An asymptotic estimate of Hilb's type for generalized Jacobi polynomials on the unit circle

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Abstract

We consider generalized Jacobi weight functions w on the unit circle $\mathbb{T}$, i.e. w is positive and smooth everywhere except at a finite number of points $z_k \in \mathbb{T}$, such that for small t, $w(z_k e^{it})$ is equal to $|t|^{\beta_k}$ ($\beta_k > -1$) multiplied by some positive smooth function. An asymptotic estimate of Hilb's type is established for the orthogonal polynomials w.r.t. these weight functions. It gives a precise description of the asymptotic behavior of the orthogonal polynomials on the whole unit circle.

Keywords : orthogonal polynomials, asymptotics, generalized Jacobi weights, Bessel function
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1 The result

Consider the following function on the unit disk

$$\pi(z) = \psi(z) \prod_k (1 - \bar{z}_k z)^{\alpha_k}, \tag{1.1}$$

where

- the z_k represent a finite number of points on the unit circle,
- $\psi(z)$ is analytic on the open unit disk and all its derivatives extend to a continuous function on the closed unit disk,
- $\psi(z)$ has no zeros in the closed unit disk and $\psi(0) = 1$,
- $\alpha_k < \frac{1}{2}$.

We are interested in the asymptotic behavior of the orthogonal polynomials for the measure

$$d\mu(z) = \frac{1}{|\pi(z)|^2} d\lambda(z) \tag{1.2}$$

on the unit circle, where $d\lambda(e^{it}) = \frac{1}{2\pi} dt$ represents the uniform probability measure on the unit circle. Hence $\pi(z)$ is the inverse of the Szegö function of the weight and from Szegö's theory it follows that the weight functions described in the abstract can be put in this form up to a normalizing constant.

Let $p_n(z)$ denote the polynomial of degree $\leq n$ such that $p_n(0) > 0$ and such that for each polynomial $q(z)$ of degree $\leq n$, we have

$$\int \overline{p_n(z)} q(z) d\mu(z) = \frac{q(0)}{p_n(0)}. \tag{1.3}$$

Then it is easily verified that its reciprocal

$$p_n^*(z) := z^n \overline{p_n(\frac{1}{\bar{z}})} \tag{1.4}$$

is an orthonormal polynomial of degree n for the measure μ on the unit circle. In this paper the reciprocals of the orthonormal polynomials play a more central role than the orthonormal polynomials themselves. Therefore this unconventional reversion of the notation of p_n and p_n^* was preferred.

Orthogonal polynomials w.r.t. generalized Jacobi weights have been studied extensively both on the unit circle and on a finite interval (generally under weaker smoothness assumptions than here). We have

$$\lim_{n\to\infty} p_n(z) = \pi(z) \tag{1.5}$$

uniformly for $|z| \leq r < 1$ [9] and uniformly on every closed arc on the unit circle that contains no singular point z_k (cf. e.g. [3]). The behavior near the singular points is more subtle. We have the following bounds holding on the whole unit circle [5, 2]

$$C_1 \prod_k \left(|z - z_k| + \frac{1}{n}\right)^{\alpha_k} \leq |p_n(z)| = |p_n^*(z)| \leq C_2 \prod_k \left(|z - z_k| + \frac{1}{n}\right)^{\alpha_k} \quad (z \in \mathbb{T}), \tag{1.6}$$

where C_1 and C_2 are independent of n and z. Other results include the asymptotics of the reflection coefficients, of the Christoffel function and of the mean convergence of Fourier series. Some references to these topics are [1, 6, 8].
In this paper it is shown that the asymptotic behavior of $p_n(z)$ on the unit circle near a singular point is governed by an entire function $P_\alpha(z)$ that can be expressed in terms of Bessel functions as follows

$$P_\alpha(2z) = 2^{\alpha-\frac{1}{2}}\sqrt{\pi}e^{iz}\left(j_{-\alpha-\frac{1}{2}}(z) - izj_{-\alpha+\frac{1}{2}}(z)\right), \tag{1.7}$$

where

$$j_p(z) = z^{-p}J_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+p+k)k!}\left(\frac{z}{2}\right)^{2k}. \tag{1.8}$$

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose Δ_k is a closed interval containing* 0 *as an interior point such that*

$$z = z_k e^{ix} \quad (x \in \Delta_k)$$

is the parametrization of an arc containing z_k but no other singular point. Write the function $\pi(z)$ on this arc as follows

$$\pi(z_k e^{ix}) = (-ix)^{\alpha_k}\chi_k(x) \quad (x \in \Delta_k).$$

Then we have, uniformly for $x \in \Delta_k$, that

$$p_n(z_k e^{ix}) = n^{-\alpha_k}P_{\alpha_k}(nx)\chi_k(x) + \mathcal{O}\left(\frac{\log(n)}{n}\left(\frac{1}{n}+|x|\right)^{\alpha_k}\right) \tag{1.9}$$

as $n \to \infty$.

The proof consists of 4 steps. First a sequence of auxiliary polynomials q_n is constructed. These polynomials converge to π, except near the singular points, where they satisfy an asymptotic estimate like (1.9) but with P_{α_k} replaced by some other function Q_{α_k}. Secondly, with q_n the Bernstein-Szegö integral equation on the unit circle is constructed whose unique solution is p_n. Then, similarly, an integral equation on the real line is constructed with Q_{α_k}. It is shown that this integral equation has a unique solution and that this solution satisfies an orthogonality condition w.r.t. the weight $\frac{dt}{|t|^{2\alpha_k}}$ on the real line. The solution of this integral equation is P_{α_k}. This orthogonality of entire functions is related to Krein's theory concerning the continuous analog of the orthogonal polynomials on the unit circle [7]. Finally, it is shown that under proper scaling the integral equations on the real line (one for each singular point z_k) form in some sense the limit of the integral equation on the unit circle. This permits then to establish the above theorem.
The smoothness assumptions on $\psi(z)$ are stronger than necessary for theorem 1.1 to hold (conditions on the derivatives of order higher than 2 are superfluous). These strong assumptions however permit to obtain in theorem 2.2 for the auxiliary polynomials q_n a complete asymptotic expansion that is termwise differentiable. This result suggests that maybe also for the polynomials p_n more precise asymptotic estimates can be obtained than in theorem 1.1.

2 Polynomial approximations of $\pi(z)$

2.1 A sequence of convolution kernels

We choose a smooth even function $\phi(t)$ with support in $[-1,1]$ such that

$$\phi(0) = \frac{1}{2\pi} \tag{2.1}$$

and

$$\phi^{(k)}(0) = 0 \qquad k = 1,2,3,\ldots. \tag{2.2}$$

Let Φ_n denote the following trigonometric polynomial of degree $< n$

$$\Phi_n(x) = \sum_{k=-n}^{n} \phi(\frac{k}{n})e^{ikx}. \tag{2.3}$$

For a 2π-periodic function f, the convolution of Φ_n with f is

$$(\Phi_n * f)(x) = \int_{-\pi}^{+\pi} f(t)\Phi_n(x-t)dt = \sum_{k=-n}^{n} \phi(\frac{k}{n})e^{ikx}\int_{-\pi}^{+\pi} f(t)e^{-ikt}dt.$$

Observe that if $f(x) = g(e^{ix})$, where $g(z)$ is analytic inside the open unit disk and continuous in its closure, then $(\Phi_n * f)(x) = g_n(e^{ix})$, where $g_n(z)$ is a polynomial of degree $< n$ and $g_n(0) = g(0)$.
In order to estimate $\Phi_n * f$ for large values of n we use the Poisson summation formula

$$\Phi_n(x) = n\sum_{m=-\infty}^{\infty} \hat{\phi}(n(x+2\pi m)) \tag{2.4}$$

where

$$\hat{\phi}(x) = \int_{-\infty}^{+\infty} \phi(t)e^{-ixt}dt, \tag{2.5}$$

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\phi}(x)e^{ixt}dx. \tag{2.6}$$

$\hat{\phi}(x)$, as well as its derivatives, decreases faster at infinity than any negative power of x (integrate by parts (2.5)) and it has vanishing moments

$$\int_{-\infty}^{+\infty} \hat{\phi}(x)dx = 1, \tag{2.7}$$

$$\int_{-\infty}^{+\infty} \hat{\phi}(x)x^k dx = 0 \quad k = 1,2,3,\ldots. \tag{2.8}$$

The convolution of Φ_n with f can now be expressed as

$$\begin{aligned}(\Phi_n * f)(x) &= \int_{-\pi}^{+\pi} \Phi_n(t)f(x+t)dt \\ &= \sum_{m=-\infty}^{+\infty} n\int_{-\pi}^{+\pi} \hat{\phi}(n(t+2\pi m))f(x+t)dt \\ &= \int_{-\infty}^{+\infty} n\hat{\phi}(nt)f(x+t)dt \\ &= \int_{-\infty}^{+\infty} \hat{\phi}(s)f(x+\frac{s}{n})ds. \end{aligned} \tag{2.9}$$

2.2 The polynomials q_n

Let a be a fixed positive number that will be specified later.

Definition 2.1 *q_n is defined as the polynomial of degree $< n$ such that $q_n(e^{ix})$ is the convolution of Φ_n with the function*

$$\pi(e^{-\frac{a}{n}}e^{ix}) = \pi(e^{i(x+i\frac{a}{n})}). \tag{2.10}$$

Note that $q_n(0) = 1$.
Fix a singular point z_k. Let Δ_k be a closed interval containing 0 as an interior point and such that

$$z = z_k e^{ix} \quad (x \in \Delta_k) \tag{2.11}$$

is the parametrization of an arc containing no singular point except z_k. Let

$$\sigma_\alpha(z) = (-iz)^\alpha \quad (\Im(z) \geq 0, z \neq 0) \tag{2.12}$$

and write

$$\pi(z_k e^{iz}) = \sigma_{\alpha_k}(z)\chi_k(z). \tag{2.13}$$

Here $\chi_k(z) = \chi_k(x+iy)$ is analytic in the upper half-plane $y > 0$ and all its derivatives have a limit as $y \to 0+$ when x is in Δ_k or close to Δ_k.

Theorem 2.2 *For $x \in \Delta_k$, we have*

$$q_n(z_k e^{ix}) = \sum_{l=0}^{p-1} \frac{Q_{\alpha_k,l}(nx)}{n^{\alpha_k+l}} \frac{\chi_k^{(l)}(x)}{l!} + R_{k,p,n}(x), \tag{2.14}$$

where

$$Q_{\alpha,l}(x) = \int_{-\infty}^{+\infty} \hat{\phi}(s)\sigma_\alpha(x+s+ia)(s+ia)^l ds \tag{2.15}$$

and for all $m \in \mathbb{N}$, there exists an $M_{k,p,m} > 0$ such that

$$\forall x \in \Delta_k : |R_{k,p,n}^{(m)}(x)| \leq \frac{M_{k,p,m}}{n^p}\left(\frac{1}{n} + |x|\right)^{\alpha_k - m}. \tag{2.16}$$

In the sequel we will often write shortly Q_α instead of $Q_{\alpha,0}$.
Let $F_\alpha(z)$ denote the entire function

$$F_\alpha(z) = \frac{2\pi}{\Gamma(-\alpha)}\int_0^1 e^{izt}t^{-\alpha-1}\phi(t)dt \quad (\alpha < 0) \tag{2.17}$$

$$= -\frac{2\pi}{\Gamma(1-\alpha)}\int_0^1 t^{-\alpha}e^{izt}(iz\phi(t)+\phi'(t))dt \quad (\alpha < 1). \tag{2.18}$$

The second expression is obtained by integrating by parts the first expression and represents its analytic continuation w.r.t. α.

Lemma 2.3 *We have*

$$Q_\alpha(x) = F_\alpha(x + ia). \tag{2.19}$$

For all $q > 0$, we have

$$F_\alpha(z) = (-iz)^\alpha + \mathcal{O}(|z|^{-q}) \quad \text{as } |z| \to \infty \tag{2.20}$$

uniformly for $\Im(z) \geq 0$, and

$$F_\alpha(z) = \mathcal{O}(|z|^\alpha e^{-\Im(z)}) \quad \text{as } |z| \to \infty \tag{2.21}$$

uniformly for $\Im(z) \leq 0$.

Corollary 2.4 *If a is chosen sufficiently large, then, for all sufficiently large n, all zeros of $q_n(z)$ lie in the exterior of the unit circle.*

3 Integral equations

3.1 The Bernstein-Szegö integral equation

We give here the well-known Bernstein-Szegö integral equation with its derivation that is written so as to stress the link with the integral equation on the real line of the next section. We assume that a is chosen sufficiently large and that n is sufficiently large so that, by corollary 2.4, $q_n(z)$ has no zeros in the closed unit disk. We consider the measure μ_n on the unit circle, with

$$d\mu_n(z) = \frac{d\lambda(z)}{|q_n(z)|^2}. \tag{3.1}$$

Then, for all polynomials $f(z)$, we have

$$\int \overline{q_n(z)} f(z) d\mu_n(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{q_n(z)} \frac{dz}{z} = \frac{f(0)}{q_n(0)} = f(0). \tag{3.2}$$

Hence, it is easily verified that

$$q_n^*(z) = z^n \overline{q_n\left(\frac{1}{\bar{z}}\right)} \tag{3.3}$$

is an orthonormal polynomial of degree n for the measure μ_n. Let

$$k_n(z, \zeta) = -z\bar{\zeta} \, \frac{q_n(z)\overline{q_n(\zeta)} - q_n^*(z)\overline{q_n^*(\zeta)}}{z\bar{\zeta} - 1}. \tag{3.4}$$

For fixed ζ, $k_n(z, \zeta)$ is a polynomial of degree $\leq n$ in z and

$$\overline{k_n(z, \zeta)} = k_n(\zeta, z). \tag{3.5}$$

For every polynomial f of degree $\leq n$, we have

$$\int k_n(z, \zeta) f(\zeta) d\mu_n(\zeta) = -\frac{z q_n(z)}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{q_n(\zeta)(z - \zeta)\zeta} d\zeta \tag{3.6}$$

$$+\frac{z q_n^*(z)}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{q_n^*(\zeta)(z - \zeta)\zeta} d\zeta. \tag{3.7}$$

If z happens to lie on the unit circle, the contour integrals on the right-hand side are in the sense of the Cauchy principal value. Applying the residue theorem for the first integral on the right-hand side with respect to the interior of the unit circle and for the second integral on the right-hand side with respect to the exterior of the unit circle, we have

$$\int k_n(z,\zeta)f(\zeta)d\mu_n(\zeta) = -f(0)q_n(z) + f(z). \tag{3.8}$$

In particular, we have

$$p_n(z) = p_n(0)q_n(z) + \int k_n(z,\zeta)p_n(\zeta)d\mu_n(\zeta). \tag{3.9}$$

Using (1.3), we have

$$\int k_n(z,\zeta)p_n(\zeta)d\mu(\zeta) = \overline{\int \overline{p_n(\zeta)}k_n(\zeta,z)d\mu(\zeta)} = \overline{\left(\frac{k_n(0,z)}{p_n(0)}\right)} = 0. \tag{3.10}$$

Subtracting (3.10) from (3.9) and dividing the resulting equation by $p_n(0)$ gives the Berstein-Szegö integral equation

$$\frac{p_n(z)}{p_n(0)} = q_n(z) + \int k_n(z,\zeta)\frac{p_n(\zeta)}{p_n(0)}d(\mu_n - \mu)(\zeta). \tag{3.11}$$

It is not hard to prove that $\frac{p_n(z)}{p_n(0)}$ is actually the unique solution to this integral equation. It will be convenient to consider the ratio

$$u_n(z) = \frac{p_n(z)}{p_n(0)q_n(z)}. \tag{3.12}$$

The integral equation (3.11) transforms into the integral equation for u_n

$$u_n(z) = 1 + \int \kappa_n(z,\zeta)u_n(\zeta)d\lambda(\zeta), \tag{3.13}$$

where

$$\kappa_n(z,\zeta) = \frac{q_n(\zeta)}{q_n(z)}k_n(z,\zeta)\left(\frac{1}{|q_n(\zeta)|^2} - \frac{1}{|\pi(\zeta)|^2}\right) \tag{3.14}$$

3.2 An integral equation on the real line

In the same way as the Bernstein-Szegö integral equation was constructed with the polynomial q_n, we here construct an integral equation on the real line with the function Q_α ($\alpha < \frac{1}{2}$), where we assume a sufficiently large so that by lemma 2.3 all zeros of Q_α lie in the lower half-plane. We show that this integral equation has a unique solution, which is characterized by an orthogonality property, and give an explicit expression for this solution.

Definition 3.1 *Let $\mathcal{V}_{\alpha-1}$ denote the space of the entire functions $v(z)$ such that*

$$\sup_{z\in\mathbb{C}} \frac{|v(z)|}{(1+|e^{iz}|)(1+|z|)^{\alpha-1}} < \infty. \tag{3.15}$$

The analogue of (3.2) is the following proposition.

Proposition 3.2 *Suppose $v \in \mathcal{V}_{\alpha-1}$ and*

$$\forall \theta \in (0,\pi) : \lim_{R\to+\infty} \frac{v(Re^{i\theta})}{(-iRe^{i\theta})^{\alpha-1}} = c. \tag{3.16}$$

Then

$$\lim_{R\to+\infty} \frac{1}{\pi} \int_{-R}^{+R} \overline{Q_\alpha(t)} v(t) \frac{dt}{|Q_\alpha(t)|^2} = c. \tag{3.17}$$

Let

$$Q_\alpha^*(z) = e^{iz} \overline{Q_\alpha(\bar{z})}, \tag{3.18}$$

and

$$k_\alpha(z,t) = -\frac{1}{2\pi i} \frac{Q_\alpha(z)\overline{Q_\alpha(t)} - Q_\alpha^*(z)\overline{Q_\alpha^*(t)}}{z-\bar{t}}. \tag{3.19}$$

For fixed t, $k_\alpha(z,t) \in \mathcal{V}_{\alpha-1}$ and

$$\overline{k_\alpha(z,t)} = k_\alpha(t,z). \tag{3.20}$$

Lemma 3.3 *There exists a constant $M_\alpha > 0$ such that for all $z \in \mathbb{C}$ and all $t \in \mathbb{R}$, we have*

$$|k_\alpha(z,t)| \le M_\alpha \frac{(1+|e^{iz}|)(1+|z|)^\alpha(1+|t|)^\alpha}{1+|z-t|}.$$

The next theorem states that $k_\alpha(z,t)$ is actually a reproducing kernel for $\mathcal{V}_{\alpha-1}$.

Theorem 3.4 *If $v \in \mathcal{V}_{\alpha-1}$, then*

$$v(z) = \int_{-\infty}^{+\infty} k_\alpha(z,t)v(t)\frac{dt}{|Q_\alpha(t)|^2} \tag{3.21}$$

Lemma 3.5 $$\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}} = \mathcal{O}\left(\frac{1}{|t|^{2\alpha+2}}\right) \quad \text{as} \quad t \to \pm\infty.$$

Definition 3.6 *Let $\mathcal{Y}_\alpha$ denote the Banach space of the functions y on $\mathbb{R}$ such that*

$$\tilde{y}(t) = \frac{y(t)}{(1+|t|)^\alpha} \in L^\infty(\mathbb{R}), \tag{3.22}$$

with the norm

$$\|y\| = \|\tilde{y}\|_\infty = ess\ sup_{t\in\mathbb{R}} |\tilde{y}(t)|. \tag{3.23}$$

For $y \in \mathcal{Y}_\alpha$, let

$$(K_\alpha y)(z) = \int_{-\infty}^{+\infty} k_\alpha(z,t)y(t)\left(\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}}\right) dt. \tag{3.24}$$

Theorem 3.7 *K_α is a compact linear operator on $\mathcal{Y}_\alpha$.*

Theorem 3.8 *If $y \in \mathcal{Y}_\alpha$, then $K_\alpha y \in \mathcal{V}_{\alpha-1}$.*

Lemma 3.9 *Suppose $v \in \mathcal{V}_{\alpha-1}$ and*

$$\forall\, x \in \mathbb{R} : \quad \int_{-\infty}^{+\infty} k_\alpha(x,t)v(t)\frac{dt}{|t|^{2\alpha}} = 0. \tag{3.25}$$

Then $v = 0$.

Theorem 3.10 *Suppose $y \in \mathcal{Y}_\alpha$ and $y = K_\alpha y$. Then $y = 0$.*

Theorem 3.11 *The integral equation*

$$y = Q_\alpha + K_\alpha y \tag{3.26}$$

has a unique solution.

Theorem 3.12 *Suppose $y(z)$ is an entire function whose restriction to $\mathbb{R}$ is in $\mathcal{Y}_\alpha$. Then y is the solution of the integral equation (3.26) if and only if the following conditions are satisfied*

1. $y - Q_\alpha \in \mathcal{V}_{\alpha-1}$,
2. *If $v \in \mathcal{V}_{\alpha-1}$ and*

$$\forall\theta \in (0,\pi) : \lim_{R\to+\infty} \frac{v(Re^{i\theta})}{(-iRe^{i\theta})^{\alpha-1}} = c, \tag{3.27}$$

then

$$\lim_{R\to+\infty} \frac{1}{\pi}\int_{-R}^{+R} \overline{y(t)}v(t)\frac{dt}{|t|^{2\alpha}} = c. \tag{3.28}$$

The following property of the Bessel function $J_{-\alpha-\frac{1}{2}}$ will permit to express the solution of the integral equation (3.26) in terms of Bessel functions.

Lemma 3.13 *Suppose f is an even entire function such that*

$$\sup \frac{|f(z)|}{\cosh(\Im z)(1+|z|)^{\alpha-1}} < \infty. \tag{3.29}$$

Then

$$I(\alpha,f,R) := \int_0^R J_{-\alpha-\frac{1}{2}}(x)f(x)x^{-\alpha+\frac{1}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{C_R} e^{iz}\frac{f(z)}{(-iz)^{\alpha-1}}\frac{dz}{iz} + \mathcal{O}(\frac{1}{R}), \tag{3.30}$$

where $C_R = \{z = Re^{it} : t \in [0,\pi]\}$.

Theorem 3.14 *The solution of the integral equation (3.26) is $P_\alpha(z)$, where*

$$P_\alpha(2z) = 2^{\alpha-\frac{1}{2}}\sqrt{\pi}e^{iz}\left(j_{-\alpha-\frac{1}{2}}(z) - izj_{-\alpha+\frac{1}{2}}(z)\right). \tag{3.31}$$

Another expression for $P_\alpha(z)$ is

$$P_\alpha(z) = \frac{1}{\Gamma(-\alpha)}\int_0^1 e^{izt}t^{-\alpha-1}(1-t)^{-\alpha}dt \qquad (\alpha < 0). \tag{3.32}$$

3.3 Asymptotics of the solution of the Bernstein-Szegö integral equation

We use discrete convergence theory as in [10]. We first give a short outline of this theory.

3.3.1 Discrete convergence theory

Consider Banach spaces E and E_n $(n \in \mathbb{N})$ as well as bounded linear maps $\pi_n : E \to E_n$ such that

$$\forall u \in E : \lim \|\pi_n u\| = \|u\|. \tag{3.33}$$

It can be shown that this implies that

$$\limsup \|\pi_n\| < \infty. \tag{3.34}$$

Definition 3.15 *A sequence $(u_n)_{n\in\mathbb{N}}$, where $u_n \in E_n$, is called* **bounded** *if $(\|u_n\|)_{n\in\mathbb{N}}$ is bounded. We say that $(u_n)_{n\in\mathbb{N}}$ is* **convergent** *with limit $\lim u_n = u \in E$ if $\lim \|u_n - \pi_n u\| = 0$. We say that $(u_n)_{n\in\mathbb{N}}$ is* **compact** *if for every infinite subset N of $\mathbb{N}$ there exists an infinite subset N' of N such that the subsequence $(u_n)_{n\in N'}$ is convergent.*

Definition 3.16 *Consider a sequence $(T_n)_{n\in\mathbb{N}}$, where T_n is a bounded linear operator on E_n. Then we say that $(T_n)_{n\in\mathbb{N}}$* **converges compactly** *to the bounded linear operator T on E and write*

$$T_n \to T \quad \textit{compactly},$$

if the following three conditions are satisfied:

1. $\limsup \|T_n\| < \infty$.
2. $\forall u \in E : \lim \|T_n \pi_n u - \pi_n T u\| = 0$.
3. *If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence, then $(T_n u_n)_{n\in\mathbb{N}}$ is a compact sequence.*

Theorem 3.17 *Assume $T : E \to E$ and $T_n : E_n \to E_n$ are compact linear operators and*

$$T_n \to T \quad \textit{compactly}.$$

Suppose $f \in E$ and the equation

$$u = Tu + f$$

has a unique solution $u \in E$. Then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the equation

$$u_n = T_n u_n + \pi_n f$$

has a unique solution $u_n \in E_n$ and

$$\lim u_n = u.$$

Moreover, there exist two positive constants c_1 and c_2, independent of f and n, such that

$$c_1 \|T_n \pi_n u - \pi_n T u\| \leq \|u_n - \pi_n u\| \leq c_2 \|T_n \pi_n u - \pi_n T u\|.$$

This theorem is e.g. proved in [10].

3.3.2 Application of discrete convergence theory

We consider a partition of the unit circle $\mathbb{T}$ in arcs Γ_k, each containing exactly one singular point z_k (not as an endpoint). Then there exist intervals Δ_k, containing 0 as an interior point, such that the map

$$\Delta_k \to \Gamma_k : x \mapsto z_k e^{ix} \tag{3.35}$$

is bijective.
We consider the Banach space $E = \prod_k L^\infty(\mathbb{R})$, with the norm

$$\|(u_k)\| := \max_k \|u_k\|_\infty$$

and the Banach space $E_n = L^\infty(\mathbb{T})$.

Definition 3.18 *Let $\pi_{n,k} : L^\infty(\mathbb{R}) \to L^\infty(\Gamma_k)$ be defined by*

$$(\pi_{n,k}u)(z_k e^{ix}) = u(nx) \qquad (x \in \Delta_k), \tag{3.36}$$

and $\pi_n : E \to E_n : u = (u_k) \mapsto \pi_n u$ by

$$(\pi_n u)(z) = (\pi_{n,k} u_k)(z) \qquad (z \in \Gamma_k). \tag{3.37}$$

Verification of (3.33) is straightforward.

Definition 3.19 *Let $T_n : E_n \to E_n$ be defined as*

$$(T_n u)(z) = \int \kappa_n(z,\zeta) u(\zeta) d\lambda(\zeta), \tag{3.38}$$

where κ_n is as in (3.14), and let $T : E \to E : u = (u_k) \mapsto Tu = (S_k u_k)$, where

$$(S_k y)(x) = \int_{-\infty}^{+\infty} \kappa_{\alpha_k}(x,t) y(t) dt, \tag{3.39}$$

$$\kappa_\alpha(x,t) = \frac{Q_\alpha(t)}{Q_\alpha(x)} k_\alpha(x,t) \left(\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha_k}} \right), \tag{3.40}$$

and $k_\alpha(x,t)$ is as in (3.19).

Proposition 3.20 *The operators T_n and T are compact.*

Theorem 1.1 is essentially a corollary of theorem 3.17 and theorem 3.21 below.

Theorem 3.21 $\qquad T_n \to T \qquad$ *compactly, and* $\qquad \|T_n\pi_n - \pi_n T\| = \mathcal{O}\left(\frac{\log n}{n}\right).$

The proof of this theorem is based on the lemmas 3.23 and 3.24 below. We define an auxiliary operator $\tilde{T}_n$ on E_n as follows.

Definition 3.22 *Let $i_n : L^\infty(\mathbb{T}) \to \prod_k L^\infty(\mathbb{R})$ be defined as follows*

$$(i_n u)_k(x) = \begin{cases} u(z_k e^{i\frac{x}{n}}) & \text{if } x \in n\Delta_k \\ 0 & \text{otherwise} \end{cases}, \tag{3.41}$$

and let $\tilde{T}_n = \pi_n T i_n$.

We have the following integral representation for $\tilde{T}_n$

$$(\tilde{T}_n u)(z_k e^{ix}) = \int_{\Delta_k} n\kappa_{\alpha_k}(nx, nt) u(z_k e^{it}) dt \qquad (x \in \Delta_k). \tag{3.42}$$

On one hand we have

Lemma 3.23 $\tilde{T}_n \to T$ *compactly, and* $\|\tilde{T}_n \pi_n - \pi_n T\| = \mathcal{O}\left(\frac{1}{n}\right)$.

On the other hand we have

Lemma 3.24

$$\begin{aligned} &\|T_n - \tilde{T}_n\| \\ &= \max_k \sup_{x \in \Delta_k} \left(\int_{\Delta_k} \left|\frac{1}{2\pi}\kappa_n(z_k e^{ix}, z_k e^{it}) - n\kappa_{\alpha_k}(nx, nt)\right| dt + \sum_{l \neq k} \int_{\Gamma_l} |\kappa_n(z_k e^{ix}, \zeta)| d\lambda(\zeta) \right) \\ &= \mathcal{O}\left(\frac{\log n}{n}\right). \end{aligned}$$

The first equality in this lemma follows immediately from the integral representations of T_n and $\tilde{T}_n$. The asymptotic bound is based on asymptotic estimates of the factors of the kernel κ_n, which are elaborated below. All these estimates are based on theorem 2.2.

Proposition 3.25 *Suppose* $p \in \mathbb{N}$, $x, t \in \Delta_k$ *and* $z = z_k e^{ix}, \zeta = z_k e^{it}$. *Then*

$$k_n(z, \zeta) = \sum_{l=0}^{p-1} \frac{1}{n^l} k_{n,k,l}(z, \zeta) + \mathcal{O}\left(\frac{\left(\frac{1}{n} + |z - z_k|\right)^{\alpha_k} \left(\frac{1}{n} + |\zeta - z_k|\right)^{\alpha_k}}{n^p \left(\frac{1}{n} + |z - \zeta|\right)} \right), \tag{3.43}$$

uniformly for $z, \zeta \in \Gamma_k$, *where*

$$k_{n,k,l}(z, \zeta) = -\frac{z}{z - \zeta} \left(\frac{G_{k,l}(nx, x, nt, t) - e^{in(x-t)} G_{k,l}(nt, t, nx, x)}{n^{2\alpha_k}} \right), \tag{3.44}$$

and

$$G_{k,l}(u, x, v, t) = \sum_{j=0}^{l} Q_{\alpha_k, j}(u) \frac{\chi_k^{(j)}(x)}{j!} \overline{Q_{\alpha_k, l-j}(v) \frac{\chi_k^{(l-j)}(t)}{(l-j)!}}. \tag{3.45}$$

Lemma 3.26 *Suppose* $t \in \Delta_k$ *and let*

$$f_{n,k}(t) = \frac{1}{|q_n(z_k e^{it})|^2} - \frac{1}{|\pi(z_k e^{it})|^2} \tag{3.46}$$

and

$$f_{n,k,0}(t) = \left(\frac{1}{|Q_{\alpha_k}(nt)|^2} - \frac{1}{|nt|^{2\alpha_k}} \right) \frac{n^{2\alpha_k}}{|\chi_k(t)|^2}. \tag{3.47}$$

Then

$$f_{n,k}(t) = f_{n,k,0}(t) + \mathcal{O}\left(\frac{1}{n} \left(\frac{1}{n} + |t| \right)^{-2\alpha_k} \right) \tag{3.48}$$

uniformly for $t \in \Delta_k$.

Lemma 3.27 *Suppose $k \neq l$. Then*

$$\sup_{z\in\Gamma_k}\int_{\Gamma_l}|\kappa_n(z,\zeta)|d\lambda(\zeta)=\mathcal{O}\left(\frac{\log n}{n}\right). \tag{3.49}$$

Lemma 3.28

$$\sup_{x\in\Delta_k}\int_{\Delta_k}|\kappa_n(z_ke^{ix},z_ke^{it})-\frac{Q_{\alpha_k}(nt)\chi_k(t)}{Q_{\alpha_k}(nx)\chi_k(x)}k_{n,k,0}(z_ke^{ix},z_ke^{it})f_{n,k,0}(t)|dt=\mathcal{O}\left(\frac{\log n}{n}\right). \tag{3.50}$$

Lemma 3.29

$$\sup_{x\in\Delta_k}\int_{\Delta_k}|\frac{Q_{\alpha_k}(nt)\chi_k(t)}{Q_{\alpha_k}(nx)\chi_k(x)}k_{n,k,0}(z_ke^{ix},z_ke^{it})f_{n,k,0}(t)-2\pi n\kappa_{\alpha_k}(nx,nt)|dt=\mathcal{O}\left(\frac{1}{n}\right). \tag{3.51}$$

4 Proofs

4.1 Proofs of section 2

Proof of theorem 2.2. Let

$$f_n(x)=\pi(e^{i(x+\frac{ia}{n})}).$$

Then, with $\alpha=\min_k\alpha_k$, we have for all $m\in\mathbb{N}$ that

$$\sup_{x\in\mathbb{R}}|f_n^{(m)}(x)|=\mathcal{O}\left(\left(\frac{1}{n}\right)^{\alpha-m}+1\right). \tag{4.1}$$

Now, fix k and choose $\epsilon>0$ sufficiently small so that for all $x\neq 0$ at a distance at most ϵ from Δ_k, we have that z_ke^{ix} is not a singular point. Then, for $x\in\Delta_k$,

$$q_n(z_ke^{ix})=\int_{-\infty}^{+\infty}\hat{\phi}(s)f_n(x+\frac{s}{n})ds=g_{n,1}(x)+g_{n,2}(x), \tag{4.2}$$

where

$$\begin{aligned} g_{n,1}(x) &= \int_{-n\epsilon}^{+n\epsilon}\hat{\phi}(s)\sigma_{\alpha_k}(x+\frac{s+ia}{n})\chi_k(x+\frac{s+ia}{n})ds \\ g_{n,2}(x) &= \int_{|s|\geq n\epsilon}\hat{\phi}(s)f_n(x+\frac{s}{n})ds. \end{aligned}$$

By the fast decay of $\hat{\phi}(s)$ and by (4.1), we have that for all $m\in\mathbb{N}$ and all $q>0$

$$g_{n,2}^{(m)}(x)=\mathcal{O}\left(\frac{1}{n^q}\right). \tag{4.3}$$

uniformly for all $x\in\Delta_k$. For $x\in\Delta_k$ we expand χ_k about x

$$\chi_k(x+h)=\sum_{l=0}^{p-1}\frac{\chi_k^{(l)}(x)}{l!}h^l+r_p(\chi_k,x,h).$$

Then

$$
\begin{aligned}
g_{n,1}(x) &= \sum_{l=0}^{p-1}\left(\int_{-n\epsilon}^{+n\epsilon}\hat{\phi}(s)\sigma_{\alpha_k}(x+\frac{s+ia}{n})\left(\frac{s+ia}{n}\right)^l ds\right)\frac{\chi_k^{(l)}(x)}{l!} \\
&\quad + \int_{-n\epsilon}^{+n\epsilon}\hat{\phi}(s)\sigma_{\alpha_k}(x+\frac{s+ia}{n})r_p(\chi_k, x, \frac{s+ia}{n})ds \\
&= \sum_{l=0}^{p-1}\frac{Q_{\alpha_k,l}(nx)}{n^{\alpha_k+l}}\frac{\chi_k^{(l)}(x)}{l!} - \sum_{l=0}^{p-1}\rho_{l,n}(x)\frac{\chi_k^{(l)}(x)}{l!} + r_{p,n}(x), \qquad (4.4)
\end{aligned}
$$

where

$$
\rho_{l,n}(x) = \int_{|s|\geq n\epsilon}\hat{\phi}(s)\left(\frac{s+ia}{n}\right)^l \sigma_{\alpha_k}(x+\frac{s+ia}{n})ds \qquad (4.5)
$$

and

$$
r_{p,n}(x) = \int_{-n\epsilon}^{n\epsilon}\hat{\phi}(s)\sigma_{\alpha_k}(x+\frac{s+ia}{n})r_p(\chi_k, x, \frac{s+ia}{n})ds. \qquad (4.6)
$$

We have

$$
\begin{aligned}
|\rho_{l,n}^{(m)}(x)| &= \left|\int_{|s|\geq n\epsilon}\hat{\phi}(s)\left(\frac{s+ia}{n}\right)^l \sigma_{\alpha_k}^{(m)}(x+\frac{s+ia}{n})ds\right| \\
&\leq \frac{1}{n^{l+\alpha_k-m}}\int_{|s|\geq n\epsilon}|\hat{\phi}(s)(s+ia)^l||\sigma_{\alpha_k}^{(m)}(nx+s+ia)|ds. \qquad (4.7)
\end{aligned}
$$

For $x \in \Delta_k$ and $|s| \geq n\epsilon$, we have

$$
|a| \leq |nx+s+ia| \leq \left(\frac{\mathrm{diam}(\Delta_k)}{\epsilon}+1\right)|s|+|a|.
$$

Hence,

$$
|\sigma_{\alpha_k}^{(m)}(nx+s+ia)| = \mathcal{O}(\max(1,|s|^{\alpha_k-m}))
$$

uniformly for $x \in \Delta_k$ and $|s| \geq n\epsilon$. Substituting this result in (4.7), we have by the fast decay of $\hat{\phi}$ that for all $q > 0$,

$$
\rho_{l,n}^{(m)}(x) = \mathcal{O}\left(\frac{1}{n^q}\right) \qquad (4.8)
$$

uniformly for $x \in \Delta_k$. As $R_{k,p,n}$ is composed of $g_{n,2}$, the $\rho_{l,n}$ and $r_{p,n}$, it remains to show that the derivatives of $r_{p,n}$ are suitably bounded. We have

$$
\frac{\partial^m}{\partial x^m}r_p(\chi_k, x, \frac{s+ia}{n}) = r_p(\chi_k^{(m)}, x, \frac{s+ia}{n}) = \frac{\chi_k^{(m+p)}(z)}{p!}\frac{(s+ia)^p}{n^p},
$$

for some z between x and $x+\frac{s+ia}{n}$. Hence,

$$
\frac{\partial^m}{\partial x^m}r_p(\chi_k, x, \frac{s+ia}{n}) = \mathcal{O}\left(\frac{(1+|s|)^p}{n^p}\right) \qquad (4.9)
$$

uniformly for $x \in \Delta_k$ and $|s| \le n\epsilon$. The derivatives of $r_{p,n}(x)$ are obtained by applying the Leibniz rule of differentiation to the integrand in (4.6). By (4.9), these derivatives are suitably bounded if for all m we have

$$\int_{-n\epsilon}^{+n\epsilon} |\hat{\phi}(s)||\sigma_{\alpha_k}^{(m)}(nx+s+ia)|(1+|s|)^p ds = \mathcal{O}((1+|nx|)^{\alpha_k - m}),$$

uniformly for $x \in \Delta_k$. By the fast decay of $\hat{\phi}$, this holds if for sufficiently large q

$$\int_{-\infty}^{+\infty} (1+|s|)^{-q}(1+|u-s|)^{\alpha_k - m} ds = \mathcal{O}((1+|u|)^{\alpha_k - m}).$$

This is proved in the next lemma. $\square$

Lemma 4.1 *Suppose* $\beta \in \mathbb{R}$ *and* $\gamma > \max(\beta, 0)$. *Then*

$$\int_{-\infty}^{+\infty} (1+|t|)^{-\gamma-1}(1+|t-x|)^{\beta} dt = \mathcal{O}((1+|x|)^{\beta}).$$

Proof. Put

$$g(x,t) = (1+|t|)^{-\gamma-1}(1+|t-x|)^{\beta}.$$

We distinguish between 2 cases.

1. $\beta \ge 0$.
   We split the integration region in 2 subregions and prove the bound for the integral on each of the subregions. For $|t-x| \le 2|x|$, we have

   $$(1+|t-x|)^{\beta} \le (2+2|x|)^{\beta} = 2^{\beta}(1+|x|)^{\beta},$$

   whence,

   $$\int_{|t-x|\le 2|x|} g(x,t)dt \le 2^{\beta}(1+|x|)^{\beta} \int_{-\infty}^{+\infty} (1+|t|)^{-\gamma-1} dt = \mathcal{O}((1+|x|)^{\beta}).$$

   For $|t-x| \ge 2|x|$ we have $|t| \ge \frac{|t-x|}{2}$, whence

   $$(1+|t|)^{-\gamma-1} \le (\frac{1}{2} + \frac{|t-x|}{2})^{-\gamma-1} = 2^{\gamma+1}(1+|t-x|)^{-\gamma-1}.$$

   It follows that

   $$\int_{|t-x|\ge 2|x|} g(x,t)dt \le 2^{\gamma+1} \int_{-\infty}^{+\infty} (1+|t-x|)^{\beta-\gamma-1} dt < \infty.$$

2. $\beta < 0$.
For $|t-x| \le \frac{|x|}{2}$, we have $|t| \ge \frac{|x|}{2}$ and thus

$$(1+|t|)^{-\gamma-1} \le 2^{\gamma+1}(1+|x|)^{-\gamma-1}.$$

Hence, for $\beta \neq -1$,

$$\begin{aligned}
\int_{|t-x|\le\frac{|x|}{2}} g(x,t)dt &\le 2^{\gamma+1}(1+|x|)^{-\gamma-1}\int_{|t-x|\le\frac{|x|}{2}}(1+|t-x|)^{\beta}dt \\
&\le 2^{\gamma+1}(1+|x|)^{-\gamma-1}\frac{2}{\beta+1}(1+\frac{|x|}{2})^{\beta+1} \\
&= \mathcal{O}((1+|x|)^{\beta-\gamma}) \\
&= \mathcal{O}((1+|x|)^{\beta}),
\end{aligned}$$

and it is easily verified that this last bound also holds when $\beta = -1$.
For $|t-x| \ge \frac{|x|}{2}$, we have

$$(1+|t-x|)^{\beta} \le (1+\frac{|x|}{2})^{\beta} \le 2^{-\beta}(1+|x|)^{\beta}.$$

Hence,

$$\int_{|t-x|\ge\frac{|x|}{2}} g(x,t)dt \le 2^{-\beta}(1+|x|)^{\beta}\int_{-\infty}^{+\infty}(1+|t|)^{-\gamma-1}dt = \mathcal{O}((1+|x|)^{\beta}).$$

□

Proof of lemma 2.3. If $\alpha < 0$, we have by the definition of the Gamma function that

$$(-ix+a)^{\alpha} = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty} e^{ixt}e^{-at}t^{-\alpha-1}dt. \tag{4.10}$$

As $Q_\alpha(x)$ is the convolution of this function with the Fourier transform of $\phi(t)$, (2.19) results from the fact that multiplication in the t-space corresponds to convolution in the x-space. For $0 \le \alpha < \frac{1}{2}$, (2.19) is obtained by analytic continuation w.r.t. α.
We now show (2.20) by an argument that reduces essentially to Watson's lemma. Suppose first that $\Im(z) > 0$ and $\alpha < 0$. Then the decomposition of ϕ

$$\phi(t) = \frac{1}{2\pi} + \left(\phi(t) - \frac{1}{2\pi}\right)$$

implies the following decomposition of F_α

$$F_\alpha(z) = (-iz)^{\alpha} + H_\alpha(z), \tag{4.11}$$

where

$$H_\alpha(z) = \frac{2\pi}{\Gamma(-\alpha)}\int_0^{\infty} e^{izt}t^{-\alpha-1}\left(\phi(t) - \frac{1}{2\pi}\right)dt = \int_0^{\infty} e^{izt}\tilde{\phi}_\alpha(t)dt. \tag{4.12}$$

Here $\tilde{\phi}_\alpha(t)$ is smooth on $[0,+\infty)$, at $t=0$ all its derivatives vanish and for $t \geq 1$ it equals $-\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}$. The expression (4.12) is an entire function of α, whence, by analytic continuation, (4.11) also holds for $\alpha < 1$. After m successive integrations by parts, (4.12) gives

$$H_\alpha(z) = (-iz)^{-m} \int_0^\infty e^{izt} \tilde{\phi}_\alpha^{(m)}(t) dt. \tag{4.13}$$

For $m > 1$, we have

$$|H_\alpha(z)| \leq |z|^{-m} \int_0^\infty |\tilde{\phi}_\alpha^{(m)}(t)| dt \qquad (\Im(z) \geq 0). \tag{4.14}$$

As m can be chosen arbitrarily large, (2.20) is established. It is easily verified, using the Leibniz rule of differentiation and the Taylor expansion, that actually

$$\begin{aligned} \int_0^\infty |\tilde{\phi}_\alpha^{(m)}(t)| dt &= \int_0^1 |\tilde{\phi}_\alpha^{(m)}(t)| dt + \int_1^\infty \frac{t^{-\alpha-m-1}}{|\Gamma(-\alpha-m)|} dt \\ &\leq c_{\alpha,m} \max_{0 \leq t \leq 1} |\phi^{(m+1)}(t)| + \frac{1}{|\Gamma(1-\alpha-m)|}, \end{aligned} \tag{4.15}$$

where $c_{\alpha,m}$ is some constant. This result will be used later on.
Suppose $y \geq 0$. Then

$$e^{-y} F_\alpha(x - iy) = \frac{2\pi}{\Gamma(-\alpha)} \int_{-\infty}^{+\infty} e^{ixt} t^{-\alpha-1} \phi_y(t) dt, \qquad (\alpha < 0) \tag{4.16}$$

where

$$\phi_y(t) = e^{-(1-t)y} \phi(t). \tag{4.17}$$

Let $H_{\alpha,y}$ be defined by formula (4.12) but with ϕ replaced by ϕ_y. Then

$$e^{-y} F_\alpha(x - iy) = e^{-y}(-ix)^\alpha + H_{\alpha,y}(x) \tag{4.18}$$

is valid for $x \in \mathbb{R}$ and $x \neq 0$. By (4.14) and (4.15) the bound

$$H_{\alpha,y}(x) = \mathcal{O}(|x|^{-q}) \tag{4.19}$$

holds uniformly for $y \geq 0$ provided the derivatives of $\phi_y(t)$ admit a bound that is independent of y. This is indeed true, since

$$\begin{aligned} \sup_{y \geq 0,\, 0 \leq t < 1} |\phi_y^{(m)}(t)| &\leq \sum_{k=0}^m \binom{m}{k} \sup_{y \geq 0,\, 0 \leq t < 1} |y^k e^{-(1-t)y} \phi^{(m-k)}(t)| \\ &\leq \sum_{k=0}^m \binom{m}{k} \sup_{y \geq 0,\, 0 \leq t < 1} |((1-t)y)^k e^{-(1-t)y}| \, |(1-t)^{-k} \phi^{(m-k)}(t)| \\ &\leq \sum_{k=0}^m \binom{m}{k} \left(\sup_{u \geq 0} u^k e^{-u} \right) \left(\sup_{0 \leq t < 1} (1-t)^{-k} \phi^{(m-k)}(t) \right) \\ &< \infty, \end{aligned}$$

where in the last step, we used the fact that at $t = 1$ all derivatives of $\phi(t)$ vanish. In the region $|x| \geq y$, we have

$$(-ix)^\alpha = \mathcal{O}(|x - iy|^\alpha) \quad \text{and} \quad |x|^{\alpha + c'} = \mathcal{O}(|x - iy|^{\alpha + c'}).$$

Hence, in this region (2.21) follows from (4.18) and (4.19).
Now suppose $|x| \le y$. Using a partition of unity on the interval $[0,1]$, we can write

$$F_\alpha(z) = F_{\alpha,1}(z)+F_{\alpha,2}(z) = \frac{1}{\Gamma(-\alpha)}\int_0^\epsilon e^{izt}t^{-\alpha-1}\psi_1(t)dt+\int_0^1 e^{izt}\psi_2(t)dt, \qquad (\alpha<0) \tag{4.20}$$

where $0<\epsilon<1$, $\psi_1(t)$ is a smooth function that vanishes for $t \ge \epsilon$ and coincides with $2\pi\phi(t)$ in a neighborhood of 0 and $\psi_2(t)$ is a smooth function with support in $[0,1]$. In order to obtain a formula for $F_{\alpha,1}(z)$ that is also valid when $0<\alpha<\frac{1}{2}$, we integrate by parts

$$F_{\alpha,1}(z) = \frac{1}{\Gamma(1-\alpha)}\int_0^\epsilon e^{izt}t^{-\alpha}(iz\psi_1(t)+\psi_1'(t))dt. \tag{4.21}$$

This directly yields the asymptotic bound

$$F_{\alpha,1}(x-iy) = \mathcal{O}(|x-iy|e^{y\epsilon}) = \mathcal{O}(|x-iy|^{-m}e^y) \tag{4.22}$$

where m is arbitrary. By m successive integrations by parts, we also obtain

$$F_{\alpha,2}(x-iy) = \mathcal{O}(|x-iy|^{-m}e^y). \tag{4.23}$$

Hence, (2.21) is now also established in the region $|x| \le y$ and the proof is completed. □

Proof of corollary 2.4. The function $\pi(e^{-\frac{a}{n}}z)$ has no zeros in the closed unit disk. If its approximation $q_n(z)$ has a relative error smaller than 1 on the unit circle, in other words, if

$$\sup_{x\in\mathbb{R}}\left|\frac{q_n(e^{ix})}{\pi(e^{-\frac{a}{n}}e^{ix})}-1\right|<1, \tag{4.24}$$

then it follows from Rouché's theorem that neither $q_n(z)$ has any zero in the closed unit disk. We cover the unit circle by a finite number of arcs, each containing exactly one singular point z_k. It is then sufficient to check the condition on the relative error on each of the arcs separately. For the arc containing the singularity z_k we have by theorem 2.2 (with p = 1) that the condition

$$\sup_{x\in\Delta_k}\left|\frac{q_n(z_k e^{ix})}{\pi(e^{-\frac{a}{n}}z_k e^{ix})}-1\right|<1, \tag{4.25}$$

is satisfied for sufficiently large n if

$$\sup_{x\in\mathbb{R}}\left|\frac{Q_{\alpha_k}(x)}{(-ix+a)^{\alpha_k}}-1\right|<1. \tag{4.26}$$

By (2.19) and (2.20), (4.26) is satisfied for sufficiently large a. □

4.2 Proofs of subsection 3.2

Proof of Proposition 3.2. The integral in (3.17) is

$$\frac{1}{\pi}\int_{-R}^{+R}\frac{v(t)}{Q_\alpha(t)}dt.$$

As the integrand is analytic in the upper half-plane, we can replace the integral by the integral on the semicircle joining $-R$ to $+R$

$$-\frac{1}{\pi}\int_{0}^{\pi}\frac{v(Re^{i\theta})}{Q_\alpha(Re^{i\theta})}Rie^{i\theta}d\theta.$$

From the asymptotic behavior of $Q_\alpha(z)$ in the upper half-plane (cf. proposition 2.3) and the fact that $v \in \mathcal{V}_{\alpha-1}$, the integrand admits a bound independent of R and θ. Application of the dominated convergence theorem now proves the theorem. $\square$

Proof of Lemma 3.3. We distinguish 2 cases

1. $|z-t| \geq \frac{1}{2}$: By lemma 2.3, there exists $M_1 > 0$ such that

   $$|Q_\alpha(z)| \leq M_1(1+|e^{iz}|)(1+|z|)^\alpha$$

   and consequently also

   $$|Q^*_\alpha(z)| = |e^{iz}||Q_\alpha(\bar{z})| \leq M_1(|e^{iz}|+1)(1+|z|)^\alpha.$$

   The bound for $|k_\alpha(z,t)|$ follows readily.

2. $|z-t| < \frac{1}{2}$: Then, applying the mean value theorem, we have

   $$k_\alpha(z,t) = -\frac{1}{2\pi i}\left(Q'_\alpha(\zeta)\overline{Q_\alpha(t)} - Q^{*\prime}_\alpha(\zeta)\overline{Q^*_\alpha(t)}\right),$$

   where ζ lies somewhere on the line segment that joins t with z. Now, as

   $$Q'_\alpha(z) = F'_\alpha(z+ia) = (-i\alpha)F_{\alpha-1}(z+ia) = (-i\alpha)Q_{\alpha-1}(z),$$

   there exists a constant $M_2 > 0$ such that

   $$|Q'_\alpha(z)| \leq M_2(1+|e^{iz}|)(1+|z|)^{\alpha-1} \leq M_2(1+|e^{iz}|)(1+|z|)^\alpha$$

   and consequently

   $$|Q^{*\prime}_\alpha(z)| \leq |e^{iz}||Q'_\alpha(\bar{z})| + |e^{iz}||Q_\alpha(\bar{z})| \leq (M_2+M_1)(1+|e^{iz}|)(1+|z|)^\alpha.$$

   These bounds now easily imply the bound for $|k_\alpha(z,t)|$ in case $|z-t| < \frac{1}{2}$.

□

Proof of Theorem 3.4. We have

$$\int_{-\infty}^{+\infty} k_\alpha(z,t)v(t)\frac{dt}{|Q_\alpha(t)|^2} = \lim_{R\to+\infty} -\frac{Q_\alpha(z)}{2\pi i}\int_{-R}^{+R}\frac{v(t)}{Q_\alpha(t)(z-t)}dt \tag{4.27}$$

$$+\lim_{R\to+\infty}\frac{Q^*_\alpha(z)}{2\pi i}\int_{-R}^{+R}\frac{v(t)}{Q^*_\alpha(t)(z-t)}dt. \tag{4.28}$$

If $z \in \mathbb{R}$, the integrals on the right-hand side represent Cauchy principal values. The integrand of the first (resp. second) integral on the right-hand side is analytic on the upper (resp. lower) half-plane, except for a simple pole at $t = z$ if $\Im(z) \geq 0$ (resp. if $\Im(z) \leq 0$). We move the contour to a large semicircle in the upper (resp. lower) half-plane and apply the residue theorem. This gives

$$\begin{aligned}\int_{-\infty}^{+\infty} k_\alpha(z,t)v(t)\frac{dt}{|Q_\alpha(t)|^2} &= v(z)+\lim_{R\to+\infty}\frac{Q_\alpha(z)}{2\pi}\int_0^{\pi}\frac{v(Re^{i\theta})Re^{i\theta}}{Q_\alpha(Re^{i\theta})(z-Re^{i\theta})}d\theta \\ &\quad +\lim_{R\to+\infty}\frac{Q^*_\alpha(z)}{2\pi}\int_{-\pi}^{0}\frac{v(Re^{i\theta})Re^{i\theta}}{Q^*_\alpha(Re^{i\theta})(z-Re^{i\theta})}d\theta \\ &= v(z).\end{aligned}$$

In the last step, the integrals tend to zero, because in view of the asymptotic behavior of $Q_\alpha(z)$, $Q^*_\alpha(z)$ and $v(z) \in \mathcal{V}_{\alpha-1}$ the integrands tend uniformly to zero. □

Proof of Lemma 3.5. By lemma 2.3, we have

$$Q_\alpha(t) = (-it+a)^\alpha\left(1+\mathcal{O}\left(\frac{1}{t^2}\right)\right) \quad \text{as } t\to\pm\infty.$$

Hence,

$$\begin{aligned}|Q_\alpha(t)|^2 &= |t|^{2\alpha}\left((1+i\frac{a}{t})(1-i\frac{a}{t})\right)^\alpha\left(1+\mathcal{O}\left(\frac{1}{t^2}\right)\right) \\ &= |t|^{2\alpha}\left(1+\mathcal{O}\left(\frac{1}{t^2}\right)\right).\end{aligned}$$

The result of the lemma directly follows from the last formula. □

Proof of Theorem 3.7. Let

$$\kappa_\alpha(x,t) = k_\alpha(x,t)\left(\frac{1}{|Q_\alpha(t)|^2}-\frac{1}{|t|^{2\alpha}}\right) \tag{4.29}$$

denote the kernel of the integral operator K_α. It is convenient to subdivide the integration region into 2 parts, so that

$$(K_\alpha y)(x) = (K_{\alpha,1}y)(x)+(K_{\alpha,2}y)(x) = \int_{-\infty}^{0}\kappa_\alpha(x,t)y(t)dt+\int_0^{+\infty}\kappa_\alpha(x,t)y(t)dt. \tag{4.30}$$

We prove that $K_{\alpha,2}$ is compact. The proof that $K_{\alpha,1}$ is compact is analoguous.
Because of the bounds in lemmas 3.3 and 3.5, we have

$$\|K_{\alpha,2}\| = \sup_{x\in\mathbb{R}}\frac{1}{(1+|x|)^\alpha}\int_0^\infty|\kappa_\alpha(x,t)|(1+|t|)^\alpha dt \leq \int_0^\infty g(t)dt < \infty, \tag{4.31}$$

where

$$g(t) = 2M_\alpha(1+|t|)^{2\alpha}\left|\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}}\right|. \tag{4.32}$$

In order to show that $K_{\alpha,2}$ is compact, we show that $K_{\alpha,2}$ can be arbitrarily well approximated by a compact operator. We proceed in 2 steps.
First, let $0 < r < R$ and let $K'_{\alpha,2}$ denote the integral operator w.r.t. the kernel $\kappa'_\alpha(x,t)$, that coincides with $\kappa_\alpha(x,t)$ on the rectangle

$$D_{r,R} = \{(x,t) \in \mathbb{R}^2 : \ |x| \le R \text{ and } r \le t \le R\} \tag{4.33}$$

and vanishes elsewhere. From the bounds in the lemmas 3.3 and 3.5, it follows that

$$\begin{aligned} \|K_{\alpha,2} - K'_{\alpha,2}\| &= \sup_{x\in\mathbb{R}} \frac{1}{(1+|x|)^\alpha}\int_0^\infty |\kappa_\alpha(x,t) - \kappa'_\alpha(x,t)|(1+|t|)^\alpha dt \\ &\le \sup_{|x|\ge R}\int_0^\infty \frac{g(t)}{1+|x-t|}dt + \int_0^r g(t)dt + \int_R^\infty g(t)dt \end{aligned}$$

tends to zero as $r \to 0+$ and $R \to +\infty$. Hence, it is sufficient to show that $K'_{\alpha,2}$ is compact. On the rectangle $D_{r,R}$, $\kappa'_\alpha(x,t)$ is continuous. By the Stone-Weierstrass approximation theorem, it can therefore be arbitrarily well uniformly approximated on $D_{r,R}$ by a degenerate kernel of the form

$$\sum_{j=1}^m a_j(x)b_j(t), \tag{4.34}$$

where a_j and b_j are continuous on $|x| \le R$ and $r \le t \le R$ respectively. We extend these functions by 0 to functions on the real line. The integral operator corresponding to the degenerate kernel approximates $K'_{\alpha,2}$ arbitrarily well and is compact because it has finite dimensional range. This establishes the compactness of $K'_{\alpha,2}$. □

Proof of Theorem 3.8. Let $v(z) = (K_\alpha y)(z)$. Then, by lemma 3.3 and 3.5,

$$|v(z)| \le M_\alpha\|y\|(1+|e^{iz}|)(1+|z|)^\alpha\int_{-\infty}^{+\infty}\frac{g(t)dt}{1+|z-t|}, \tag{4.35}$$

where

$$g(t) = (1+|t|)^{2\alpha}\left|\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}}\right| \le \frac{N_\alpha}{(1+2|t|)^2} \quad (|t| \ge \frac{1}{2}), \tag{4.36}$$

for some $N_\alpha > 0$. Hence, v is suitably bounded if the function

$$h(z) = (1+|z|)\int_{-\infty}^{+\infty}\frac{g(t)dt}{1+|z-t|} \tag{4.37}$$

is bounded. Write $z = x + iy$ and distinguish between 2 cases:

1. $|y| \ge |x|$. Then $h(z)$ is bounded by

$$(1+|x|+|y|)\int_{-\infty}^{+\infty}\frac{g(t)dt}{1+|y|} \le 2\int_{-\infty}^{+\infty} g(t)dt.$$

2. $|x| \geq |y|$. Then $h(z)$ is bounded by

$$(2+2|x|)\int_{-\infty}^{+\infty} \frac{g(t)}{1+|x-t|}dt.$$

This is clearly bounded when $|x| \leq 1$. Suppose $x \geq 1$. Then

$$\begin{aligned} h(x) &\leq 2\int_{\mathbb{R}\setminus[\frac{1}{2}x,\frac{3}{2}x]} \frac{1+|x|}{1+|x-t|}g(t)dt + 2\int_{\frac{1}{2}x}^{\frac{3}{2}x} \frac{1+|x|}{1+|x-t|}g(t)dt \\ &\leq 4\int_{\mathbb{R}\setminus[\frac{1}{2}x,\frac{3}{2}x]} g(t)dt + \frac{2N_\alpha}{1+|x|}\int_{\frac{1}{2}x}^{\frac{3}{2}x} \frac{dt}{1+|x-t|} \\ &\leq 4\int_{-\infty}^{+\infty} g(t)dt + 4N_\alpha \frac{\log(1+\frac{x}{2})}{1+x} \end{aligned}$$

is bounded for $x \geq 1$. The case $x \leq -1$ is proven analoguously.

v is analytic in every $z_0 \in \mathbb{C}$ because $v(z)$ is defined as an integral whose integrand depends analytically on z and is bounded by an integrable function independent of z for z in some neighborhood of z_0. □

Proof of Lemma 3.9. By Theorem 3.4 and (3.20), we have

$$\overline{v(t)} = \int_{-\infty}^{+\infty} k_\alpha(x,t)\overline{v(x)}\frac{dx}{|Q_\alpha(x)|^2}.$$

Hence, we have

$$\int_{-\infty}^{+\infty} \frac{|v(t)|^2}{|t|^{2\alpha}}dt = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} k_\alpha(x,t)\overline{v(x)}\frac{dx}{|Q_\alpha(x)|^2}\right) v(t)\frac{dt}{|t|^{2\alpha}}.$$

Estimating the different factors of the integrand, we see that the absolute convergence of

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{|k_\alpha(x,t)\overline{v(x)}v(t)|}{|Q_\alpha(x)|^2|t|^{2\alpha}}dx\, dt < \infty$$

follows essentially from the absolute convergence of

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dx\, dt}{(1+|x|)(1+|t|)(1+|x-t|)} < \infty,$$

which can be verified by straightforward computations. Hence, we can apply Fubini's theorem, which gives

$$\int_{-\infty}^{+\infty} \frac{|v(t)|^2}{|t|^{2\alpha}}dt = \int_{-\infty}^{+\infty} \overline{v(x)}\left(\int_{-\infty}^{+\infty} k_\alpha(x,t)v(t)\frac{dt}{|t|^{2\alpha}}\right)\frac{dx}{|Q_\alpha(x)|^2} = 0.$$

This is only possible if $v = 0$. □

Proof of Theorem 3.10. By Theorem 3.8, $y \in \mathcal{V}_{\alpha-1}$. Hence, by Theorem 3.4, we have

$$y(z) - (K_\alpha y)(z) = \int_{-\infty}^{+\infty} k_\alpha(z,t)y(t)\frac{dt}{|t|^{2\alpha}}.$$

Now Lemma 3.9 implies that $y = 0$. $\square$

Proof of Theorem 3.11. By Theorem 3.10, 1 is not an eigenvalue of K_α. Since K_α is compact, this implies that 1 does not belong to the spectrum of K_α, i.e., the operator $I - K_\alpha$, where I represents the identity, is invertible. The solution of the integral equation is thus unique and is equal to

$$y = (I - K_\alpha)^{-1} Q_\alpha.$$

$\square$

Proof of Theorem 3.12. For all $y \in \mathcal{Y}_\alpha$ and $v \in \mathcal{V}_{\alpha-1}$, we have by Fubini's theorem (its applicability can be checked in the same way as was done in the proof of theorem 3.9) and by theorem 3.4 that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \overline{(K_\alpha y)(x)} v(x) \frac{dx}{|Q_\alpha(x)|^2} \\
&= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} k_\alpha(t,x) \overline{y(t)} \left(\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}} \right) dt \right) v(x) \frac{dx}{|Q_\alpha(x)|^2} \\
&= \int_{-\infty}^{+\infty} \overline{y(t)} \left(\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}} \right) \left(\int_{-\infty}^{+\infty} k_\alpha(t,x) v(x) \frac{dx}{|Q_\alpha(x)|^2} \right) dt \\
&= \int_{-\infty}^{+\infty} \overline{y(t)} v(t) \left(\frac{1}{|Q_\alpha(t)|^2} - \frac{1}{|t|^{2\alpha}} \right) dt.
\end{aligned}$$

Hence,

$$\lim_{R\to\infty} \int_{-R}^{+R} \frac{(\overline{y(t) - (K_\alpha y)(t)}) v(t)}{|Q_\alpha(t)|^2} dt = \lim_{R\to\infty} \int_{-R}^{+R} \frac{\overline{y(t)} v(t)}{|t|^{2\alpha}} dt, \tag{4.38}$$

if one of these limits exists.

Suppose y is the solution of the integral equation $y = Q_\alpha + K_\alpha y$. Then $y - Q_\alpha = K_\alpha y \in \mathcal{V}_{\alpha-1}$. Assuming v satisfies (3.27), the left-hand side of (4.38) equals πc by proposition 3.2, and thus (3.28) is satisfied.

Conversely, suppose y satisfies the 2 conditions of the theorem. Then $w := y - K_\alpha y - Q_\alpha \in \mathcal{V}_{\alpha-1}$. Assuming v satisfies (3.27), we have by (4.38) and proposition 3.2 that

$$\int_{-\infty}^{+\infty} \frac{\overline{w(t)} v(t)}{|Q_\alpha(t)|^2} dt = \lim_{R\to\infty} \int_{-R}^{+R} \frac{\overline{y(t)} v(t)}{|t|^{2\alpha}} dt - \lim_{R\to\infty} \int_{-R}^{+R} \frac{\overline{Q_\alpha(t)} v(t)}{|Q_\alpha(t)|^2} dt = 0.$$

In particular, for $v(t) = k_\alpha(t,z) = \overline{k_\alpha(z,t)}$, we have

$$0 = \int_{-\infty}^{+\infty} \frac{\overline{w(t)} k_\alpha(t,z)}{|Q_\alpha(t)|^2} dt = \overline{\int_{-\infty}^{+\infty} k_\alpha(z,t) w(t) \frac{dt}{|Q_\alpha(t)|^2}} = \overline{w(z)}.$$

Hence, $w = 0$ and y satisfies the integral equation $y = Q_\alpha + K_\alpha y$. $\square$

Proof of Lemma 3.13. Put $p = -\alpha - \frac{1}{2}$. Using the decomposition of the Bessel function in Hankel functions and then expressing the Hankel functions in terms of the Bessel functions of the third kind (cf. [4]), we have

$$
\begin{aligned}
I(\alpha, f, R) &= \frac{1}{2}\int_0^R H_p^{(1)}(x)f(x)x^{p+1}dx + \frac{1}{2}\int_0^R H_p^{(2)}(x)f(x)x^{p+1}dx \\
&= \frac{1}{\pi}\int_0^R (-ix)^{p+1}K_p(-ix)f(x)dx + \frac{1}{\pi}\int_0^R (ix)^{p+1}K_p(ix)f(x)dx \\
&= \frac{1}{\pi}\int_{-R}^{+R} (-ix)^{p+1}K_p(-ix)f(x)dx \\
&= -\frac{1}{\pi}\int_{C_R} (-ix)^{p+1}K_p(-ix)f(x)dx, \qquad (4.39)
\end{aligned}
$$

where in the last step we have applied Cauchy's theorem. We have the asymptotic estimate (cf. [4])

$$K_p(z) = \sqrt{\frac{\pi}{2z}}e^{-z}(1+\mathcal{O}(\frac{1}{|z|})) \quad |z| \to \infty,\ -\frac{3\pi}{2} < \arg(z) < \frac{3\pi}{2}. \qquad (4.40)$$

The result (3.30) follows from substituting (4.40) in (4.39), using (3.29). □

Proof of Theorem 3.14. We first show that the function P_α given by (3.31) satisfies the conditions of theorem 3.12.

That $P_\alpha - Q_\alpha \in \mathcal{V}_{\alpha-1}$, follows from the asymptotic expansion of the Bessel function (cf. [4]). Suppose $v \in \mathcal{V}_{\alpha-1}$ satisfies condition (3.27). Put

$$\tilde{v}(z) = e^{-iz}v(2z), \quad f(z) = \frac{\tilde{v}(z)+\tilde{v}(-z)}{2}, \quad g(z) = \frac{\tilde{v}(z)-\tilde{v}(-z)}{2iz}.$$

Then

$$
\begin{aligned}
\sup_{z\in\mathbb{C}} \frac{|\tilde{v}(z)|}{\cosh(\Im z)(1+|z|)^{\alpha-1}} &< \infty \\
\sup_{z\in\mathbb{C}} \frac{|f(z)|}{\cosh(\Im z)(1+|z|)^{\alpha-1}} &< \infty \\
\sup_{z\in\mathbb{C}} \frac{|g(z)|}{\cosh(\Im z)(1+|z|)^{\alpha-2}} &< \infty
\end{aligned}
$$

and f and g are even entire functions. Then

$$
\begin{aligned}
\frac{1}{\pi}\int_{-R}^{+R}\overline{P_\alpha(t)}v(t)\frac{dt}{|t|^{2\alpha}} &= \frac{2^{1-2\alpha}}{\pi}\int_{-\frac{R}{2}}^{+\frac{R}{2}}\overline{e^{-ix}P_\alpha(2x)}e^{-ix}v(2x)\frac{dx}{|x|^{2\alpha}} \\
&= \frac{2^{-\alpha+\frac{1}{2}}}{\sqrt{\pi}}\int_{-\frac{R}{2}}^{+\frac{R}{2}}(j_{-\alpha-\frac{1}{2}}(x)+ixj_{-\alpha+\frac{1}{2}}(x))(f(x)+ixg(x))\frac{dx}{|x|^{2\alpha}} \\
&= \frac{2^{-\alpha+\frac{1}{2}}}{\sqrt{\pi}}\int_{-\frac{R}{2}}^{+\frac{R}{2}}(j_{-\alpha-\frac{1}{2}}(x)f(x)-x^2j_{-\alpha+\frac{1}{2}}(x)g(x))\frac{dx}{|x|^{2\alpha}} \\
&= \frac{2^{-\alpha+\frac{3}{2}}}{\sqrt{\pi}}\int_0^{\frac{R}{2}}(J_{-\alpha-\frac{1}{2}}(x)f(x)x^{-\alpha+\frac{1}{2}} - J_{-\alpha+\frac{1}{2}}(x)g(x)x^{-\alpha+\frac{3}{2}})dx \\
&= \frac{2^{-\alpha+\frac{3}{2}}}{\sqrt{\pi}}(I(\alpha,f,\frac{R}{2}) - I(\alpha-1,g,\frac{R}{2}))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{-\alpha+1}}{\pi}\left(\int_{C_{\frac{R}{2}}} e^{iz}\frac{f(z)}{(-iz)^{\alpha-1}}\frac{dz}{iz} - \int_{C_{\frac{R}{2}}} e^{iz}\frac{g(z)}{(-iz)^{\alpha-2}}\frac{dz}{iz}\right) + \mathcal{O}(\frac{1}{R}) \\
&= \frac{2^{-\alpha+1}}{\pi}\int_{C_{\frac{R}{2}}} \frac{v(2z)}{(-iz)^{\alpha-1}}\frac{dz}{iz} + \mathcal{O}(\frac{1}{R}) \\
&= \frac{1}{\pi}\int_{C_R} \frac{v(z)}{(-iz)^{\alpha-1}}\frac{dz}{iz} + \mathcal{O}(\frac{1}{R}) \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{v(Re^{i\theta})}{(-iRe^{i\theta})^{\alpha-1}}d\theta + \mathcal{O}(\frac{1}{R}) \\
&\to c \quad \text{as} \quad R \to +\infty.
\end{aligned}
$$

The second formula (3.32) follows from the Poisson formula for the Bessel function (cf. [4])

$$\Gamma(p+\frac{1}{2})J_p(z) = \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^p \int_{-1}^{1} e^{izt}(1-t^2)^{p-\frac{1}{2}}dt \quad (p > -\frac{1}{2}).$$

This gives

$$j_{-\alpha-\frac{1}{2}}(z) = \frac{2^{\alpha+\frac{1}{2}}}{\sqrt{\pi}\Gamma(-\alpha)}\int_{-1}^{1} e^{izt}(1-t^2)^{-\alpha-1}dt, \quad (\alpha < 0)$$

and, by integration by parts,

$$
\begin{aligned}
j_{-\alpha+\frac{1}{2}}(z) &= \frac{2^{\alpha-\frac{1}{2}}}{\sqrt{\pi}\Gamma(1-\alpha)}\int_{-1}^{1} e^{izt}(1-t^2)^{-\alpha}dt, \quad (\alpha < 0) \\
&= -\frac{2^{\alpha+\frac{1}{2}}}{\sqrt{\pi}\Gamma(-\alpha)iz}\int_{-1}^{1} e^{izt}(1-t^2)^{-\alpha-1}t\,dt \quad (\alpha < 0).
\end{aligned}
$$

It follows that for $\alpha < 0$

$$
\begin{aligned}
P_\alpha(2z) &= 2^{\alpha-\frac{1}{2}}\sqrt{\pi}e^{iz}\left(j_{-\alpha-\frac{1}{2}}(z) - izj_{-\alpha+\frac{1}{2}}(z)\right) \\
&= \frac{2^{2\alpha}}{\Gamma(-\alpha)}\int_{-1}^{1} e^{iz(t+1)}(1-t)^{-\alpha}(1+t)^{-\alpha-1}dt \\
&= \frac{1}{\Gamma(-\alpha)}\int_0^1 e^{i2zu}u^{-\alpha-1}(1-u)^{-\alpha}du.
\end{aligned}
$$

In the last step, the change of variable $t = 2u-1$ was performed. The proof of formula (3.32) is now complete. □

4.3 Proofs of subsection 3.3

Proof of Proposition 3.20. T_n is compact since it has finite-dimensional range.
In order to prove that T is compact, it is sufficient to prove that each S_k is compact. As

$$\exists a_\alpha,\, b_\alpha > 0, \forall x \in \mathbb{R} : a_\alpha \le \frac{|Q_\alpha(x)|}{(1+|x|)^\alpha} \le b_\alpha, \tag{4.41}$$

the vector space isomorphism

$$\mathcal{Y}_\alpha \to L^\infty(\mathbb{R}) : y \mapsto \frac{y}{Q_\alpha},$$

where $\mathcal{Y}_\alpha$ is as in definition 3.6, is also a homeomorphism. As S_k corresponds to the compact operator K_{α_k} (cf. definition 3.6) under this isomorphism (with $\alpha = \alpha_k$), S_k is compact. □

Proof of Theorem 3.21. This follows from lemmas 3.23 and 3.24. Indeed, we check the 3 conditions for compact convergence:

1. $$\limsup \|T_n\| \leq \limsup \|\tilde{T}_n\| + \limsup \|T_n - \tilde{T}_n\| = \limsup \|\tilde{T}_n\| < \infty.$$

2. $$\begin{aligned} \|T_n\pi_n - \pi_n T\| &\leq \|T_n - \tilde{T}_n\| \sup \|\pi_n\| + \|\tilde{T}_n\pi_n - \pi_n T\| \\ &= \mathcal{O}\left(\frac{\log n}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right) \\ &= \mathcal{O}\left(\frac{\log n}{n}\right). \end{aligned}$$

3. Suppose (u_n) is a bounded sequence in E_n. Then,

   $$\lim (T_n - \tilde{T}_n) u_n = 0$$

   and $(\tilde{T}_n u_n)$ is a compact sequence so that for every infinite subset N of $\mathbb{N}$ there exists an infinite subset N' of N such that

   $$\lim_{n \in N'} \tilde{T}_n u_n = u$$

   for some $u \in E$. Hence,

   $$\lim_{n \in N'} T_n u_n = \lim_{n \in N'} \tilde{T}_n u_n + \lim_{n \in N'} (T_n - \tilde{T}_n) u_n = u + 0 = u,$$

   and $(T_n u_n)$ is thus a compact sequence.

□

Proof of lemma 3.23. The proof consists of checking the 3 conditions for compact convergence:

1. $$\limsup \|\tilde{T}_n\| \leq \limsup \|\pi_n\| \|T\| \|i_n\| \leq \|T\| < \infty.$$

2. Suppose $u = (u_k) \in \prod_k L^\infty(\mathbb{R})$. Then

   $$\|\tilde{T}_n \pi_n u - \pi_n T u\| \leq \|\pi_n\| \|T(i_n \pi_n u - u)\| \leq \max_k \|S_k y_{k,n}\|,$$

   where

   $$y_{k,n}(x) = \begin{cases} 0 & \text{if } x \in n\Delta_k \\ u_k(x) & \text{otherwise.} \end{cases}$$

Hence

$$\|S_k y_{k,n}\| \le \|u_k\| \sup_x \int_{\mathbb{R}\setminus n\Delta_k} |\kappa_{\alpha_k}(x,t)| dt.$$

Because of the bounds in the lemmas 3.3 and 3.5 and (4.41), the integral on the right-hand side is of order

$$\sup_x \int_{\mathbb{R}\setminus n\Delta_k} \frac{1}{1+|x-t|}\frac{dt}{t^2} \le \int_{\mathbb{R}\setminus n\Delta_k} \frac{dt}{t^2} = \mathcal{O}\left(\frac{1}{n}\right).$$

Hence,

$$\|\tilde{T}_n\pi_n - \pi_n T\| = \mathcal{O}\left(\frac{1}{n}\right).$$

3. Suppose (u_n) is a bounded sequence. Then $(i_n u_n)$ is a bounded sequence in E. As T is compact, there exists for every infinite subset N of $\mathbb{N}$ an infinite subset N' of N such that $(Ti_n u_n)_{n\in N'}$ converges to say y. As

$$\|\tilde{T}u_n - \pi_n y\| \le \|Ti_n u_n - y\|,$$

the sequence $(\tilde{T}u_n)_{n\in N'}$ converges to y.

□

Proof of lemma 3.24. Lemmas 3.28 and 3.29 together show that

$$\sup_{x\in\Delta_k} \int_{\Delta_k} |\frac{1}{2\pi}\kappa_n(z_k e^{ix}, z_k e^{it}) - n\kappa_{\alpha_k}(nx, nt)| dt = \mathcal{O}\left(\frac{\log n}{n}\right).$$

This result and the result of lemma 3.27 complete the proof. □

Proof of proposition 3.25. Using the expansion in theorem 2.2 for $q_n(z_k e^{ix})$, we have

$$q_n(z_k e^{ix})\overline{q_n(z_k e^{it})} = \sum_{l=0}^{p-1} \frac{G_{k,l}(nx, x, nt, t)}{n^{2\alpha_k + l}} + U_{k,p,n}(x,t),$$

where

$$\begin{aligned} U_{k,p,n}(x,t) &= \sum_{j=1}^{p} R_{k,j,n}(x)\overline{\frac{Q_{\alpha_k,p-j}(nt)\chi_k^{(p-j)}(t)}{n^{\alpha_k+p-j}(p-j)!}} + R_{k,0,n}(x)\overline{R_{k,p,n}(t)} \\ &= \sum_{j=0}^{p} R_{k,j,n}(x)\overline{R_{k,p-j,n}(t)} - \sum_{j=1}^{p} R_{k,j,n}(x)\overline{R_{k,p+1-j,n}(t)}. \end{aligned}$$

Hence, in view of (3.4) , the remainder in (3.43) can be put in the form

$$-\frac{z}{z-\zeta}(U_{k,p,n}(x,t) - e^{in(x-t)}U_{k,p,n}(t,x)).$$

It admits the desired bound if for $j = 0, 1, \ldots, p$,

$$\frac{R_{k,j,n}(x)\overline{R_{k,p-j,n}(t)} - e^{in(x-t)}R_{k,j,n}(t)\overline{R_{k,p-j,n}(x)}}{e^{ix} - e^{it}} = \mathcal{O}\left(\frac{\left(\frac{1}{n}+|x|\right)^{\alpha_k}\left(\frac{1}{n}+|t|\right)^{\alpha_k}}{n^p\left(\frac{1}{n}+|z-\zeta|\right)}\right), \quad (4.42)$$

uniformly for $x, t \in \Delta_k$ and if (4.42) also holds if p is replaced by $p+1$ and $j = 1, \ldots, p$. We prove (4.42) by distinguishing between 2 cases:

1. $|z-\zeta| > \frac{1}{2n}$. Then we bound the modulus of the left-hand side of (4.42) by replacing the denominator by a lower bound

$$|z-\zeta| > \frac{1}{3}(\frac{1}{n}+|z-\zeta|)$$

and we bound the numerator by using the bound for $R_{k,j,n}$. (4.42) follows directly.

2. $|z-\zeta| \leq \frac{1}{2n}$. Then we apply the mean value theorem, which permits us to write the left-hand side of (4.42), up to a factor of modulus 1, as

$$R'_{k,j,n}(\xi)\overline{R_{k,p-j,n}(t)} - ine^{in(\xi-t)}R_{k,j,n}(t)\overline{R_{k,p-j,n}(\xi)} - e^{in(\xi-t)}R_{k,j,n}(t)\overline{R'_{k,p-j,n}(\xi)},$$

where ξ lies between x and t. Using the bound for $R_{k,j,n}$ and its derivative, we can bound the above expression by

$$\mathcal{O}\left(n^{-p}(\frac{1}{n}+|\xi|)^{\alpha_k-1}(\frac{1}{n}+|t|)^{\alpha_k}\right) + \mathcal{O}\left(nn^{-p}(\frac{1}{n}+|\xi|)^{\alpha_k}(\frac{1}{n}+|t|)^{\alpha_k}\right)$$
$$+\,\mathcal{O}\left(n^{-p}(\frac{1}{n}+|\xi|)^{\alpha_k}(\frac{1}{n}+|t|)^{\alpha_k-1}\right).$$

As $|\xi-x| \leq \frac{\pi}{2}|e^{i\xi}-e^{ix}| \leq \frac{\pi}{4n}$, we can replace here ξ by x and since

$$\frac{1}{\frac{1}{n}+|x|}, \frac{1}{\frac{1}{n}+|t|} \leq n \leq \frac{\frac{3}{2}}{\frac{1}{n}+|z-\zeta|},$$

we obtain the bound (4.42).

The proof of the bound (4.42) when p is replaced by $p+1$ is similar and in fact easier. □

Proof of lemma 3.26. By theorem 2.2 and (4.41), we have

$$\begin{aligned} q_n(z_k e^{it}) &= \frac{Q_{\alpha_k}(nt)\chi_k(t)}{n^{\alpha_k}} + \mathcal{O}\left(\frac{1}{n}\left(\frac{1}{n}+|t|\right)^{\alpha_k}\right) \\ &= \frac{Q_{\alpha_k}(nt)\chi_k(t)}{n^{\alpha_k}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} |q_n(z_k e^{it})|^{-2} &= \left|\frac{Q_{\alpha_k}(nt)\chi_k(t)}{n^{\alpha_k}}\right|^{-2}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\ &= \left|\frac{Q_{\alpha_k}(nt)\chi_k(t)}{n^{\alpha_k}}\right|^{-2} + \mathcal{O}\left(\frac{1}{n}\left(\frac{1}{n}+|t|\right)^{-2\alpha_k}\right), \end{aligned}$$

which shows (3.48). □

Proof of lemma 3.27. We subdivide Γ_l in two parts $\Gamma_{l,1}$ and $\Gamma_{l,2}$, where $\Gamma_{l,1}$ is an arc at a positive distance from Γ_k and $\Gamma_{l,2}$ is at a positive distance from z_l. The proof then consists of 2 parts:

1. $$\sup_{z\in\Gamma_k}\int_{\Gamma_{l,1}}|\kappa_n(z,\zeta)|d\lambda(\zeta)=\mathcal{O}\left(\frac{1}{n}\right). \tag{4.43}$$

In order to prove this bound, we observe that since $\Gamma_{l,1}$ is at a positive distance from Γ_k, the denominator in (3.4) remains at a positive distance from 0 so that we have the bound

$$|\kappa_n(z,\zeta)|=\mathcal{O}(|q_n(\zeta)|^2)\left|\frac{1}{|q_n(\zeta)|^2}-\frac{1}{|\pi(\zeta)|^2}\right|=\mathcal{O}(1)\left|1-\frac{|q_n(\zeta)|^2}{|\pi(\zeta)|^2}\right|,$$

uniformly for $z\in\Gamma_k,\zeta\in\Gamma_{l,1}$. Proceeding as in the proof of lemma 3.26 we further have

$$|\kappa_n(z,\zeta)|=\mathcal{O}(1)\left|1-\frac{|Q_{\alpha_l}(nt)|^2}{|nt|^{2\alpha_l}}\right|+\mathcal{O}\left(\frac{(\frac{1}{n}+|t|)^{2\alpha_l}}{n|t|^{2\alpha_l}}\right),$$

uniformly for $z\in\Gamma_k,\zeta=z_ke^{it}\in\Gamma_{l,1}$. Hence,

$$\begin{aligned}
&\sup_{z\in\Gamma_k}\int_{\Gamma_{l,1}}|\kappa_n(z,\zeta)|d\lambda(\zeta)\\
&= \mathcal{O}(1)\int_{z_le^{it}\in\Gamma_{l,1}}\left|1-\frac{|Q_{\alpha_l}(nt)|^2}{|nt|^{2\alpha_l}}\right|dt+\mathcal{O}\left(\frac{1}{n}\int_{\Delta_l}\left(\frac{\frac{1}{n}+|t|}{|t|}\right)^{2\alpha_l}dt\right)\\
&= \mathcal{O}(\frac{1}{n})\int_{-\infty}^{+\infty}\left|1-\frac{|Q_{\alpha_l}(s)|^2}{|s|^{2\alpha_l}}\right|ds+\mathcal{O}\left(\frac{1}{n^2}\int_{n\Delta_l}\left(\frac{1+|u|}{|u|}\right)^{2\alpha_l}du\right)\\
&= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

This completes the proof of (4.43).

2. $$\sup_{z\in\Gamma_k}\int_{\Gamma_{l,2}}|\kappa_n(z,\zeta)|d\lambda(\zeta)=\mathcal{O}\left(\frac{\log n}{n}\right). \tag{4.44}$$

As $\Gamma_{l,2}$ is at a positive distance from the singularities, we have, uniformly for $\zeta\in\Gamma_{l,2}$, that

$$q_n(\zeta)=\pi(\zeta)+\mathcal{O}(\frac{1}{n}),$$

whence, again uniformly for $\zeta\in\Gamma_{l,2}$,

$$\frac{1}{|q_n(\zeta)|^2}-\frac{1}{|\pi(\zeta)|^2}=\mathcal{O}\left(\frac{1}{n}\right).$$

As $\Gamma_{l,2}$ is at a positive distance from the singularities we can join it to Γ_k in the proof of proposition 3.25 and state that

$$k_n(z,\zeta)=\mathcal{O}\left(\frac{\left(\frac{1}{n}+|z-z_k|\right)^{\alpha_k}}{\frac{1}{n}+|z-\zeta|}\right)$$

uniformly for $z \in \Gamma_k$ and $\zeta \in \Gamma_{l,2}$. It follows that over this region we have uniformly

$$\kappa_n(z,\zeta) = \mathcal{O}\left(\frac{1}{1+n|z-\zeta|}\right),$$

whence

$$\begin{aligned}
\int_{\Gamma_{l,2}} |\kappa_n(z,\zeta)| d\lambda(\zeta) &= \mathcal{O}\left(\int_{\mathbb{T}} \frac{d\lambda(\zeta)}{1+n|z-\zeta|}\right) \\
&= \mathcal{O}\left(\int_{-\pi}^{+\pi} \frac{dt}{1+n|t|}\right) \\
&= \mathcal{O}\left(\frac{\log n}{n}\right),
\end{aligned}$$

uniformly for $z \in \Gamma_k$. This shows (4.44).

□

Proof of lemma 3.28. The proof consists of 3 consecutive steps.

1. $$\sup_{z\in\Gamma_k} \int_{\Delta_k} |\frac{q_n(z_k e^{it})}{q_n(z)} k_n(z, z_k e^{it})||f_{n,k}(t) - f_{n,k,0}(t)| dt = \mathcal{O}\left(\frac{\log n}{n}\right). \tag{4.45}$$

   Using the estimates for q_n given in theorem 2.2, the bound for k_n given in proposition 3.25 and finally lemma 3.26, we show (4.45) by establishing

   $$\sup_{x\in\Delta_k} \int_{\Delta_k} \frac{1}{1+n|x-t|} dt = \mathcal{O}\left(\frac{\log n}{n}\right),$$

   which is easily verified.

2. $$\sup_{x\in\Delta_k} \int_{\Delta_k} |\frac{q_n(z_k e^{it})}{q_n(z_k e^{ix})}||k_n(z_k e^{ix}, z_k e^{it}) - k_{n,k,0}(z_k e^{ix}, z_k e^{it})||f_{n,k,0}(t)| dt = \mathcal{O}\left(\frac{\log n}{n}\right). \tag{4.46}$$

   This holds if

   $$\sup_{x\in\Delta_k} \int_{\Delta_k} \frac{\left(\frac{1}{n}+|t|\right)^{2\alpha_k}}{1+n|x-t|} |f_{n,k,0}(t)| dt = \mathcal{O}\left(\frac{\log n}{n}\right).$$

   The integral on the left-hand side can be split into 2 parts: the part $|t| \le \frac{1}{2n}$ and the part $|t| > \frac{1}{2n}$. The integral over the first part is

   $$\mathcal{O}(1) \int_{|t|\le\frac{1}{n}} \frac{|f_{n,k,0}(t)|}{n^{2\alpha_k}} dt = \mathcal{O}\left(\frac{1}{n}\right) \int_{|\theta|\le 1} \left| \frac{1}{|Q_{\alpha_k}(\theta)|^2} - \frac{1}{|\theta|^{2\alpha_k}} \right| d\theta = \mathcal{O}\left(\frac{1}{n}\right),$$

   which is suitably bounded. The integral over the second part is bounded by

   $$\mathcal{O}(1) \sup_{x\in\Delta_k} \int_{\Delta_k} \frac{dt}{1+n|x-t|} = \mathcal{O}\left(\frac{\log n}{n}\right).$$

   Hence, (4.46) is established.

3. $$\sup_{x\in\Delta_k}\int_{\Delta_k}|\frac{q_n(z_ke^{it})}{q_n(z_ke^{ix})}-\frac{Q_{\alpha_k}(nt)\chi_k(t)}{Q_{\alpha_k}(nx)\chi_k(x)}||k_{n,k,0}(z_ke^{ix},z_ke^{it})f_{n,k,0}(t)|dt=\mathcal{O}\left(\frac{\log n}{n}\right). \quad (4.47)$$

By theorem 2.2 the first factor of the integrand is

$$\mathcal{O}\left(\frac{\left(\frac{1}{n}+|t|\right)^{\alpha_k}}{n\left(\frac{1}{n}+|x|\right)^{\alpha_k}}\right),$$

so that (4.47) admits the same bound as (4.46).

□

Proof of lemma 3.29. We have for $x,t\in\Delta_k$ that

$$\begin{aligned}
&\frac{n^{2\alpha_k}}{\chi_k(x)\overline{\chi_k(t)}}k_{n,k,0}(z_ke^{ix},z_ke^{it})\\
&= -\left(\frac{ix-it}{1-e^{i(x-t)}}\right)\frac{n}{i}\left(\frac{Q_{\alpha_k}(nx)\overline{Q_{\alpha_k}(nt)}-Q^*_{\alpha_k}(nx)\overline{Q^*_{\alpha_k}(nt)}\frac{\overline{\chi_k(x)}\chi_k(t)}{\chi_k(x)\overline{\chi_k(t)}}}{nx-nt}\right)\\
&= (1+\mathcal{O}(|x-t|))\left(2\pi nk_{\alpha_k}(nx,nt)-i\left(\frac{Q^*_{\alpha_k}(nx)\overline{Q^*_{\alpha_k}(nt)}}{\chi_k(x)\overline{\chi_k(t)}}\right)\left(\frac{\overline{\chi_k(x)}\chi_k(t)-\chi_k(x)\overline{\chi_k(t)}}{x-t}\right)\right)\\
&= (1+\mathcal{O}(|x-t|))\left(2\pi nk_{\alpha_k}(nx,nt)+\mathcal{O}\left((1+|nx|)^{\alpha_k}(1+|nt|)^{\alpha_k}\right)\right).
\end{aligned}$$

By lemma 3.3, we thus have

$$\frac{n^{2\alpha_k}}{\chi_k(x)\overline{\chi_k(t)}}k_{n,k,0}(z_ke^{ix},z_ke^{it})-2\pi nk_{\alpha_k}(nx,nt)=\mathcal{O}\left((1+|nx|)^{\alpha_k}(1+|nt|)^{\alpha_k}\right),$$

uniformly for $x,t\in\Delta_k$. Hence, the left-hand side of (3.51) is bounded by some multiple of

$$\int_{\Delta_k}(1+|nt|)^{2\alpha_k}\left|\frac{1}{|Q_{\alpha_k}(nt)|^2}-\frac{1}{|nt|^{2\alpha_k}}\right|dt\le\frac{1}{n}\int_{-\infty}^{+\infty}(1+|\theta|)^{2\alpha_k}\left|\frac{1}{|Q_{\alpha_k}(\theta)|^2}-\frac{1}{|\theta|^{2\alpha_k}}\right|d\theta=\mathcal{O}\left(\frac{1}{n}\right).$$

□

4.4 Proofs of section 1

Lemma 4.2 *Under the same hyptheses as in theorem 1.1 we have*

$$\frac{p_n(z_ke^{ix})}{p_n(0)}=n^{-\alpha_k}P_{\alpha_k}(nx)\chi_k(x)+\mathcal{O}\left(\frac{\log n}{n}\left(\frac{1}{n}+|x|\right)^{\alpha_k}\right). \quad (4.48)$$

uniformly for $x\in\Delta_k$.

Proof. By theorem 3.17 and theorem 3.21, we have

$$\frac{p_n(z_ke^{ix})}{p_n(0)q_n(z_ke^{ix})}=\frac{P_{\alpha_k}(nx)}{Q_{\alpha_k}(nx)}+\mathcal{O}\left(\frac{\log n}{n}\right),$$

uniformly for $x\in\Delta_k$. (4.48) follows by multiplying this result with $q_n(z_ke^{ix})$ using theorem 2.2. □

Proposition 4.3

$$p_n(0) = 1 + \mathcal{O}\left(\frac{\log n}{n}\right). \tag{4.49}$$

Proof. We have by (1.3) that

$$p_n(0)^{-2} = \int \left|\frac{p_n(z)}{p_n(0)}\right|^2 d\mu(z) = \frac{1}{2\pi}\sum_k \int_{\Delta_k} \left|\frac{p_n(z_k e^{ix})}{p_n(0)}\right|^2 \frac{dx}{|\pi(z_k e^{ix})|^2}. \tag{4.50}$$

Using (4.48), we estimate these integrals

$$\begin{aligned}
&\int_{\Delta_k} \left|\frac{p_n(z_k e^{ix})}{p_n(0)}\right|^2 \frac{dx}{|\pi(z_k e^{ix})|^2} \\
&= \int_{\Delta_k} \frac{|P_{\alpha_k}(nx)|^2}{|nx|^{2\alpha_k}} dx + \mathcal{O}\left(\frac{\log n}{n} \int_{\Delta_k} \frac{\left(\frac{1}{n} + |x|\right)^{2\alpha_k}}{|x|^{2\alpha_k}} dx\right) \\
&= \frac{1}{n}\int_{n\Delta_k} \frac{|P_{\alpha_k}(u)|^2}{|u|^{2\alpha_k}} du + \mathcal{O}\left(\frac{\log n}{n^2} \int_{n\Delta_k} \frac{(1+|u|)^{2\alpha_k}}{|u|^{2\alpha_k}} du\right) \\
&= \text{length}(\Delta_k) + \mathcal{O}\left(\frac{\log n}{n}\right),
\end{aligned} \tag{4.51}$$

since

$$\frac{|P_{\alpha_k}(u)|^2}{|u|^{2\alpha_k}} = 1 + \mathcal{O}\left(\frac{1}{|u|}\right) \quad \text{as } u \to \pm\infty.$$

Substituting (4.51) in (4.50) gives

$$p_n(0)^{-2} = 1 + \mathcal{O}\left(\frac{\log n}{n}\right),$$

which implies (4.49). □

Proof of theorem 1.1. Multiply (4.48) with (4.49). □

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