{k-1}$ be an orthonormal basis for $\mathcal{L}_n$, then the reproducing kernels $k_n(z,w) = \sum_{k=0}^n \phi_k(z)\overline{\phi_k(w)}$ for $\mathcal{L}_n$ satisfy*

$$A_{n+1} \frac{z - \overline{w}}{z\overline{w}} k_n(z,w) = \left[\overline{\begin{pmatrix} \phi_n(w) \\ \zeta_n(w) \end{pmatrix}} \begin{pmatrix} \phi_{n+1}(z) \\ \zeta_{n+1}(z) \end{pmatrix} - \begin{pmatrix} \phi_n(z) \\ \zeta_n(z) \end{pmatrix} \overline{\begin{pmatrix} \phi_{n+1}(w) \\ \zeta_{n+1}(w) \end{pmatrix}} \right].$$

Like in the polynomial case, one can prove [9]

Theorem 5.7 *The zeros of the numerators of ϕ_n are simple and are in $\hat{\mathbb{R}}$. If ϕ_n is regular, then the numerators of ϕ_n and ϕ_{n-1} have no common zeros.*

Note that the presence of poles can disrupt the classical property for polynomials which says that the zeros of the orthogonal polynomials are real and interlace with the zeros of the neighboring orthogonal polynomials. In the rational case, the numerators of the orthogonal functions have simple zeros which are all in $\hat{\mathbb{R}}$, but it is not a priory sure that none of the zeros of the numerators coincides with with one of the numbers in $\hat{\mathfrak{P}} = \mathfrak{P} \cup \{\infty\} = \{1/\alpha_0, 1/\alpha_1, \ldots\}$. A zero at ∞ means here that the polynomial has a defective degree. Therefore, one should consider zeros of quasi-orthogonal functions to construct quadrature formulas.

Quasi-orthogonal functions are defined by

$$Q_n(z, \tau_n) = \phi_n(z) + \tau_n \frac{\zeta_n(z)}{\zeta_{n-1}(z)} \phi_{n-1}(z) \in \mathcal{L}_n$$

where $\tau_n \in \hat{\mathbb{R}}$. For $\tau_n = \infty$, this should be read as $Q_n(z, \tau_n) = [\zeta_n(z)/\zeta_{n-1}(z)]\phi_{n-1}(z)$. The numerator of this function, will have n simple zeros in $\mathbb{R} \setminus \mathfrak{P}_n$ with $\mathfrak{P}_n = \{1/\alpha_1, \ldots, 1/\alpha_n\}$, except for at most n values of $\tau_n \in \mathbb{R}$. Let us denote by $\mathfrak{E}_n$ the set of these exceptional points for τ_n. We call τ_n regular if $\tau_n \in \mathbb{R} \setminus \mathfrak{E}_n$. We call $Q_n(z, \tau_n)$ regular if ϕ_n is regular and τ_n is regular. Note that if $Q_n(z, \tau_n)$ is regular, then $Q_n(z, \tau_n)$ has n simple real zeros, which are not in $\{1/\alpha_1, \ldots, 1/\alpha_n\}$. Also observe that $\tau_n = \infty$ can never be a regular value because $z = 1/\alpha_{n-1}$ will always be a zero of the numerator of $Q_n(z, \infty)$ by construction. In fact the exceptional points $\mathfrak{E}_{n+1}$ are among the points of the form

$$\mathfrak{E}_{n+1} \subseteq \left\{ -\frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)}, \quad w \in \mathfrak{P}_{n+1} \right\}.$$

We have with $\mathfrak{P}_{n+1} = \{1/\alpha_1, \ldots, 1/\alpha_{n+1}\}$

Theorem 5.8 *If $Q_{n+1}(z, \tau_{n+1})$ is regular, then it has $n+1$ simple and real zeros in $\mathbb{R} \setminus \mathfrak{P}_{n+1}$. Let us denote them by $\{\xi_{n,k}\}_{k=0}^n$. Moreover, defining $\lambda_{nk} = [k_n(\xi_{nk}, \xi_{nk})]^{-1}$, with $k_n(z,w)$ the reproducing kernel for $\mathcal{L}_n$, then equality holds in*

$$\langle f, g \rangle = \sum_{k=0}^{n} \lambda_{nk} f(\xi_{nk}) g_*(\xi_{nk}), \quad \forall f, g \in \mathcal{L}_n.$$

We know that in the polynomial case $\tau_{n+1} = 0$ is a regular value since the $n+1$ zeros of ϕ_{n+1} are simple and real. Moreover we have the Gaussian quadrature formula

$$\int_{\hat{\mathbb{R}}} f(x) \mathrm{d}\mathring{\mu}(x) = \sum_{k=0}^{n} \lambda_{nk} f(\xi_{nk}), \quad \forall f \in \Pi_{2n+1}.$$

Also this can be generalized to the rational case: if $\tau_{n+1} = 0$ is a regular value, then the zeros of ϕ_{n+1} can be used in a quadrature formula which will be exact in a slightly larger space $\mathcal{L}_{n+1} \cdot \mathcal{L}_n$, and not just in $\mathcal{L}_n \cdot \mathcal{L}_n$.

Concerning the zeros $\{\xi_{nk}\}$, we can show an analog of Theorem 5.3. Namely ξ_{n0} can be chosen arbitrarily in $\mathbb{R} \setminus \mathfrak{P}_{n+1}$ and the other zeros are then given by the zeros of $k_n(z, \xi_{n0})$.

Theorem 5.9 *Let $k_n(z,w)$ be the reproducing kernel for $\mathcal{L}_n$ and assume that ϕ_{n+1} is regular. Choose w arbitrary in $\mathbb{R} \setminus \mathfrak{P}_{n+1}$ and define*

$$\tau_{n+1} = -\frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)}. \tag{5.1}$$

Then, if τ_{n+1} is finite (i.e. w is not a zero of the numerator of ϕ_n), then defining $\xi_{n0} = w$ and $\{\xi_{nk}\}_{k=1}^n$ as the n zeros of $k_n(z,w)$, we have that the numbers $\{\xi_{nk}\}_{k=0}^n$ are the zeros of the regular quasi-orthogonal function $Q_{n+1}(z,\tau_{n+1})$.

Conversely, if $\{\xi_{nk}\}_{k=0}^n$ are the zeros of some regular quasi-orthogonal function $Q_{n+1}(z,\tau_{n+1})$, then there exists a number $w \in \mathbb{R} \setminus \mathfrak{P}_{n+1}$, such that $\xi_{n0} = w$ and τ_{n+1} is given by (5.1), while $\{\xi_{nk}\}_{k=1}^n$ are the n zeros of $k_n(z,w)$.

Proof. Note that $w \notin \mathfrak{P}_{n+1}$ implies $\tau_{n+1} \notin \mathfrak{E}_{n+1}$ unless $\tau_{n+1} = \infty$. In that case w should be a zero of the numerator of ϕ_n, but since it is not one of the α_k, it has to be a zero of ϕ_n. In that case, it follows from the Christoffel-Darboux formula that, as a function of z, $k_n(z,w) \in \mathcal{L}_{n-1}$, and thus, it can never have $n+1$ zeros. However, if τ_{n+1} is finite, and thus regular, then it follows easily from the Christoffel-Darboux relation that the n zeros of $k_n(z,w)$ coincide with n zeros of $Q_{n+1}(z,\tau_{n+1})$. The remaining zero of $Q_n(z,\tau_{n+1})$ is obviously $z = w$.

The converse statement follows along the same lines. By Lemma 12.7 in Appendix A, there are $n+1$ values of w for which τ_{n+1} as defined in (5.1) will give the value of τ_{n+1} Because $Q_{n+1}(z,\tau_{n+1})$ is regular, w has to be in $\mathbb{R} \setminus \mathfrak{P}_{n+1}$ and because $\tau_{n+1} \neq \infty$, w will not be a zero of ϕ_n. Now the Christoffel-Darboux relation can be applied again to give the result. □

6 The ORK basis

We discussed the ORF basis for $\mathcal{L}_n$, but this has in general not the property of being a local basis. For a wavelet analysis, one would rather have a basis of kernel functions, which, if possible, should be chosen orthogonal.

By Corollary 3.3, we know that $\{k_n(z,x_j)\}_{j=0}^n$ forms a basis for $\mathcal{L}_n$ for almost any set of distinct points $\mathbf{x} = \{x_j\}$. The question is whether it is possible to choose the points in $\mathbf{x}$ such that this basis is orthogonal. In that case we would have a basis of orthogonal rational kernels (ORK).

6.1 The circle case

It turns out that if we choose $x_j = \xi_{nj}$, $j = 0,\ldots,n$ the zeros of a para-orthogonal function $Q_{n+1}(z,\tau_{n+1})$, then the basis $\{\varphi_{nj}(z) = k_n(z,\xi_{nj})\}_{j=0}^n$ is orthogonal.

Theorem 6.1 *Let $k_n(z,w)$ be the reproducing kernel for $\mathcal{L}_n$ and let $\boldsymbol{\xi}_n = \{\xi_{nj}\}_{j=0}^n$ be the zeros of a para-orthogonal function $Q_{n+1}(z,\tau_{n+1})$ for some $\tau_{n+1} \in \mathbb{T}$. Then the basis for $\mathcal{L}_n$ defined by*

$$\varphi_{nj}(z) = k_n(z,\xi_{nj}), \quad j = 0,1,\ldots,n$$

is orthogonal.

Proof. Let $\Phi_n = \Phi_n(\boldsymbol{\xi}_n)$ be defined by (3.1) then it follows from (3.2) and from

$$\langle \varphi_{ni}, \varphi_{nj} \rangle = \langle k_n(z,\xi_{ni}), k_n(z,\xi_{nj}) \rangle = k_n(\xi_{nj},\xi_{ni}), \quad i,j = 0,\ldots,n$$

that $\{\varphi_{nj}\}_{j=0}^n$ will be an orthogonal basis if and only if

$$k_n(\xi_{nj},\xi_{ni}) = \delta_{ij} k_n(\xi_{ni},\xi_{ni}), \quad i,j = 0,\ldots,n.$$

In terms of the matrix (3.1) this reads

$$\Phi_n^H \Phi_n = \Lambda_n^{-1} \quad \text{or equivalently} \quad \Phi_n \Lambda_n \Phi_n^H = I$$

with $\Lambda_n = \text{diag}\,(\lambda_{n0}, \ldots, \lambda_{nn})$ a diagonal matrix with $\lambda_{ni} = 1/k_n(\xi_{ni}, \xi_{ni})$. Writing $\Phi_n \Lambda_n \Phi_n^H = I$ explicitly gives

$$\sum_{k=0}^{n} \lambda_{nk} \phi_i(\xi_{nk}) \overline{\phi_j(\xi_{nk})} = \delta_{ij}, \quad i,j = 0, \ldots, n.$$

Because also $\langle \phi_i, \phi_j \rangle = \delta_{ij}$ for $i,j = 0, \ldots, n$, this means that the above quadrature formula is exact for the inner product of all basis functions in $\mathcal{L}_n$, hence for the inner product of any two functions in $\mathcal{L}_n$. By Theorem 5.2, this means that it can only be the rational Szegő quadrature formula. Thus the theorem follows. □

6.2 The case of the real line

For the real line, a similar development can be given. The proof of the following theorem is exactly like the proof of Theorem 6.1.

Theorem 6.2 *Let $k_n(z,w)$ be the reproducing kernel for $\mathcal{L}_n$ and let $\{\xi_{nj}\}_{j=0}^{n}$ be the zeros of a regular quasi-orthogonal function $Q_{n+1}(z, \tau_{n+1})$ for some regular $\tau_{n+1} \in \mathbb{R} \setminus \mathfrak{E}_{n+1}$. Then the basis for $\mathcal{L}_n$ defined by*

$$\varphi_{nj}(z) = k_n(z, \xi_{nj}), \quad j = 0, 1, \ldots, n$$

is orthogonal.

7 The WRK basis

Now suppose that we know the function at resolution level $2n$, i.e., we know $f_{2n} \in \mathcal{L}_{2n}$. The problem is to decompose the function into two orthogonal functions

$$f_{2n} = f_n + g_n, \quad f_n \in \mathcal{L}_n, \quad g_n \in \mathcal{K}_n,$$

where $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$ is the wavelet space. This is a trivial problem if we know the decomposition with respect to the ORF basis: if $f_{2n} = \sum_{k=0}^{2n} c_k \phi_k$, then $f_n = \sum_{k=0}^{n} c_k \phi_k$ while $g_n = \sum_{k=n+1}^{2n} c_k \phi_k$. However, for reasons that have been explained, we prefer not to use the ORF basis, but we use the ORK basis instead. We know how to express the function f_n in terms of an ORK basis. The remaining problem is to write g_n in terms of a basis which is generated from the reproducing kernel $l_n(z,w)$ for $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$. Obviously the reproducing kernel for $\mathcal{K}_n$ is

$$l_n(z,w) = k_{2n}(z,w) - k_n(z,w) = \sum_{k=n+1}^{2n} \phi_k(z) \overline{\phi_k(w)}.$$

The main question in this respect is: Can we find n numbers $\{\eta_{nj}\}_{j=0}^{n-1}$ such that the functions $\{\psi_{nj}(z) = l_n(z, \eta_{nj})\}_{j=0}^{n-1}$ form a basis for $\mathcal{K}_n$ and if possible, can it be made orthogonal?

This problem for the basis of $\mathcal{K}_n$ is not as trivial as it was in Corollary 3.3 for the basis of $\mathcal{L}_n$. There is indeed no guarantee that for an arbitrary set of distinct points $\mathbf{y} = \{y_k\}_{k=0}^{n-1}$ on $\partial\mathbb{O}$ or not, the matrix

$$\Psi_n(\mathbf{y}) = \begin{bmatrix} \phi_{n+1}(y_0) & \cdots & \phi_{n+1}(y_{n-1}) \\ \vdots & & \vdots \\ \phi_{2n}(y_0) & \cdots & \phi_{2n}(y_{n-1}) \end{bmatrix} \tag{7.1}$$

would be regular. However, using the following lemma, it is possible to prove that there always exists a set of points $\{y_j\}_{j=0}^{n-1}$ on $\partial\mathbb{O}$ which make this matrix regular.

Lemma 7.1 *Assume that $Q \in \mathbb{C}^{(2n+1)\times(2n+1)}$ is a square matrix such that $Q^H Q = D$ with D regular and diagonal. Assume moreover that this Q is subdivided as*

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$$

with $Q_1 \in \mathbb{C}^{(n+1)\times(n+1)}$ and hence $Q_4 \in \mathbb{C}^{n\times n}$. If Q_1 is regular and if $Q_3 \in \mathbb{C}^{n\times n+1}$ is of full rank n, then Q_4 is regular.

Proof. Let D be subdivided into the two parts $D_1 \in \mathbb{R}^{(n+1)\times(n+1)}$ and $D_2 \in \mathbb{R}^{n\times n}$. We know from $Q^H Q = D$ that

$$Q_2^H Q_1 + Q_4^H Q_3 = 0$$

so that $\operatorname{rank}(Q_2^H Q_1) = \operatorname{rank}(Q_4^H Q_3)$. Because Q_1 is regular and $\operatorname{rank}(Q_2) = n$, we find that $\operatorname{rank}(Q_4^H Q_3) = n$. Now if Q_4 were singular, then there would exist a nonzero vector $\mathbf{c} \in \mathbb{C}^{1\times n}$ such that $\mathbf{c}Q_4^H = 0$, hence also $\mathbf{c}Q_4^H Q_3 = 0$. In other words, $\operatorname{rank}(Q_4^H Q_3) < n$, which is a contradiction. □

7.1 The circle case

In the circle case we can prove

Theorem 7.2 *Consider the zeros $\boldsymbol{\xi}_{2n} = \{\xi_{2n,k} : k = 0, 1, \ldots, 2n\}$ of the para-orthogonal function $Q_{2n+1}(z, \tau_{2n+1})$ for some $\tau_{2n+1} \in \mathbb{T}$. If we select $\mathbf{y}_n = \{y_k : k = 0, \ldots, n-1\}$ to be any n out of the $2n+1$ zeros in $\boldsymbol{\xi}_{2n}$, then the matrix $\Psi_n(\mathbf{y}_n)$ as defined in (7.1) will be regular.*

Proof. Since the ordering of the zeros $\xi_{2n,k}$ is completely arbitrary, we can always assume that we select the y_k to be the last elements of $\boldsymbol{\xi}_{2n} = \{\xi_{2n,j} : j = 0, \ldots, 2n\}$. Now consider the matrix $\Phi_{2n} = \Phi_{2n}(\boldsymbol{\xi}_{2n})$ of (3.1) where the evaluation is in the points of $\boldsymbol{\xi}_{2n}$, then $\Psi_n(\mathbf{y}_n)$ appears as the $n \times n$ right lower part of the matrix Φ_{2n}. By Lemma 3.2, it follows that taking the block of the first $n+1$ columns in Φ_{2n}, any selection of k different rows from it will result in a matrix of rank $\min\{k, n+1\}$. Thus the conditions of the previous lemma are satisfied and the theorem follows. □

This theorem settles the question of the existence of a wavelet basis of reproducing kernels (WRK basis) of the form $\{l_n(z, y_k)\}_{k=0}^{n-1}$. It is however not clear how to choose these $\{y_k\}_{k=0}^{n-1}$ to make the WRK basis orthogonal, if it is possible at all. The same problem was encountered for the polynomial case on an interval in [20] where an orthogonal WRK basis could only be constructed for a Chebyshev weight of the second kind.

For some particular cases, it is however possible to construct such an orthogonal WRK basis. We give a trivial example. Consider again the case where all $\alpha_k = 0$ and where orthogonality is with respect to the normalized Lebesgue measure of $\mathbb{T}$. In that case

$$l_n(z,w) = \sum_{k=n+1}^{2n} (z\overline{w})^k.$$

Note that $\mathcal{L}_n = \Pi_n$ and $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n = z^{n+1}\Pi_{n-1}$. Furthermore, for some distinct η_{nk}, $k = 0,\ldots,n-1$, all on $\mathbb{T}$ define the functions $\psi_{nk}(z) = l_n(z,\eta_{nk})$, $k = 0,\ldots,n-1$. In analogy with Theorem 6.1, for these to be an orthogonal basis of $\mathcal{K}_n$, we need the existence of numbers λ_{nk}, $k = 0,\ldots,n-1$ such that

$$\sum_{k=0}^{n-1} \lambda_{nk}\phi_i(\eta_{nk})\overline{\phi_j(\eta_{nk})} = \delta_{ij}, \quad i,j = n+1,\ldots,2n. \tag{7.2}$$

In other words, we need a rational Szegő quadrature formula which is exact in $\mathcal{K}_n = z^{n+1}\Pi_{n-1}$. Because $f(z) = z^{n+1}p_{n-1}(z)$ and $g(z) = z^{n+1}q_{n-1}(z)$ are both in $\mathcal{K}_n$ if $p_{n-1}, q_{n-1} \in \Pi_{n-1}$, we have

$$\langle f,g\rangle = \langle p_{n-1}, q_{n-1}\rangle.$$

The quadrature formula will be exact if and only if the nodes η_{nk} are the zeros of the para-orthogonal polynomial $Q_n(z,\tau) = z^n + \tau$, $\tau \in \mathbb{T}$ and the weights are given by

$$\lambda_{nk} = \frac{1}{k_{n-1}(\eta_{nk},\eta_{nk})} = \frac{1}{\sum_{k=0}^{n-1}|\eta_{nk}|^{2k}} = \frac{1}{n}.$$

Thus in this case, one can take for example the nth roots of unity for η_{nk} and the basis ψ_{nk} will be an orthogonal WRK basis.

We can push this example a bit further in two directions. First by considering orthogonal polynomials with respect to a rational modification of the Lebesgue measure. A rational modification of the Lebesgue measure is defined as a measure μ satisfying for some $a_j \in \mathbb{D}$, $j = 1,\ldots,m$

$$\mathrm{d}\mu(t) = \frac{\mathrm{d}\lambda(t)}{|h(t)|^2}, \quad h(t) = \prod_{j=1}^{m}(1-\overline{a}_j t), \quad t = e^{i\theta}$$

where λ is the normalized Lebesgue measure. Orthogonal polynomials for such a measure and corresponding quadrature formulas were considered in [21], see also [28, p.289-290]. The orthogonal polynomials for such a measure are given by $\phi_k(z) = z^{k-m}h^*(z) = z^k h_*(z)$ for $k \geq m$ (see [28, Thm. 11.2, p.289]). For the polynomial $h(z)$ of degree m we have used $h^*(z) = z^m h_*(z) = \prod_{j=1}^{m}(z-a_j)$. Thus for $n+1 \geq m$, it follows that $\mathcal{K}_n = \operatorname{span}\{\phi_{n+1},\ldots,\phi_{2n}\} = z^{n+1-m}h^*(z)\Pi_{n-1}$. Therefore, if $f,g \in \mathcal{K}_n$, then they can be represented as $f(z) = z^{n+1-m}h^*(z)p_{n-1}(z)$ and $g(z) = z^{n+1-m}h^*(z)q_{n-1}(z)$ with $p_{n-1}, q_{n-1} \in \Pi_{n-1}$ and we have $\langle f,g\rangle_\mu = \langle p_{n-1}, q_{n-1}\rangle_\lambda$.

If we want $\{\psi_{nk}(z) = l_n(z,\eta_{nk})\}$ to be an orthogonal WRK basis, then, as shown before, the numbers η_{nk} should be chosen such that there exist positive numbers λ_{nk} such that (7.2) holds, hence such that

$$\sum_{k=0}^{n-1} \lambda_{nk}\eta_{nk}^{i-j}|h^*(\eta_{nk})|^2 = \langle \phi_i,\phi_j\rangle_\mu = \left\langle z^{i-n-1}, z^{j-n-1}\right\rangle_\lambda.$$

This is clearly possible by choosing η_{nk} to be the n zeros of the para-orthogonal polynomial $z^n + \tau$, $\tau \in \mathbb{T}$ (and the weights are $\lambda_{nk} = |h(\eta_{nk})|^2/n$).

A second generalization of the simple case which was first considered is to consider orthogonal rational functions with respect to the Lebesgue measure. Such a situation was considered in [7]. In this case the orthogonal rational functions are given by

$$\phi_k(z) = \sqrt{1-|\alpha_k|^2}\frac{zB_k(z)}{z-\alpha_k}.$$

Therefore $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n = \operatorname{span}\{\phi_{n+1},\ldots,\phi_{2n}\}$ is given by

$$\mathcal{K}_n = \beta_{n+1}\tilde{\mathcal{L}}_{n-1}, \quad \beta_{n+1}(z) = \frac{zB_{n+1}(z)}{z-\alpha_{n+1}}$$

and where $\tilde{\mathcal{L}}_{n-1}$ is the space of rational functions associated with the points $\{\tilde{\alpha}_j = \alpha_{n+1+j}\}_{j=1}^{n-1}$. Thus, if $f, g \in \mathcal{K}_n$, then there exist $\tilde{p}_{n-1}, \tilde{q}_{n-1} \in \tilde{\mathcal{L}}_{n-1}$ such that

$$f(z) = \beta_{n+1}(z)\tilde{p}_{n-1}(z), \quad g(z) = \beta_{n+1}(z)\tilde{q}_{n-1}(z),$$

and therefore

$$\langle f,g\rangle_\lambda = \int_{\mathbb{T}} f(t)\overline{g(t)}\mathrm{d}\lambda(t) = \int_{\mathbb{T}} \tilde{p}_{n-1}\overline{\tilde{q}_{n-1}(t)}\mathrm{d}\mu(t) = \langle \tilde{p}_{n-1}, \tilde{q}_{n-1}\rangle_\mu \tag{7.3}$$

with $\mathrm{d}\mu(t) = |t-\alpha_{n+1}|^{-2}\mathrm{d}\lambda(t)$. Note that this is a rational modification of the Lebesgue measure which is, up to the factor $1-|\alpha_{n+1}|^2$ equal to the Poisson kernel $P(t,\alpha_{n+1}) = (1-|\alpha_{n+1}|^2)/|t-\alpha_{n+1}|^2$. In [7, Section 3] weights and abscisses were given for this case which give an n-point rational Szegő quadrature formula which is exact in $\tilde{\mathcal{L}}_{(n-1)*}\cdot\tilde{\mathcal{L}}_{n-1}$, hence for which the inner product (7.3) is evaluated exactly. Thus also in this case we can construct an orthogonal wavelet basis $\{\psi_k(z) = l_n(z,\eta_{nk})\}_{k=0}^{n-1}$.

In Figure 2 we give an example where the α_k are located near the unit circle and where orthogonality is with respect to the normalized Lebesgue measure.

In general however, for an arbitrary positive measure μ on $\mathbb{T}$, it is not clear how one should construct an orthogonal wavelet basis of reproducing kernels.

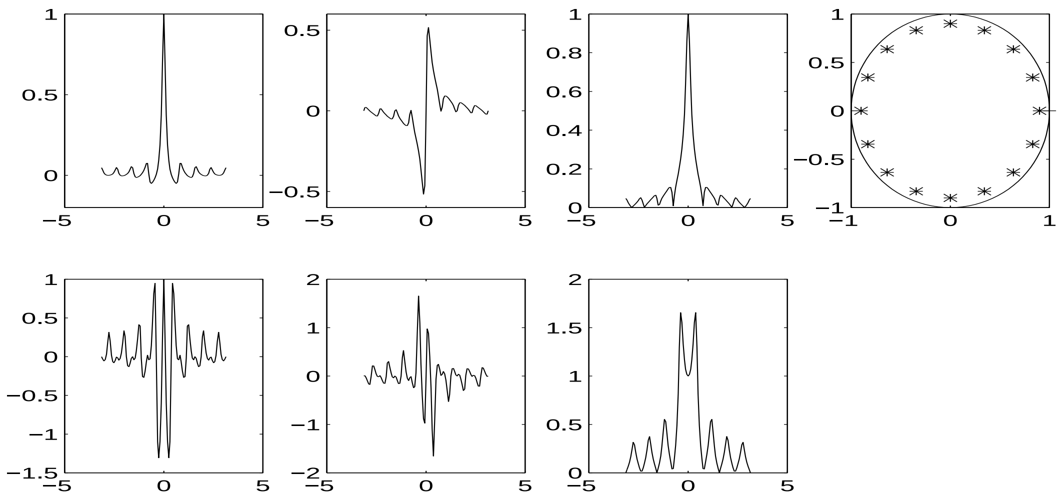
7.2 The case of the real line

To construct a wavelet basis of reproducing kernels in the case of the real line, we can copy the proofs of Section 7.1. So we have

Theorem 7.3 *Consider the zeros* $\boldsymbol{\xi}_{2n} = \{\xi_{2n,k} : k = 0,1,\ldots,2n\}$ *of the regular quasi-orthogonal function* $Q_{2n+1}(z,\tau_{2n+1})$ *for some* $\tau_{2n+1} \in \mathbb{R}\setminus\mathfrak{E}_{2n+1}$*. If we select* $\mathbf{y}_n = \{y_k : k = 0,\ldots,n-1\}$ *to be any* n *out of the* $2n+1$ *zeros in* $\boldsymbol{\xi}_{2n}$*, then the matrix* $\Psi_n(\mathbf{y}_n)$ *as defined in (7.1) will be regular and hence* $\{\psi_{nj}(z) = k_{2n}(z,y_j) - k_n(z,y_j)\}_{j=0}^{n-1}$ *will form a basis for the wavelet space* $\mathcal{L}_{2n}\ominus\mathcal{L}_n$*.*

In the polynomial case, another possibility exists to construct a WRK basis by choosing $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$ to be the zeros of the polynomial ϕ_n. This was proved [20]. We give an alternative proof.

Figure 2: Circular case. The real and imaginary part and the modulus of the functions $k_8(z,w)$ (top) and $l_8(z,w)$ (bottom) for $z = e^{i\theta}$ and $w = 1$, plotted as functions of θ. The measure is the normalized Lebesgue measure and the α_k are chosen as $\alpha_k = 0.9e^{i2k\pi/8}$ for $k = 1,\ldots,8$ and $\alpha_{k+8} = 0.9e^{i(2k+1)\pi/8}$ for $k = 1,\ldots,8$. These α_k's are plotted in the figure on the right.



Theorem 7.4 *Assume that* $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$ *are the zeros of the orthogonal polynomial* ϕ_n*, then* $\{l_n(z,y_k)\}_{k=0}^{n-1}$ *forms a basis for* $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$*.*

Proof. If it were not a basis, then the matrix $\Psi_n(\mathbf{y}_n)$ would be singular, and thus there would exist a nonzero vector $\mathbf{c} = [c_1,\ldots,c_n]$ such that $\Psi_n(\mathbf{y}_n)\mathbf{c}^T = 0$. Consider the function $\phi(z) = \sum_{k=1}^n c_k\phi_{n+k}$. This function vanishes in the zeros of ϕ_n. Thus it has to be of the form $\phi = \phi_n p_r$ where p_r is a polynomial of strict degree r with $0 \le r \le n$. On the other hand $\langle\phi,\phi_k\rangle = 0$ for any $k = 0,\ldots,n$. In particular $\langle\phi,\phi_{n-r}\rangle = \langle p_r\phi_{n-r},\phi_n\rangle = 0$. This is however impossible because $p_r\phi_{n-r}$ is of strict degree n and can thus not be orthogonal to ϕ_n. Thus we have a contradiction and the $l_n(z,y_k)$ have to be independent. □

Let us give an example. The Hermite polynomials (all $\alpha_k = \infty$) are given by the recurrence relation

$$\phi_n(z) = (A_n z + B_n)\phi_{n-1}(z) + C_n\phi_{n-2}(z), \quad n = 2,3,\ldots$$

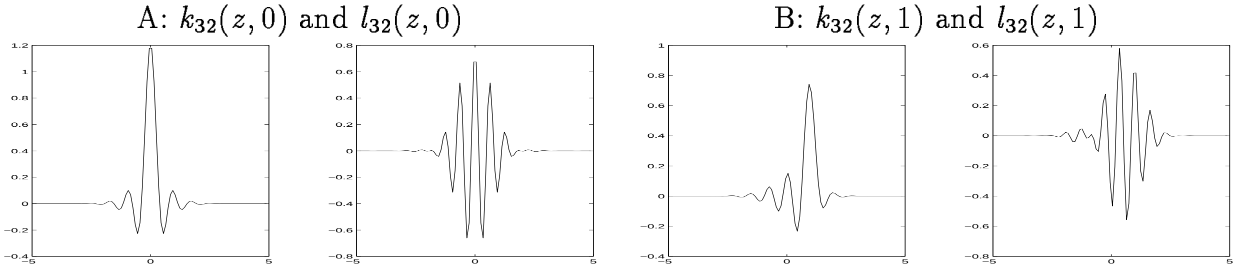
with $\phi_0 = 1$, $\phi_1(z) = \sqrt{2}z$, and

$$A_n = \sqrt{\frac{2}{n}}, \quad B_n = 0, \quad C_n = -\sqrt{\frac{n-1}{n}}, \quad n = 2,3,\ldots.$$

These are orthogonal with respect to the weight $w(x) = \pi^{-1/2}e^{-x^2}$, thus in our notation

$$\mathrm{d}\mathring{\mu}(x) = \sqrt{\pi}(1+x^2)e^{-x^2}\frac{\mathrm{d}x}{\pi(1+x^2)}.$$

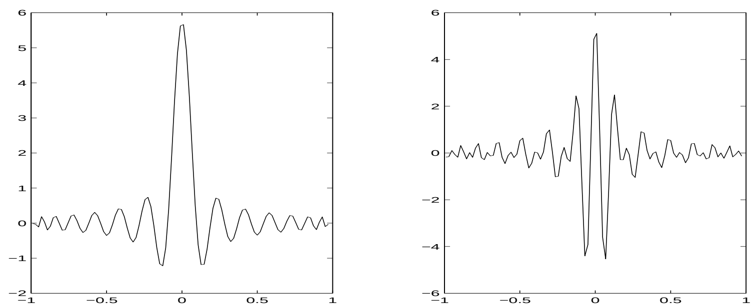
Figure 3: Case of the real line. The scaling and wavelet functions $k_{32}(z, w)$ (left) and $l_{32}(z, w)$ (right) multiplied by the weight function in the case of Hermite polynomials. In Figure A: $w = 0$, in Figure B: $w = 1$.



The scaling functions and wavelet functions multiplied by the weight are plotted in Figure 3.

For the Chebyshev polynomials of the first kind we have plotted Figure 4. Of course this corresponds to the real part of the polynomial circular case where the weight is the Lebesgue weight on the circle. Indeed, the real part of $\sum_{k=0}^{n} z^k$ is given by $\sum_{k=0}^{n} \cos k\theta$ and this is a transformation of $\sum_{k=0}^{n} T_k(x)$ where T_k is the normalized Chebyshev polynomial. Of course this is directly related to the trigonometric bases studied in [11, 25].

Figure 4: Case of the real line. The scaling and wavelet functions $k_{32}(z, 0)$ (left) and $l_{32}(z, 0)$ (right) in the case of Chebyshev polynomials.



8 Interpolation and biorthogonal bases

Recall the definition $\partial\mathbb{O}$ which means the boundary of $\mathbb{O}$, that is in our case $\mathbb{T}$ or $\mathbb{R}$. The pole set for $\mathcal{L}_n$ is given by $\mathfrak{P}_n = \{1/\overline{\alpha}_1, \ldots, 1/\overline{\alpha}_n\}$. None of these is on $\mathbb{T}$ for the circle, while they are all real for the case of the line.

Let us define the Lagrange polynomials in Π_n for the interpolation points $\mathbf{x}_n = \{x_k\}_{k=0}^n$ (all distinct and on $\partial\mathbb{O} \setminus \mathfrak{P}_n$) by

$$l_{nk}(z) = \frac{\prod_{j=0,j\neq k}^n (z - x_j)}{\prod_{j=0,j\neq k}^n (x_k - x_j)}, \quad k = 0, \dots, n$$

and define

$$L_{nk}(z) = l_{nk}(z)\frac{\pi_n(x_k)}{\pi_n(z)}, \quad k = 0, \dots, n$$

with

$$\pi_n = \prod_{j=0}^n (1 - \overline{\alpha}_j z).$$

Then obviously $L_{nk}(x_j) = \delta_{kj}$, $k, j = 0, \dots, n$ while $L_{nk} \in \mathcal{L}_n$. We call these L_{nk} the fundamental Lagrange interpolating functions (FLIF) of $\mathcal{L}_n$ for the points $\mathbf{x}_n = \{x_k\}_{k=0}^n$. It immediately follows that for any function $f \in \mathcal{L}_n$ we may write

$$f(z) = \sum_{k=0}^n f(x_k) L_{nk}(z).$$

Defining the discrete inner product in $\mathcal{L}_n$

$$\langle f, g\rangle_{\mathbf{x}_n} = \sum_{k=0}^n f(x_k)\overline{g(x_k)},$$

it is directly seen that $\langle L_{nk}, L_{nj}\rangle_{\mathbf{x}_n} = \delta_{kj}$. Thus the FLIF are an orthonormal basis for $\mathcal{L}_n$ with respect to this inner product. So, by the general theory of reproducing kernels, it follows that the solution of the problem

$$\min\{\|f\|^2_{\mathbf{x}_n} : f \in \mathcal{L}_n; f(x_k) = 1\}$$

is given by $\sum_{j=0}^n L_{nj}(z)\overline{L_{nj}(x_k)} = L_{nk}(z)$. Moreover, if we set $\varphi_{nk}(z) = k_n(z, x_k)$ and $\tilde{\varphi}_{nk}(z) = L_{nk}(z)$, then because of the reproducing property of φ_{nk}

$$\langle \tilde{\varphi}_{nk}, \varphi_{nj}\rangle = \delta_{kj}.$$

In other words, φ_{nk} and $\tilde{\varphi}_{nk}$ are biorthogonal bases for $\mathcal{L}_n$.

Note that we can characterize $\tilde{\varphi}_{nk}$ in another way. We may write $\tilde{\varphi}_{nk} = \sum_{j=0}^n c_j^{(k)}\phi_j$ where $\mathbf{c}^{(k)} = [c_0^{(k)} \dots c_n^{(k)}]$ is defined by $\mathbf{c}^{(k)}\Phi_n = \mathbf{e}_k$ where $\mathbf{e}_k = [0 \dots 0\ 1\ 0 \dots 0]$ is the kth unit vector and $\Phi_n = \Phi_n(\mathbf{x}_n)$ is the matrix (3.1). Thus $\mathbf{c}^{(k)}$ is row k in the inverse of Φ_n, so that we have

$$\begin{bmatrix} \tilde{\varphi}_{n0} \\ \vdots \\ \tilde{\varphi}_{nn} \end{bmatrix} = \begin{bmatrix} c_0^{(0)} & \cdots & c_n^{(0)} \\ \vdots & & \vdots \\ c_0^{(n)} & \cdots & c_n^{(n)} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_n \end{bmatrix} = \Phi_n^{-1} \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_n \end{bmatrix}. \tag{8.1}$$

It is clear that if the x_k are the zeros of a para-orthogonal polynomial (the case of the circle) or the regular quasi-orthogonal (the case of the real line) $Q_{n+1}(z, \tau_{n+1})$, then the φ_{nk} are orthogonal and $\langle \varphi_{ni}, \varphi_{nj}\rangle = \delta_{ij}\varphi_{ni}(x_i)$. In that case we have of course $L_{nk}(z) = \varphi_{nk}(z)/\varphi_{nk}(x_k)$.

Similarly, given $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$, one can construct a biorthogonal basis for $\psi_{nk}(z) = l_n(z, y_k)$, on condition that the matrix $\Psi_n = \Psi_n(\mathbf{y}_n)$ of (7.1) is regular. Indeed, let $\mathbf{e}_k = [0 \ldots 0\ 1\ 0 \ldots 0]$ be the kth unit vector, then if Ψ_n is regular, there is exactly one solution $\mathbf{d}^{(k)} = [d_{n+1}^{(k)} \ldots d_{2n}^{(k)}]$ to the equation $\mathbf{d}^{(k)}\Psi_n = \mathbf{e}_k$, for each $k = 0, \ldots, n-1$. The function $\tilde{\psi}_{nk}(z) = \sum_{j=n+1}^{2n} d_j^{(k)} \psi_{nj}(z)$ is obviously in $\mathcal{K}_n$ and we have $\tilde{\psi}_{nk}(y_j) = \delta_{kj}$, $k, j = 0, \ldots, n-1$. Thus these $\tilde{\psi}_{nk}$ form the FLIF of $\mathcal{K}_n$ for the interpolation points $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$. We can write for any function $f \in \mathcal{K}_n$ that

$$f(z) = \sum_{k=0}^{n-1} f(y_k) \tilde{\psi}_{nk}(z).$$

The $\tilde{\psi}_{nk}$ are orthonormal with respect to the discrete inner product

$$\langle f, g \rangle_{\mathbf{y}_n} = \sum_{k=0}^{n-1} f(y_k) \overline{g(y_k)}$$

and $\tilde{\psi}_{nk}$ is the solution to the problem

$$\min\{\|f\|_{\mathbf{y}_n}^2 : f(y_k) = 1; f \in \mathcal{K}_n\}.$$

Moreover, they form a biorthogonal basis for the ψ_{nk} in $\mathcal{K}_n$ because by the reproducing property of ψ_{nk} we have $\left\langle \tilde{\psi}_{nk}, \psi_{nj} \right\rangle = \delta_{kj}$, $k, j = 0, \ldots, n-1$. The relation between the bases $\{\phi_k\}_{k=n+1}^{2n}$ and $\{\tilde{\psi}_{nk}\}_{k=0}^{n-1}$ is given by

$$\begin{bmatrix} \tilde{\psi}_{n0} \\ \vdots \\ \tilde{\psi}_{n,n-1} \end{bmatrix} = \begin{bmatrix} d_{n+1}^{(0)} & \cdots & d_{2n}^{(0)} \\ \vdots & & \vdots \\ d_{n+1}^{(n-1)} & \cdots & d_{2n}^{(n-1)} \end{bmatrix} \begin{bmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{bmatrix} = \Psi_n^{-1} \begin{bmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{bmatrix} \tag{8.2}$$

with $\Psi_n = \Psi_n(\mathbf{y}_n)$.

9 Decomposition and reconstruction

To do the wavelet analysis of a function we should be able to decompose a function $f_{2n} \in \mathcal{L}_{2n}$ into a sum $f_{2n} = f_n + g_n$ where $f_n \in \mathcal{L}_n$ and $g_n \in \mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$. This is a matter of a change of basis. Assume that for each n, we select a number of distinct points $\mathbf{x}_n = \{x_{nk}\}_{k=0}^{n}$ on $\partial\mathbb{O} \setminus \mathfrak{P}_n$ and a number of points $\mathbf{y}_n = \{y_{nk}\}_{k=0}^{n-1}$ distinct and on $\partial\mathbb{O} \setminus \mathfrak{P}_{2n}$ such that $\Psi_n(\mathbf{y}_n)$ of (7.1) is regular. We write in short hand $\Phi_n = \Phi_n(\mathbf{x}_n)$ and $\Psi_n = \Psi_n(\mathbf{y}_n)$. In that case $\varphi_{nk}(z) = k_n(z, x_{nk})$ forms a basis for $\mathcal{L}_n$ and $\psi_{nk}(z) = k_{2n}(z, y_{nk}) - k_n(z, y_{nk})$ is a basis for $\mathcal{K}_n$ when $k_n(z, w)$ is the reproducing kernel for $\mathcal{L}_n$. Let the coefficients with respect to the appropriate bases be defined by

$$f_{2n} = \sum_{k=0}^{2n} p_{2n,k} \varphi_{2n,k}; \quad f_n = \sum_{k=0}^{n} p_{nk} \varphi_{n,k}; \quad g_n = \sum_{k=0}^{n-1} q_{nk} \psi_{n,k}.$$

Then setting $\mathbf{p}_{2n} = [p_{2n,0} \ldots p_{2n,2n}]$, $\mathbf{p}_n = [p_{n0} \ldots p_{n,n}]$, and $\mathbf{q}_n = [q_{n0} \ldots q_{n,n-1}]$, we get

$$f_{2n} = \mathbf{p}_{2n} \Phi_{2n}^H \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{2n} \end{bmatrix}, \quad f_n = \mathbf{p}_n \Phi_n^H \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_n \end{bmatrix}, \quad g_n = \mathbf{q}_n \Psi_n^H \begin{bmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{bmatrix}.$$

Equating coefficients of the corresponding basis functions in $f_{2n} = f_n + g_n$ leads to

$$\mathbf{p}_{2n}\Phi_{2n}^H = [\mathbf{p}_n \quad \mathbf{q}_n] \left[\begin{array}{cc} \Phi_n^H & 0 \\ 0 & \Psi_n^H \end{array} \right].$$

This relation allows us to compute $\mathbf{p}_n$ and $\mathbf{q}_n$ from $\mathbf{p}_{2n}$ and conversely to reconstruct $\mathbf{p}_{2n}$ from $\mathbf{p}_n$ and $\mathbf{q}_n$.

For example, using the relation $p_{nr} = \langle f_{2n}, \tilde{\varphi}_{nr} \rangle$ and the biorthogonality relations, we get

$$p_{nr} = \sum_{k=0}^{2n} p_{2n,k} \overline{\tilde{\varphi}_{nr}(x_{2n,k})}, \quad r = 0, \ldots, n.$$

Similarly, it is seen from $q_{nr} = \left\langle f_{2n}, \tilde{\psi}_{nr} \right\rangle$ that

$$q_{nr} = \sum_{k=0}^{2n} p_{2n,k} \overline{\tilde{\psi}_{nr}(x_{2n,k})}, \quad r = 0, \ldots, n-1.$$

In the special case where the $\varphi_{n,k}$ are orthogonal, i.e., when the $x_{n,k}$ are the zeros $\xi_{n,k}$ of $Q_{n+1}(z, \tau_{n+1})$, then $\tilde{\varphi}_{n,r} = \varphi_{n,r}/\varphi_{n,r}(\xi_{n,r})$ we get for the first of the analysis formulas

$$p_{nr} = \frac{1}{\varphi_{n,r}(\xi_{n,r})} \sum_{k=0}^{2n} p_{2n,k} \overline{\varphi_{nr}(x_{2n,k})}, \quad r = 0, \ldots, n.$$

In general, the ψ_{nk} are not orthogonal, but if we choose $\{y_{nr}\}_{r=0}^{n-1}$ to be the last n zeros $\{\xi_{2n,n+1+r}\}_{r=0}^{n-1}$, then by definition $\tilde{\psi}_{nr}(y_{ns}) = \delta_{rs}$, $r, s = 0, \ldots, n-1$. Thus, the second analysis formula reduces to

$$q_{nr} = \sum_{k=0}^{n} p_{2n,k} \overline{\tilde{\psi}_{nr}(\xi_{2n,k})} + p_{2n,n+1+r}, \quad r = 0, \ldots, n-1.$$

For the reconstruction formula, we have by $p_{2n,r} = \langle f_n + g_n, \tilde{\varphi}_{2n,r} \rangle$,

$$p_{2n,r} = \sum_{k=0}^{n} p_{nk} \langle \varphi_{nk}, \tilde{\varphi}_{2n,r} \rangle + \sum_{l=0}^{n-1} q_{nl} \langle \psi_{nl}, \tilde{\varphi}_{2n,r} \rangle, \quad r = 0, \ldots, 2n.$$

When the $\varphi_{2n,r}$ are orthogonal, i.e., when the $x_{2n,k}$ are the zeros $\xi_{2n,k}$ of $Q_{2n+1}(z, \tau_{2n+1})$, then $\tilde{\varphi}_{2n,r} = \varphi_{2n,r}/\varphi_{2n,r}(\xi_{2n,r})$, and the reconstruction formula becomes as in [20],

$$p_{2n,r} = \frac{1}{\varphi_{2n,r}(\xi_{2n,r})} \left(\sum_{k=0}^{n} p_{nk} \varphi_{nk}(\xi_{2n,r}) + \sum_{l=0}^{n-1} q_{nl} \psi_{nl}(\xi_{2n,r}) \right), \quad r = 0, \ldots, 2n.$$

10 Dilation and translation

The nested spaces $\{\mathcal{V}_s\}_{s=-1}^{\infty}$ are a special case of a second generation multiresolution analysis [27], but they can be interpreted in a way which is much closer to the classical definition of a multiresolution analysis. This is what we shall do here.

Before we start checking the MRA properties, we should first adapt the notion of a shift which is essential in the definition of an MRA.

10.1 Generalized shift

Recall that several of the properties in definition of classical MRA refer to shifted functions. In our case, this shift has to be given a more general meaning. It was also introduced in the polynomial case [20], and we follow here the same lines.

To introduce the idea, it is easiest to consider the circle case and write $F(\theta)$ for $f(e^{i\theta})$. Then $F(\theta)$ is a 2π-periodic function. In classical MRA for 2π periodic functions, the orthogonal basis is the Fourier basis $\phi_k(t) = e^{ikt}$, a shift $F(\theta+\tau)$ has the effect that the kth Fourier coefficient of F is multiplied with $\overline{\phi_k(\tau)/\phi_k(0)} = e^{-ik\tau}$. Translating this to the functions f defined on $\mathbb{T}$ with orthogonal basis $\phi_k(z) = z^k$, then a shift has the effect that the kth Fourier coefficient is multiplied with $\overline{\phi_k(\xi)/\phi_k(1)} = \xi^{-k}$ where $\xi = e^{i\tau}$. Similarly, for the continuous Fourier transform, the basis functions are $\phi_\omega(t) = e^{i\omega t}$ and a shift $f(t+\tau)$ has the effect that the Fourier transform $f^\wedge(\omega)$ is multiplied by $\overline{\phi_\omega(\tau)/\phi_\omega(0)} = e^{-i\omega\tau}$.

The generalization we need is to consider a shift operator which is defined as above but now with respect to our orthonormal basis ϕ_k which in general is not the basis of classical Fourier analysis. So we consider the Fourier transform with respect to the basis ϕ_k namely $\mathcal{F}(f) = \{f_k^\wedge\}_{k=0}^\infty$ with $f_k^\wedge = \langle f, \phi_k\rangle$. A shift operator S_τ will be defined as the operator whose effect is that

$$(S_\tau f)_k^\wedge = f_k^\wedge \cdot \overline{\phi_k(\tau)}/\overline{\phi_k(\tau_0)},$$

where $\tau \in \mathbb{T}$ and $\tau_0 = 1$ for the circle, while $\tau \in \mathbb{R}$ and $\tau_0 = 0$ for the real line. In fact the choice of a specific $\tau_0 \in \partial\mathbb{O}$ is not really crucial and if another point is more appropriate, one can use it. This τ_0 is just a reference point with respect to which the shift is taken.

If we consider $\mathcal{F}(f)$ for $f \in \mathcal{L}_n$ as a finite dimensional vector of dimension $n+1$, then we can describe the shift operator restricted to $\mathcal{L}_n$ as a multiplication with a finite dimensional diagonal matrix: So we define the shift restricted to $\mathcal{L}_n$ as $S_{n,\tau}$, then for $f \in \mathcal{L}_n$

$$g = S_{n,\tau} f \quad \Leftrightarrow \quad \mathcal{F}(g) = \mathcal{F}(f) D_{n,\tau}$$

where $D_{n,\tau}$ is a diagonal matrix defined in terms of the orthogonal basis functions $\{\phi_k\}_{k=0}^n$ of $\mathcal{L}_n$.

$$D_{n,\tau} = \text{diag}\left(\frac{\phi_0(\tau)}{\phi_0(\tau_0)}, \ldots, \frac{\phi_n(\tau)}{\phi_n(\tau_0)}\right)^H$$

Thus if $\mathbf{a}_n = [a_0, \ldots, a_n]$ and $f = \mathbf{a}_n[\phi_0, \ldots, \phi_n]^T \in \mathcal{L}_n$, then $S_{n,\tau} f = \mathbf{b}_n[\phi_0, \ldots, \phi_n]^T$ with $\mathbf{b}_n = [b_0, \ldots, b_n] = \mathbf{a}_n D_{n,\tau}$. Note that this corresponds to the classical shift if the ϕ_k are the classical Fourier basis of complex exponentials.

Now we turn to the MRA properties.

10.2 Nesting property

This one is trivial to verify. Defining $\mathcal{V}_{-1} = \mathcal{L}_0$, it is immediately seen that

$$\mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_s \subset \mathcal{V}_{s+1} \subset \cdots.$$

10.3 The completeness condition

If $\mathcal{K}$ is a subspace of the Hilbert space $\mathcal{H}$, then for the $\{\mathcal{V}_k\}$ to form a MRA of $\mathcal{K}$, one should have that

$$\text{clos}_{\mathcal{H}}\left(\bigcup_{s=-1}^{\infty} \mathcal{V}_s\right) = \mathcal{K}.$$

In our case, this means that we should verify in which spaces the system $\{B_k\}_{k=0}^{\infty}$ is complete. There are several possibilities.

In the circle case we have in this respect the following properties.

Theorem 10.1 ([1, p.244-246], [9, Corollary 7.2.4])

If $\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty$ then the system $\{B_k\}_{k=0}^{\infty}$ is complete in $H_2(\mathbb{T})$ (with respect to the Lebesgue measure) as well as in $H_2(\mathbb{T},\mu)$.

If $\int \log \mu'(t)dt=-\infty$ and $\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty$ then the system $\{B_k\}_{k=0}^{\infty}$ is complete in $L_2(\mathbb{T},\mu)$.

$\int \log \mu'(t)dt>-\infty$ is known as Szegő's condition while $\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty$ means that the Blaschke product $B(z)=\prod_{k=1}^{\infty}\zeta_k(z)$ diverges to zero. Thus the α_k should not approach the boundary $\mathbb{T}$ too fast.

In the case of the real line we will avoid a long discussion of whether the rational functions considered are are not dense in $H_2(\hat{\mathbb{R}},\mathring{\mu})$ or $L_2(\hat{\mathbb{R}},\mathring{\mu})$, and we will just define

$$\mathcal{L}=\mathrm{clos}_{L_2(\hat{\mathbb{R}},\mathring{\mu})}\left(\bigcup_{s=-1}^{\infty}\mathcal{L}_n\right).$$

We then have a multiresolution of the space $\mathcal{L}$.

10.4 Scaling property

The scaling property in a classical MRA says that if one doubles the frequency then one moves from $\mathcal{V}_s$ to $\mathcal{V}_{s+1}$. Interpreting frequency again in a generalized sense as being the Fourier coefficients with respect to the basis ϕ_k, then this can be reformulated as follows. If $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}(f)=\{f_k^{\wedge}\}_{k=0}^{\infty},\quad f_k^{\wedge}=\langle f,\phi_k\rangle,$$

then $f\in\mathcal{V}_s \Leftrightarrow \mathrm{supp}\,\mathcal{F}(f)=\{0,1,\ldots,n=2^s\}$. Thus moving from $\mathcal{V}_s$ to $\mathcal{V}_{s+1}$ is equivalent to doubling the support of this Fourier transform

10.5 Shift invariance

The shift invariance, in the classical MRA definition says that a shifted version of a function remains at the same resolution scale, i.e., stays in the same space $\mathcal{V}_s$: $f\in\mathcal{V}_s \Rightarrow S_\tau f\in\mathcal{V}_s$ for all shifts τ. With our definition of general shift, this is obviously true because ($n=2^s$)

$$f=\sum_{k=0}^{n} f_k^{\wedge}\phi_k\in\mathcal{V}_s \quad\Rightarrow\quad S_\tau f=\sum_{k=0}^{n}\left(\frac{\overline{\phi_k(\tau)}}{\overline{\phi_k(\tau_0)}}f_k^{\wedge}\right)\phi_k\in\mathcal{V}_s.$$

10.6 Riesz basis

The final condition for a stationary multiresolution is that there should exist a Riesz basis for $\mathcal{V}_0$ which consists of translates of one scaling function. By the scaling property, one can then generate bases at all resolution levels. Here we shall have one scaling function per resolution level. Indeed, using again the general shift operator defined as above, then it is clear that if $\varphi_{nk}(z) = k_n(z, x_{nk})$, $k = 0, \ldots, n = 2^s$ is a basis of reproducing kernel scaling functions (orthogonal or not) for $\mathcal{V}_s$, then we can write them as $\varphi_{nk}(z) = S_{x_{nk}}\varphi_n(z)$ with

$$\varphi_n(z) = \sum_{k=0}^{n} \overline{\phi_k(\tau_0)}\phi_k(z) = k_n(z, \tau_0).$$

Thus all the scaling basis functions φ_{nk} are generalized translates of a unique (for $\mathcal{V}_s$) scaling function φ_n.

Similarly $\psi_{nk} = S_{y_{nk}}\psi_n$ where

$$\psi_n(z) = \sum_{k=n+1}^{2n} \overline{\phi_k(\tau_0)}\phi_k(z) = l_n(z, \tau_0).$$

The bases used also have a Riesz property, i.e. there are constants $A \neq 0$ and B such that

$$A\|\mathbf{p}_n\| \leq \|f\| \leq B\|\mathbf{p}_n\|$$

where the norms are 2-norms and $f \in \mathcal{V}_s$ is given by $f = \mathbf{p}_n[\varphi_{n0}, \ldots, \varphi_{nn}]^T$. It follows from (3.2) and Parseval's equality that

$$\|f\|^2 = \|\mathbf{p}_n\Phi_n^H[\phi_0, \ldots, \phi_n]^T\|^2 = \|\mathbf{p}_n\Phi_n^H\|^2$$

and this implies that

$$\frac{1}{\|\Phi_n^{-1}\|}\|\mathbf{p}_n\| \leq \|f\| \leq \|\Phi_n\|\|\mathbf{p}_n\|.$$

Similarly, it holds that

$$\frac{1}{\|\Psi_n^{-1}\|}\|\mathbf{q}_n\| \leq \|f\| \leq \|\Psi_n\|\|\mathbf{q}_n\|$$

when $f \in \mathcal{W}_s$ with $f = \mathbf{q}_n[\psi_{n0}, \ldots, \psi_{n,n-1}]^T$.

10.7 Consequences of the generalized shift

This generalized shift operator implies that the functions in the ORK basis look like (cyclic) shifts of each other. The same holds for the functions in the WRK basis. This can be verified in Figure 5. If one introduces poles which are close to the unit circle $\mathbb{T}$, then this will have an influence on the form of the shifted ORK and WRK functions. For example in Figure 6, $\alpha_1 = 0.9$ while all other $\alpha_k = 0$. This gives a pole near $z = 1$, i.e., near $\theta = 0$. This pole forces the peak near $\theta = 0$ to be more pronounced, but it also implies oscillations of the ORK and WRK functions in the neighborhood of that pole (here near $\theta = 0$). These oscillations disappear for $w = 1$ in $k_n(z, w)$ and $l_n(z, w)$, but they do show when w moves away from the pole. This is almost not visible in Figure 6. However this oscillation effect increases when the pole is given more weight by making it a multiple pole. This implies a visual distortion of the simple shift-like property as in Figure 7 where we took all $\alpha_k = 0.9$, $k \geq 1$. The peak at $\theta = 0$ is now more pronounced than in Figure 6, and the shifted versions are heavily perturbed and do not look much like being shifted now.

Figure 5: Circular case. The real and imaginary part of the functions $k_n(z,w) = \sum_{k=0}^{n}(z/w)^k$ (top) and $l_n(z,w) = \sum_{k=n+1}^{2n}(z/w)^k$ (bottom) for $n = 16$. All $\alpha_k = 0$. In Figure A, we took $w = 1$, and in Figure B, we took $w = \exp(i4\pi/17)$.

A: $k_{16}(z,1)$ and $l_{16}(z,1)$ B: A "shifted" version of A

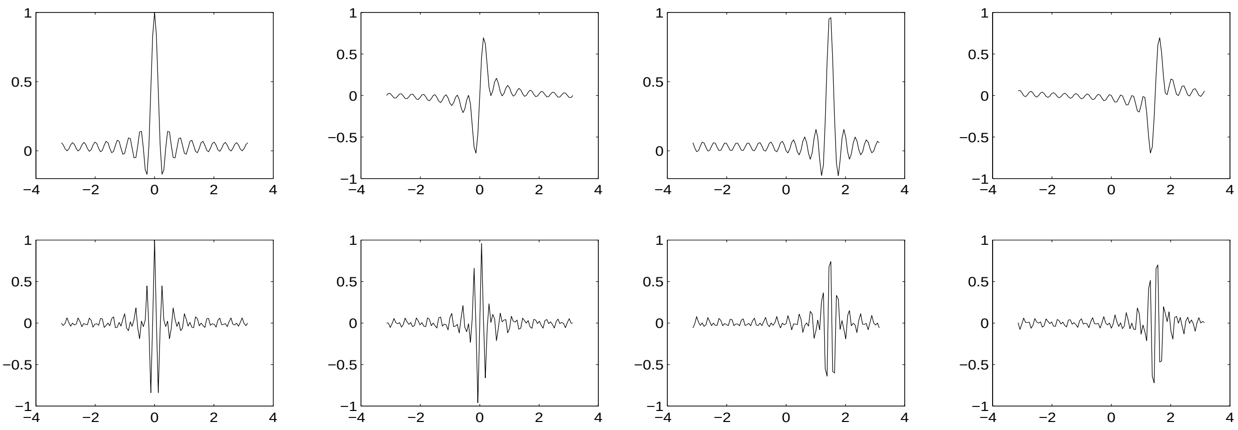


Figure 6: Circular case. The real and imaginary part of the functions $k_n(z,w)$ (top) and $l_n(z,w)$ (bottom) for $n = 16$. Here $\alpha_1 = 0.9$ while all other $\alpha_k = 0$. In Figure A, we took $w = 1$, and in Figure B, we took one of its shifts: $w = e^{i4\pi/17}$.

A: $k_{16}(z,1)$ and $l_{16}(z,1)$ B: A "shifted" version of A

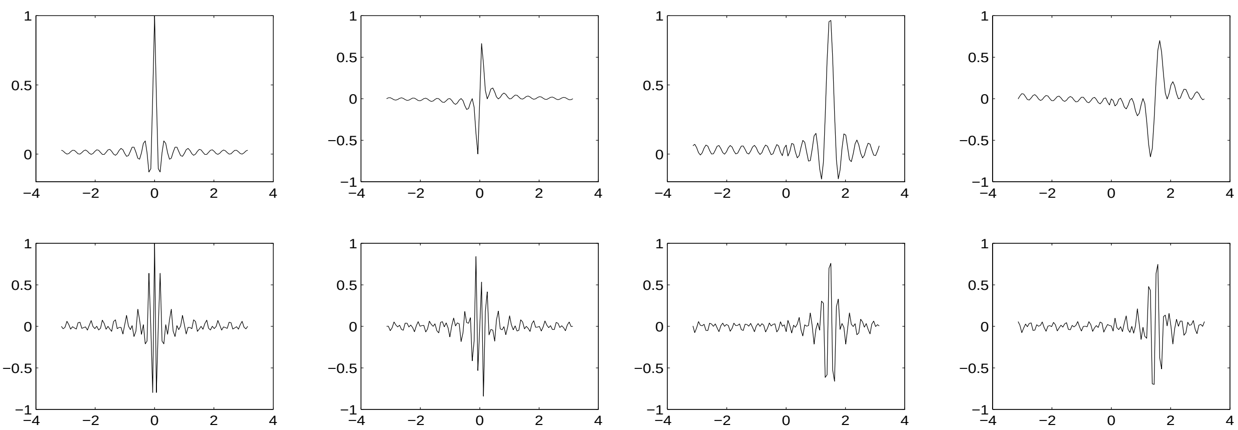
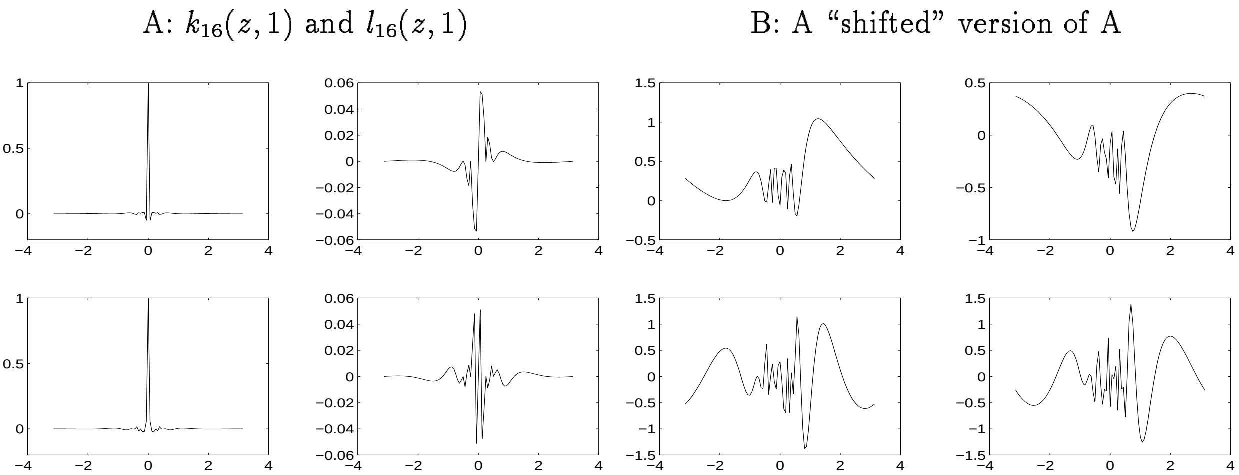


Figure 7: Circular case. The real and imaginary part of the functions $k_n(z,w)$ (top) and $l_n(z,w)$ (bottom) for $n = 16$. Here $\alpha_k = 0.9$ for $k \geq 1$. In Figure A, we took $w = 1$, and in Figure B, we took one of its shifts: $w = e^{i4\pi/17}$.



11 Symmetry

11.1 The circle case

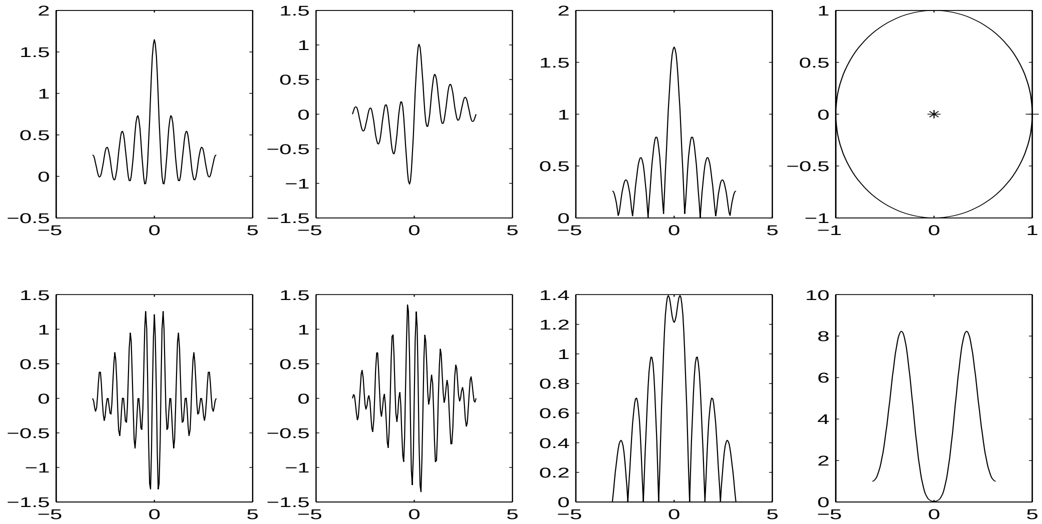
We have observed in several of the examples that the real part of the scaling and wavelet functions were symmetric while the imaginary parts showed an antisymmetric property. We can explain this as follows. Suppose that the measure μ is real and symmetric in the sense that $\mu(S) = \mu(\overline{S})$ where $S = \{e^{i\theta} : 0 \leq \theta_0 \leq \theta \leq \theta_1 \leq \pi\}$ is an arc of the upper half of the unit circle and $\overline{S}$ is the corresponding arc on the lower half circle: $\overline{S} = \{z : \overline{z} \in S\}$. Moreover assume that the poles $1/\overline{\alpha}_k$ are chosen symmetrically, that is the poles used in $\mathcal{L}_n$ are either real or they appear in complex conjugate pairs. In that case the symmetry that was observed will take place.

Theorem 11.1 *Under the above conditions about symmetry of the measure and of the poles, the kernels $k_{2^s}(z,1)$ and therefore also the kernels $l_{2^s}(z,1) = k_{2^{s+1}}(z,1) - k_{2^s}(z,1)$ have a symmetric real part and an antisymmetric imaginary part.*

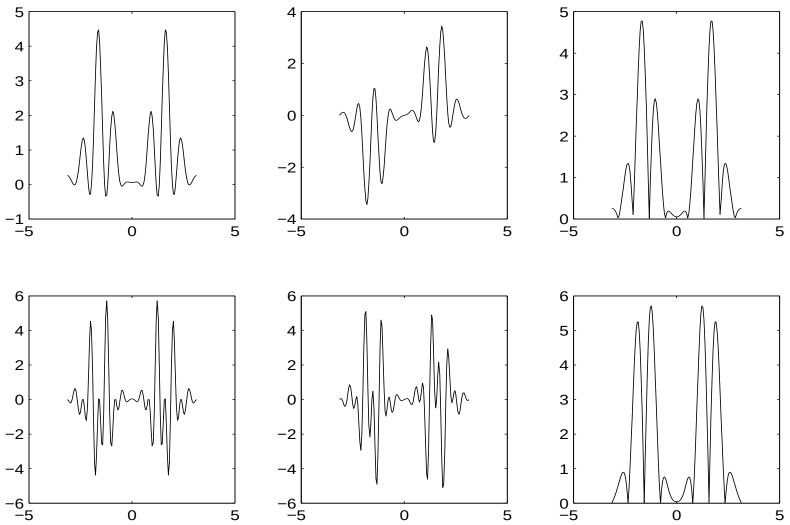
Proof. Suppose $n = 2^s$. We first observe that $k_n(z,w)$ is given by $\sum_{k=0}^n f_k(z)\overline{f_k(w)}$ where $\{f_k\}_{k=0}^n$ is any basis of orthogonal functions such that $\mathcal{L}_n = \operatorname{span}\{f_k : k = 0,\ldots,n\}$. Now consider the measure $\mathrm{d}\mu_n(t) = \mathrm{d}\mu(t)/|\pi_n(t)|^2$ where $t = e^{i\theta}$ and $\pi_n(t) = \prod_{k=1}^n (1 - \overline{\alpha}_k t)$. Define the polynomials q_k, $k = 0, 1, \ldots, n$ by orthogonalizing the functions $\{1, z, \ldots, z^n\}$ with respect to μ_n. Because μ_n is real and symmetric on $\mathbb{T}$, the coefficients of the p_k will be real. Hence $p_k(\overline{z}) = \overline{p_k(z)}$. In particular $p_k(1) \in \mathbb{R}$. The reproducing kernel $\tilde{k}_n(z,1) = \sum_{k=0}^n p_k(z)\overline{p_k(1)}$ for the polynomial space Π_n with respect to μ_n will thus satisfy $\tilde{k}_n(\overline{z},1) = \overline{\tilde{k}_n(z,1)}$. Furthermore, because of the symmetry of the α_k, it holds that $\pi_n(\overline{z}) = \overline{\pi_n(z)}$. The theorem now follows by observing that the kernel $\tilde{k}_n(z,1)$ can be transformed into a

Figure 8: Circular case. The real and imaginary part and the modulus of the functions $k_8(z,1)$ (top) and $l_8(z,1)$ (bottom) where all $\alpha_k = 0$ and $\rho_1 = \rho_2 = -0.4$ and all other $\rho_k = 0$. This corresponds to a weight which is plotted in the lower right corner. In Figure B, we plotted the same functions (real and imaginary parts and modulus) multiplied by the weight.

A: $k_8(z,1)$ and $l_8(z,1)$



B: Same functions as in A, multiplied by the weight



reproducing kernel for $\mathcal{L}_n$ with respect to μ by setting $k_n(z,1) = \tilde{k}_n(t,1)/[\pi_n(t)\overline{\pi_n(1)}]$ and obviously $k_n(\overline{z},1) = \overline{k_n(z,1)}$. □

11.2 The case of the real line

Here we can proceed in a similar manner. Assume again that the measure is symmetric with respect to $x = 0$ and let the poles defining $\mathcal{L}_n$ be chosen symmetric with respect to $x = 0$, thus they appear in pairs $(1/\alpha_k, -1/\alpha_k)$.

Theorem 11.2 *Under the above conditions about symmetry of the measure and of the poles, the kernels $k_{2^s}(z,0)$ and therefore also the kernels $l_{2^s}(z,0) = k_{2^{s+1}}(z,0) - k_{2^s}(z,0)$ are symmetric w.r.t. $x = 0$.*

Proof. As in the case of the circle, we can write the reproducing kernel $k_n(z,0)$ as $\tilde{k}_n(z,0)$ where $\tilde{k}_n(z,0)$ is the reproducing kernel for the space of polynomials Π_n with respect to the measure $\mathrm{d}\mathring{\mu}_n(x) = \mathrm{d}\mathring{\mu}(x)/|\pi_n(x)|^2$. Because this measure is symmetric, it follows that the orthogonal polynomials for this space are even for an even index and odd for an odd index. In particular for odd k, we have $p_k(0) = 0$. Hence $\tilde{k}_n(x,0) = \sum_{k=0}^{n} p_k(x)p_k(0) = \sum_{k \text{ even}} p_k(x)p_k(0)$ is an even function. Therefore $k_n(x,0)$ is even and hence also $l_n(x,0)$ as a difference of even functions. □

12 Computation

12.1 The circle case

To compute the kernels, one can make use of the recurrence relation that exists for the orthonormal functions. We have [4, 5]

Theorem 12.1 *The orthonormal functions ϕ_k are given by $\phi_0 = \phi_0^* = 1$ and for $n \geq 1$. they satisfy the recurrence relation*

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = e_n \frac{1 - \overline{\alpha}_{n-1} z}{1 - \overline{\alpha}_n z} \begin{bmatrix} 1 & \overline{\rho}_n \\ \rho_n & 1 \end{bmatrix} \begin{bmatrix} \tilde{\zeta}_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix}$$

with

$$\tilde{\zeta}_{n-1}(z) = \frac{z_n}{z_{n-1}} \zeta_{n-1}(z) = z_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha}_{n-1} z}, \quad z_n = -\frac{\overline{\alpha}_n}{|\alpha_n|}, \text{ for } \alpha_n \neq 0, \quad z_n = 1, \text{ for } \alpha_n = 0.$$

The number e_n is positive and satisfies

$$e_n^2 = \frac{1 - |\alpha_n|^2}{1 - |\alpha_{n-1}|^2} \frac{1}{1 - |\rho_n|^2}.$$

The reflection coefficients ρ_n are in $\mathbb{D}$.

Conversely, for arbitrary $\rho_n \in \mathbb{D}$, the functions generated by the previous recurrence relation will be orthogonal with respect to some probability measure on the unit circle.

We note that there also exists a recurrence relation of a similar form for the reproducing kernels $k_n(z,w)$. However, this will degenerate as soon as $w \in \mathbb{T}$, so that it can not be applied for practical computations in the present context.

Furthermore, defining $K_n(z) = k_n(z,0)/\sqrt{k_n(0,0)}$, then we have [9]

Theorem 12.2 *With* $K_n(t) = k_n(t,0)/\sqrt{k_n(0,0)}$ *we have*

$$\langle f,g\rangle = \int_{\mathbb{T}} f(t)\overline{g(t)}\mathrm{d}\mu(t) = \int_{\mathbb{T}} f(t)\overline{g(t)}\mathrm{d}\mu_n(t), \quad \forall f,g \in \mathcal{L}_n$$

where

$$\mathrm{d}\mu_n(t) = \frac{\mathrm{d}\theta}{2\pi|K_n(t)|^2}, \quad t = e^{i\theta}$$

Moreover, if $\mathrm{d}\mu(t) = \frac{w(t)}{2\pi}\mathrm{d}\theta$, $t = e^{i\theta}$ and $\log w$ is integrable over $\mathbb{T}$ with respect to the Lebesgue measure, then $|K_n(t)|^{-2}$ converges to $w(t)$ [3]. Thus in our examples for the circular case, one can plot $|K_n(t)|^{-2}$ for $t \in \mathbb{T}$ and n sufficiently large and this will give an idea about the underlying weight. This is used in Figure 8. We took all $\alpha_k = 0$ and chose $\rho_1 = \rho_2 = -0.4$ and all other $\rho_k = 0$. We have plotted $k_8(t,1)$ and $l_8(t,1)$ and their modulus. The rightmost pictures show the unit circle with the position of the $\alpha_k = 0$ and below it, we have plotted the approximation for the weight. In the figure B below, we plotted $k_8(t,1)$ and $l_8(t,1)$ multiplied by the weight (pictures at the bottom).

If one wants to start from a given positive weight and compute the corresponding kernels from it, thus compute the reflection coefficients, then the procedure is as follows. First one should compute the positive real part of that weight, i.e., the Carathéodory function

$$\Omega(z) = \int \frac{t+z}{t-z}\mathrm{d}\mu(t), \quad t \in \mathbb{T},$$

and then it is in principle possible to compute by a Nevanlinna-Pick algorithm [4] all the reflection coefficients ρ_n. Computing these numerically is not as simple as it seems. Suppose that the α_k are mutually different from each other. The inner product in $\mathcal{L}_n$, and therefore also the reflection coefficients ρ_k, $k = 1,\ldots,n$ are completely defined by the values of $\Omega(\alpha_k)$, $k = 0,\ldots,n$, or equivalently by the numbers $\Gamma_0(\alpha_k)$, $k = 0,\ldots,n$ with $\Gamma_0 = (\Omega-1)/(\Omega+1)$. So one should start with these numbers and transform them by the successive steps of the Nevanlinna-Pick algorithm. To compute $k_n(z,w)$, one should perform a general step as follows: Compute $\gamma_k = \Gamma_{k-1}(\alpha_k)$ and set $\Gamma_k'(z) = (\Gamma_{k-1}(z) - \gamma_k)/(1-\overline{\gamma}_k z)$. Then "deflate" α_k by setting $\Gamma_k''(z) = \Gamma_k'(z)/\zeta_k(z)$ and finally compute $\rho_k = \Gamma_k''(w)$ and set $\Gamma_k(z) = (\Gamma_k''(z) - \rho_k)/(1-\overline{\rho}_k z)$. The deflation step discontinues the successive transformations of the data $\Gamma_0(\alpha_k)$, so that the number of data to be transformed decreases by one in every step, resulting in a triangular table of numbers. Such an implementation can be found in [10]. If however some of the α_k are repeated, then the algorithm becomes intractable, since one has to give values of the derivatives of Ω in α_k and if $\Omega(z)$ is only known numerically, then this results in a much more complicated algorithm. The deflation step would indeed require the evaluation of a division $[f(z) - f(\alpha)]/[z-\alpha]$ for $z = \alpha$ which is only possible if one can evaluate the derivative.

12.2 The case of the real line

For reasons similar to the ones given in the circle case, it will be difficult to write for example a general matlab code to generate the coefficients A_n, B_n and C_n given the measure and the points α_k. However, given the coefficients A_1, B_n and C_n and the points α_n, it is relatively simple to generate the ORF basis and with these, the kernels and hence also the scaling functions and the wavelets can be generated.

Like in the circular case, one can define the functions $K_n(x) = k_n(x,i)/\sqrt{k_n(i,i)}$ with $k_n(z,w)$ the reproducing kernels and i the imaginary unit and it holds that

Theorem 12.3 *With $K_n(x) = k_n(x,i)/\sqrt{k_n(i,i)}$ we have*

$$\langle f,g\rangle = \int_{\hat{\mathbb{R}}} f(x)\overline{g(x)}\mathrm{d}\mathring{\mu}(x) = \int_{\hat{\mathbb{R}}} f(x)\overline{g(x)}\mathrm{d}\mathring{\mu}_n(t), \quad \forall f,g \in \mathcal{L}_n$$

where

$$\mathrm{d}\mathring{\mu}_n(x) = \frac{\mathrm{d}\mathring{\lambda}(x)}{|K_n(x)|^2} = \frac{\mathrm{d}x}{\pi|K_n(x)|^2(1+x^2)}.$$

Moreover, if all the generalized moments $m_n = \int x^n/\pi_n(x)d\mathring{\mu}(x)$ with $\pi_n(x) = \prod_{k=1}^n(1-\alpha_k x)$ are finite for all n and if $\log\mu'$ is integrable with respect to the normalized Lebesgue measure $\mathrm{d}\mathring{\lambda}(x) = [\pi(1+x^2)]^{-1}\mathrm{d}x$, then $1/K_n(z)$ will converge to the spectral factor $\sigma(z)$ of the measure μ uniformly in compact subsets of the upper half plane, but in general a reasonable convergence then also holds for $z \in \mathbb{R}$. This is for example the case when we consider the case of the Hermite polynomials. However there are many important examples where the measure is only supported on a half line or on a finite interval. In that case, $\log\mu'$ will not be integrable and simple convergence will not hold in the support of the measure. Fortunately, if one replaces $K_n(x)$ by the $(C,1)$ Cesàro sums

$$S_n(x) = \frac{1}{n+1}\sum_{j=0}^n K_j(x)$$

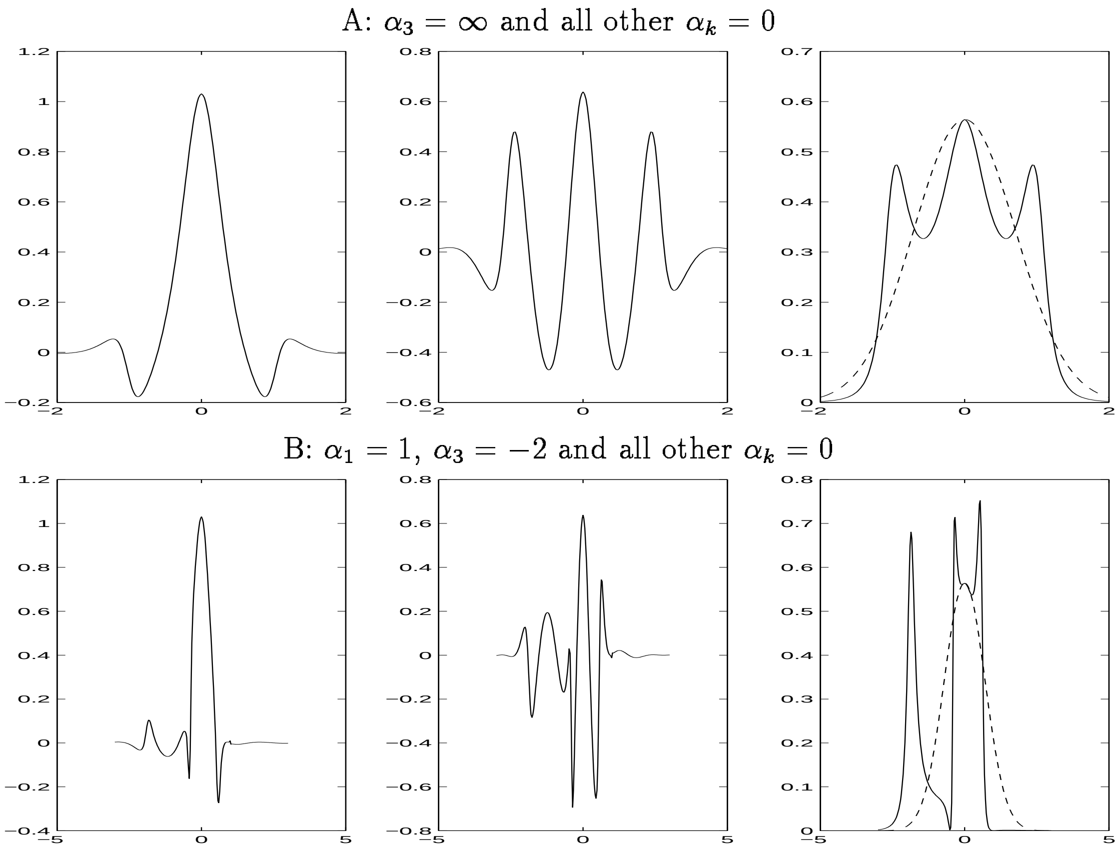
then we do have convergence and we can get a reasonable picture of the weight function $w(x)$ in $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$ by plotting

$$\frac{1}{\pi(1+x^2)}\frac{1}{|S_n(x)|^2}$$

for n sufficiently large.

This strategy is used for example to see what the influence is when poles are introduced. It is illustrated in Figure 9. We took the recurrence relation for the Hermite polynomials and introduced poles. In figure A, the only pole was $1/\alpha_1 = 0$ and in figure B we chose two poles $1/\alpha_1 = 1$ and $1/\alpha_3 = -0.5$. The corresponding weight is plotted together with the scaling functions $k_{16}(x,0)$ and $l_{16}(x,0)$ both multiplied by the weight function. For comparison, the original Hermite weight is plotted in dashed lines.

Figure 9: Case of the real line. The functions $k_{16}(z,0)$ and $l_{16}(z,0)$ multiplied by the weight function and the weight function itself. They are generated by the recurrence relation of the Hermite polynomials, but now with $\alpha_3 = \infty$ and all other $\alpha_k = 0$ (figure A) or $\alpha_1 = 1$, $\alpha_3 = -2$ and all other $\alpha_k = 0$ (figure B). The dashed line shows the Hermite weight function.



Appendix A: Proofs of some auxiliary results

In this appendix we prove some of the results for the case of the real line that were used in the text. First we prove

Lemma 12.4 *Consider the orthogonal rational function on the real line* $\phi_{n+1} = p_{n+1}/\pi_{n+1}$ *with* $\pi_{n+1}(z) = \prod_{k=1}^{n+1}(1-\alpha_k z)$. *Then* ϕ_{n+1} *will be singular, i.e.* $p_{n+1}(1/\alpha_n) = 0$ *iff* $A_{n+1} = 0$.

Proof. First assume $A_{n+1} = 0$. Define $f_k(z) = \phi_k(z)/\zeta_k(z)$. Then it follows from the Christoffel-Darboux relation that

$$\frac{zw}{(z-w)}[f_{n+1}(w)f_n(z) - f_{n+1}(z)f_n(w)] = A_{n+1}k_n(z,w) = 0.$$

Therefore $f_{n+1}(w)/f_n(w) = f_{n+1}(z)/f_n(z)$ for all z and w, so that $f_{n+1}(w)/f_n(w)$ is a constant. Thus

$$\tau(w) = \frac{f_{n+1}(w)}{f_n(w)} = \frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)} = \frac{(1-\alpha_{n+1}w)\phi_{n+1}(w)}{(1-\alpha_n w)\phi_n(w)} = \frac{p_{n+1}(w)}{(1-\alpha_n w)p_n(w)}$$

is a constant. Taking $w = 1/\alpha_n$, it follows $p_{n+1}(1/\alpha_n) = 0$. Thus $A_{n+1} = 0$ implies that ϕ_{n+1} is singular.

Conversely, assume ϕ_{n+1} is singular. It follows from the recurrence relation that

$$p_{n+1}(z) = [A_{n+1}z + B_{n+1}(1-\alpha_n z)]p_n(z) + C_{n+1}(1-\alpha_n z)(1-\alpha_{n-1}z)p_{n-2}(z).$$

Substituting $z = 1/\alpha_n$ and using $p_{n+1}(1/\alpha_n) = 0$, this implies

$$A_{n+1}/\alpha_n \cdot p_n(1/\alpha_n) = 0.$$

Because $p_n(1/\alpha_n)$ can not be zero, we must have $A_{n+1} = 0$. □

Lemma 12.5 *Let* ϕ_k *be the orthogonal rational functions on the real line. Define* $f_n(z) = \phi_n(z)/\zeta_n(z)$ *and* $\tau = f_{n+1}/f_n$. *If* $A_{n+1} \neq 0$ *then the derivative* τ' *does not change sign on the real line.*

Proof. First note that $f_n(x)$ is real for $x \in \mathbb{R} \setminus \mathfrak{P}_{n+1}$, where $\mathfrak{P}_{n+1} = \{1/\alpha_1, \ldots, 1/\alpha_{n+1}\}$. So when in the Christoffel-Darboux formula we take $w \in \mathbb{R} \setminus \mathfrak{P}_{n+1}$ we get

$$\frac{zw}{A_{n+1}(z-w)}[f_{n+1}(w)f_n(z) - f_{n+1}(z)f_n(w)] = k_n(z,w).$$

Note that $A_{n+1} \neq 0$ means that ϕ_{n+1} is regular. Now let $z \to w$, then

$$\frac{w^2}{A_{n+1}}[f_{n+1}(w)f_n'(w) - f_{n+1}'(w)f_n(w)] = k_n(w,w) > 0.$$

On the other hand

$$\tau' = \left(\frac{f_{n+1}}{f_n}\right)' = \frac{f_n f_{n+1}' - f_n' f_{n+1}}{f_n^2}.$$

Thus it follows that depending on the sign of A_{n+1}, $\tau'(w)$ is positive or negative for all real values of w such that $f_n(w) \neq 0$. □

Lemma 12.6 *Let ϕ_k be the orthogonal functions on the real line. Define the function h_k by $h_k(z) = (1 - \alpha_k z)\phi_k(z)$. If ϕ_{n+1} is regular, then the functions h_{n+1} and h_n can have no common zeros.*

Proof. Like in the proof of the previous lemma, we get from the confluent form of the Christoffel-Darboux formula that

$$\frac{1}{A_{n+1}}[h'_{n+1}(w)h_n(w) - h'_n(w)h_{n+1}(w)] = k_n(w,w) > 0.$$

Therefore a common zero of h_n and h_{n+1} is impossible. □

Lemma 12.7 *Let $\phi_n = p_n/\pi_n$ with $\pi_n(z) = \prod_{k=1}^{n}(1 - \alpha_k z)$ be the orthogonal rational functions on the real line. Define*

$$\tau(w) = \frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)} = \frac{(1-\alpha_{n+1}w)\phi_{n+1}(w)}{(1-\alpha_n w)\phi_n(w)} = \frac{p_{n+1}(w)}{(1-\alpha_n w)p_n(w)}.$$

Then if ϕ_{n+1} is a regular, $\tau(w)$ will run $n+1$ times through all values of $\mathbb{R}$ as w runs over all values in $\mathbb{R}$.
Hence, for a given value $\tilde{\tau} \in \mathbb{R}$, there are $n+1$ values of $w_k \in \mathbb{R}$, $k = 0, \ldots, n$ such that $\tilde{\tau} = \tau(w_k)$.

Proof. Because the numerator and the denominator of τ will have no zeros in common by the previous lemma, and because $p_n(z)$ has at least n simple zeros in $\mathbb{R}$, and because $\tau'(w) \neq 0$ wherever the derivative exists, the conclusion of the lemma follows. □

Appendix B: Matlab code

We include the matlab code for the routines that compute the scaling functions $K_n(z,w) = k_n(z,w)/\sqrt{k_n(w,w)}$ and the wavelet function $L_n(z,w) = l_n(z,w)/\sqrt{l_n(w,w)}$ in the case of the circle and in the case of the real line.

In the case of the circle, the weight $w(t)$ in $\mathrm{d}\mu(t) = w(t)\frac{\mathrm{d}\theta}{2\pi}$, $t = e^{i\theta}$, can be approximated by $1/|K_n(t,0)|^2$, it can be computed by the same routine.

In the case of the real line, assuming that

$$\mathrm{d}\mathring{\mu}(x) = \mathring{w}(x)\mathrm{d}\mathring{\lambda}(x) = \mathring{w}(x)\frac{\mathrm{d}x}{\pi(1+x^2)} = w(x)\mathrm{d}x,$$

then the weight $\mathring{w}(x)$ can be approximated by $1/|K_n(x,i)|^2$, but because this gives problems when μ' is not in $L^1(\hat{\mathbb{R}})$, we have replaced $1/|K_n(x,i)|^2$ by Cesàro means to give the approximants $\mathring{w}_n(x)$ for $\mathring{w}(x)$. Obviously, the weight $w(x)$ can then be approximated by $\mathring{w}_n(x)/[\pi(1+x^2)]$.

Code for the circle:

```
function [kns,knw]=k2n(z,w,n)

% function [kns,knw]=k2n(z,w,n)
% computes kernel k_n(z,w) and wavelet k_{2n}(z,w)-k_n(z,w)
% in the case of the circle.
% If w==0 then compute the weight as 1/|K_n(z,0)|^2 (normalized kernel)
% z may be a vector, w is scalar
%
% Uses recursion
%  |fie_k(z)|         |1  rho(k)'| |Z_{k-1}(z) 0| |fie_{k-1}(z) |
%  |         |= E_k(z)|          | |            | |             |
%  |fi*_k(z)|         |rho(k)   1| |0          1| |fi*_{k-1}(z) |
%
% E_k(z)=e(k)*(1-z*alpha(k-1))/(1-z*alpha(k)')
% e(k)= (1-|alpha(k)|^2)/(1-|alpha(k-1)|^2)/(1-|rho(k)|^2)
% Z_n(z)=bz(n,z)=sz(n+1)*(z-alpha(n))/(1-z*alpha(n)')
% sz(n)=1 if alpha(n)==0; sz(n)=-alpha(n)'/|alpha(n)| otherwise

nz=size(z);
compute_weight= (w==0); % if w==0 then computation of the weight

knzw=ones(nz);knww=1;
fienz=ones(nz); fienzs=ones(nz); fienw=1; fienws=1;
if(compute_weight) cknz0=ones(nz); end;

for k=1:2*n
  hz0=fienz.*bz(k-1,z);  hw0=fienw.*bz(k-1,w);
  hz1=hz0+rho(k)'*fienzs; hw1=hw0+rho(k)'*fienws;
  hz2=hz0*rho(k) +fienzs; hw2=hw0*rho(k) +fienws;
```

```
  facz=e(k)*(ones(nz)-alpha(k-1)'*z)./(ones(nz)-alpha(k)'*z);
  facw=e(k)*(1       -alpha(k-1)'*w) /(1       -alpha(k)'*w);
  fienz=hz1.*facz; fienzs=hz2.*facz; fienw=hw1*facw; fienws=hw2*facw;
%
  knzw=knzw+fienz.*fienw';
  knww=knww+fienw *fienw';
  if(compute_weight) cknz0=cknz0+ones(nz)*knww./(abs(knzw).^2); end;
%
  if( k==n & ~compute_weight)
    kns=knzw/sqrt(knww);     % scaling fct
    knzw=zeros(nz); knww=0;  % reinitialize for wavelet
  end;

end

if (compute_weight)           % compute weight by
  kns=cknz0/(2*n+1);          % Cesaro mean
% kns=ones(nz)*knww./(abs(knzw).^2); %no Cesaro mean
  knw=0;    % not used
else % compute the wavelet
  knw=knzw/sqrt(knww);
end
```

Code for the real line:

```
function [kns,knw]=rk2n(x,w,n)

% function [kns,knw]=rk2n(x,w,n)
% computes kernel kns=k_n(x,w) and wavelet knw=k_{2n}(x,w)-k_n(x,w)
% x can be a vector; w is scalar
% If w==i then compute the weight by Cesaro mean of normalized 1/|K_n(x,i)|^2
%
% Uses recursion for ORFs
% fie_{k+1}(x) = [Ak Z_k(x) + Bk Z_k(x)/Z_{k-1}(x)] fie_k(x)
%                            + Ck Z_k(x)/Z_{k-2}(x)  fie_{k-1}(x)
%
% Z_k(x)=x/(1-x*ralpha(k))
% rb(k)=Bk, rc(k)=Ck, ra(1)=A1, Ak=-Ck*A{k-1}

nx=size(x);
compute_weight= (w==i);          % if w==i then compute weight

% fie_0(x), fie_0(w), k_0(x,w), k_0(w,w)

fienx=ones(nx); fienw=1;
knxw =ones(nx); knww =1;
if(compute_weight) cknxi=ones(nx); end;
```

```
% fie_1(x), fie_1(w), k_1(x,w), k_1(w,w)

Ak=ra(1); Bk=rb(1);
fienx1=(Ak*x./(ones(nx)-x*ralpha(1))+Bk*ones(nx)).*fienx;
fienw1=(Ak*w /(1       -w*ralpha(1))+Bk         ) *fienw;
knxw=knxw+fienx1*fienw1';
knww=knww+fienw1*fienw1';
if(compute_weight) cknxi=cknxi+ones(nx)*knww./(abs(knxw).^2); end;

for k=2:2*n
  Bk=rb(k); Ck=rc(k); Ak=-Ck*Ak;
  fienx2=(Ak*x+Bk*(ones(nx)-x*ralpha(k-1)))./(ones(nx)-x*ralpha(k)).*fienx1...
        +(Ck*(ones(nx)-x*ralpha(k-2))./(ones(nx)-x*ralpha(k))     ).*fienx;
  fienw2=(Ak*w+Bk*(1       -w*ralpha(k-1))) /(1       -w*ralpha(k)) *fienw1...
        +(Ck*(1       -w*ralpha(k-2)) /(1       -w*ralpha(k))     ) *fienw;
  fienx=fienx1;fienw=fienw1;
  fienx1=fienx2;fienw1=fienw2;
%
  knxw=knxw+fienx2*fienw2';
  knww=knww+fienw2*fienw2';
  if(compute_weight) cknxi=cknxi+ones(nx)*knww./(abs(knxw).^2); end;
%
  if( k==n & ~compute_weight)
    kns=knxw/sqrt(knww);            % the scaling function
    knxw=zeros(nx); knww=0;         % reinitialize for wavelet
  end;

end

if(compute_weight)              % Compute weight by
  kns=cknxi/(2*n+1);            % Cesaro mean
% kns=ones(nx)*knww./(ans(knxw).^2); % no Cesaro mean
  kns=kns./(1+x.^2)/pi;
  knw=zeros(nx); % not used
else
  knw=knxw/sqrt(knww);          % the wavelet function
end;
```

Appendix C: The recurrence in the circular case

This appendix serves to give a note of warning about the recurrence relation in the circular case.

Assume that the orthonormal functions ϕ_n are normalized such that their expansion in the basis B_k is given by

$$\phi_n(z) = a_{n0} + a_{n1}B_1(z) + \cdots + a_{nn}B_n(z).$$

We call $a_{nn} = \kappa_n$ the leading coefficient. The function ϕ_n can be made unique by imposing the condition $\kappa_n > 0$. With this normalization, we shall denote the functions with a hat. These $\hat{\phi}_n$ satisfy the recursion

$$\begin{bmatrix} \hat{\phi}_n(z) \\ \hat{\phi}_n^*(z) \end{bmatrix} = e_n \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z} \begin{bmatrix} \eta_{n1} & 0 \\ 0 & \eta_{n2} \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_{n-1}(z) \\ \hat{\phi}_{n-1}^*(z) \end{bmatrix}$$

The numbers λ_n are in $\mathbb{D}$ and η_{n1} and η_{n2} are on $\mathbb{T}$ such that the appropriate normalization is obtained and e_n is as defined in Theorem 12.1.

To get rid of these η_{n1} and η_{n2} (which depend on $\hat{\phi}_n$ itself), we can consider the "rotated" functions $\phi_n = \epsilon_n \hat{\phi}_n$ with $\epsilon_n \in \mathbb{T}$. If $\epsilon_0 = 1$ and $\epsilon_n = \epsilon_{n-1}\eta_{n2}$ for $n \geq 1$, then these ϕ_n will satisfy a recurrence relation as given in Theorem 12.1 with the relation $\rho_n = \overline{\epsilon}_{n-1}^2 \overline{z}_n z_{n-1} \lambda_n$ and where as before $z_n = 1$ is $\alpha_n = 0$ and otherwise $z_n = -\overline{\alpha}_n/|\alpha_n|$. (See [9].) Thus ρ_n is a rotated version of λ_n.

There is however a price that we pay for the simpler form of the recurrence in Theorem 12.1. If the orthogonality measure is the normalized Lebesgue measure on $\mathbb{T}$, then all $\lambda_n = 0$ and hence also all $\rho_n = 0$ no matter what α_k we choose. But if the λ_n are significantly different from zero, then we loose continuity in the neighborhood of $\alpha_n = 0$. Indeed, if $\alpha_n = \alpha_{n-1} = 0$, then $\rho_n = \overline{\epsilon}_n \lambda_n$ because $z_n = z_{n-1} = 1$. For small α_n and α_{n-1}, z_n and z_{n-1} will remain to be 1 als long as these α_n and α_{n-1} are real. However, for other small values of α_n and α_{n-1} which are not on $\mathbb{R}$, the z_n and z_{n-1} become important and they will define a quite different (rotated) ρ_n. The consequence is that the plots for small, non real α_k's may be quite different from the plots for $\alpha_n = 0$.

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