An Analysis of the Block Structure of j_{qq}-inner Functions

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An Analysis of the Block Structure of j_{qq}-inner Functions [*]

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Abstract.
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0 Introduction

The investigations of D. Z. Arov [2], [3] on Darlington synthesis mark the starting point of the systematical study of J-inner functions. After consideration of the solution set of the Nehari problem [1] he was able to conclude, that the resolvent matrix of a non-degenerate Schur-Nevanlinna-Pick problem is a J-inner function. This results led him to the inverse question of determination of all these J-inner functions which are a resolvent matrix of such a Schur-Nevanlinna-Pick problem. After this considerations, several subclasses of J-inner functions became an interesting object in his further work. Especially the j_{pq}-inner functions proved to be important. The papers [5], [6] about several completion problems of j_{pq}-inner functions give certain information about the structure of such matrix functions. The obtained results lead to a systematical and extensive investigation of the block structure of j_{qq}-inner functions. Hereby, a unique parametrization of j_{qq}-inner functions into a pair of Hardy functions and a so-called singular Carathéodory function is developed. Note that the idea of this representations goes back to Dewilde and Dym [7] and Arov

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[4]. The so-called Arov-Dewilde-Dym parametrization (or short ADD parametrization) induces immediately the associated inverse problem of construction of j_{qq}-inner functions with such terms which are satisfied by the given parametrization. A complete answer will be given in Section 5. In the last section, a first study of interrelations between left and right ADD parametrizations is started. There is a complete characterization for the case, that a given pair of pseudocontinuable Carathéodory functions with finite entropy is just the pair of the left and the right Carathéodory function generated by a certain j_{qq}-inner function.

1 Some Preliminaries And Notations

In the first section we will summarize some facts on several classes of meromorphic functions. For a detailed treatment we refer the reader to the monograph of R. Nevanlinna [15] and P. L. Duren [9]. We will start with some notations. Throughout this paper, let p and q be positive integers. We will use $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$, $\mathbb{C}_0$ and $\mathbb{E}$ to denote the set of complex numbers, the open unit disc, the unit circle, the extended complex plane and the exterior of the closed unit disc, respectively:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}\ ,\ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}\ ,\ \mathbb{C}_0 := \mathbb{C} \cup \{\infty\}\ ,\ \mathbb{E} := \mathbb{C}_0 \setminus (\mathbb{D} \cup \mathbb{T})\ .$$

If $\mathfrak{X}$ is a nonempty set, then $\mathfrak{X}^{p\times q}$ stands for the set of $p \times q$ matrices each entry of which belongs to $\mathfrak{X}$. The null matrix which belongs to $\mathbb{C}^{p\times q}$ will be denoted by $0_{p\times q}$. The identity matrix which belongs to $\mathbb{C}^{q\times q}$ will be designated by I_q. If the size of a null matrix or an identity matrix is clear then we will omit the indexes. The set of all $q \times q$ nonnegative Hermitian matrices will be denoted by $\mathbb{C}^{q\times q}_{\geq}$. A matrix $A \in \mathbb{C}^{p\times q}$ is called *contractive* (respectively, *strictly contractive*) if $I - A^*A$ is nonnegative Hermitian (respectively, positive Hermitian). We will use the notation tr A to denote the trace of a square matrix A. If A belongs to $\mathbb{C}^{q\times q}$, then let Re A and Im A be the real part of A and imaginary part of A, respectively. The linear Lebesgue-Borel measure on $\mathbb{T}$ will be designated by $\underline{\lambda}$ whereas $\mathfrak{B}_{\mathbb{T}}$ stands for the σ-algebra of all Borelian subsets of $\mathbb{T}$. If $t \in (0, \infty)$, then let $\mathcal{L}^t(\mathbb{T})$ denote the set of all Borel measurable functions $g : \mathbb{T} \longrightarrow \mathbb{C}$ for which $|g|^t$ is integrable which respect to $\underline{\lambda}$ on $\mathbb{T}$, whereas $\mathcal{L}^\infty(\mathbb{T})$ stands for the set of all functions $g : \mathbb{T} \longrightarrow \mathbb{C}$ which are bounded $\underline{\lambda}$-almost everywhere on $\mathbb{T}$.

Assume that G is a simply connected domain of $\mathbb{C}_0$. Then let $\mathcal{NM}(G)$ be the *Nevanlinna class* of all functions which are meromorphic in G and which can be represented as a quotient of two bounded holomorphic functions in G. If $g \in [\mathcal{NM}(\mathbb{D})]^{p\times q}$ (respectively, $g \in [\mathcal{NM}(\mathbb{E})]^{p\times q}$), then a well-known theorem due to Fatou implies that there exist a Borelian subset B_0 of the unit circle $\mathbb{T}$ with $\underline{\lambda}(B_0) = 0$ and a Borel measurable function $\underline{g} : \mathbb{T} \longrightarrow \mathbb{C}^{p\times q}$ such that

$$\lim_{r\to 1-0} g(rz) = \underline{g}(z) \qquad \text{(respectively.} \quad \lim_{r\to 1+0} g(rz) = \underline{g}(z)\ )$$

for all $z \in \mathbb{T} \setminus B_0$. In the following, we will continue to use the symbol $\underline{g}$ to denote a radial boundary function of a function g which belongs to $[\mathcal{NM}(\mathbb{D})]^{p\times q}$ or $[\mathcal{NM}(\mathbb{E})]^{p\times q}$.

Let $g \in [\mathcal{NM}(\mathbb{D})]^{p\times q}$. Then one says that *$g$ admits a pseudocontinuation (into $\mathbb{E}$)* if there exists a function $g^\# \in [\mathcal{NM}(\mathbb{E})]^{p\times q}$ such that the radial boundary values $\underline{g}$ and $\underline{g^\#}$ of g

and $g^{\#}$, respectively, coincide $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. It is obvious that a function $g \in [\mathcal{N}\mathcal{M}(\mathbb{D})]^{p\times q}$ admits at most one pseudocontinuation. Note that if $g \in [\mathcal{N}\mathcal{M}(\mathbb{D})]^{p\times q}$ admits a pseudocontinuation $g^{\#}$ and if, additionally, g is analytically continuable through some open arc of $\mathbb{T}$, then the analytic continuation coincides with the pseudocontinuation. In the sequel, we will continue to write $g^{\#}$ for the pseudocontinuation of g.

Let $\mathcal{X}$ be a nonempty subset of the extended complex plane $\mathbb{C}_0$, and let $f : \mathcal{X} \longrightarrow \mathbb{C}^{p\times q}$ be a matrix-valued function. Then we will use the symbol $\widehat{f}$ to denote the function $\widehat{f} : \widehat{\mathcal{X}} \longrightarrow \mathbb{C}^{q\times p}$ which is given by $\widehat{\mathcal{X}} := \{z \in \mathbb{C}_0 : 1/\overline{z} \in \mathcal{X}\}$ and $\widehat{f}(z) := [f(1/\overline{z})]^*$. The following result, which can be easily checked, will play an essential role in our further considerations.

Remark 1 *If f belongs to $[\mathcal{N}\mathcal{M}(\mathbb{D})]^{p\times q}$ (respectively, $[\mathcal{N}\mathcal{M}(\mathbb{E})]^{p\times q}$), then $\widehat{f}$ belongs to $[\mathcal{N}\mathcal{M}(\mathbb{E})]^{q\times p}$ (respectively, $[\mathcal{N}\mathcal{M}(\mathbb{D})]^{q\times p}$) and $\underline{f}^*$ is a radial boundary function of $\widehat{f}$.*

The set of all $g \in \mathcal{N}\mathcal{M}(\mathbb{D})$ which are holomorphic in $\mathbb{D}$ will be denoted by $\mathcal{N}(\mathbb{D})$. The class $\mathcal{N}(\mathbb{D})$ can be described as the set of all functions g which are holomorphic in $\mathbb{D}$ and which fulfill

$$\sup_{r\in[0,1)} \int_{\mathbb{T}} \log^+ |g(rz)| \underline{\lambda}(\mathrm{d}z) < +\infty$$

where $\log^+ x := \max(\log x, 0)$ for each $x \in [0,\infty)$. If $g : \mathbb{D} \longrightarrow \mathbb{C}$ admits a representation

$$g(w) = \alpha \cdot \exp\left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{z+w}{z-w} \log k(z) \underline{\lambda}(\mathrm{d}z) \right\} , \qquad w \in \mathbb{D} ,$$

with some $\alpha \in \mathbb{T}$ and some Borel measurable function $k : \mathbb{T} \longrightarrow [0,\infty)$ which satisfies $(1/2\pi)\int_{\mathbb{T}} |\log k| \mathrm{d}\underline{\lambda} < \infty$, then g belongs to $\mathcal{N}(\mathbb{D})$. Such functions g are called *outer*. For all $g \in \mathcal{N}(\mathbb{D})$, the inequality

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |\underline{g}(z)| \underline{\lambda}(\mathrm{d}z) \leq \lim_{r\to 1-0} \frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |g(rz)| \underline{\lambda}(\mathrm{d}z) \tag{1}$$

holds true. By the *Smirnov class* $\mathcal{N}_+(\mathbb{D})$ we will mean the set of all $g \in \mathcal{N}(\mathbb{D})$ for which equality holds true in (1). The class $\mathcal{N}_+(\mathbb{D})$ proves to be a subalgebra of $\mathcal{N}(\mathbb{D})$. If g is outer in $\mathcal{N}(\mathbb{D})$, then g necessarily belongs to $\mathcal{N}_+(\mathbb{D})$. Note that the Hardy classes $H^t(\mathbb{D})$, $t \in (0,\infty]$, are subsets of $\mathcal{N}_+(\mathbb{D})$.

A function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is called *outer* (in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$) if $\det \Phi$ is outer in $\mathcal{N}(\mathbb{D})$. Basic facts on outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ can be found in [A..]. In particular, if Φ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, then $\det \Phi(w) \neq 0$ for all $w \in \mathbb{D}$ and Φ^{-1} is also an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Conversely, if $\Phi \in [\mathcal{N}(\mathbb{D})]^{q\times q}$ satisfies $\det \Phi(w) \neq 0$ for all $w \in \mathbb{D}$ and if $\Phi^{-1} \in [\mathcal{N}(\mathbb{D})]^{q\times q}$, then Φ and Φ^{-1} are necessarily outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. If $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ and $\Psi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ are outer functions then the product $\Phi\Psi$ is also an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. An outer function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is called *normalized* if $\Phi(0)$ is nonnegative Hermitian.

A function $f : \mathbb{D} \longrightarrow \mathbb{C}^{p\times q}$ is said to be a $p \times q$ *Schur function* if f is both holomorphic and contractive in $\mathbb{D}$. The set $\mathcal{S}_{p\times q}(\mathbb{D})$ of all $p \times q$ Schur functions is obviously a subset of the Hardy class $[H^\infty(\mathbb{D})]^{p\times q}$. A function $f \in \mathcal{S}_{p\times q}(\mathbb{D})$ is called an *inner function* if f has unitary radial boundary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. If $f \in \mathcal{S}_{p\times q}(\mathbb{D})$ has even strictly contractive values $f(z)$ for all $z \in \mathbb{D}$, then f is said to be a *strictly contractive* $p \times q$ *Schur function.*

Let $f \in [\mathcal{N}\mathcal{M}(\mathbb{D})]^{p\times q}$. Then an inner function B that belongs to $\mathcal{S}_{p\times p}(\mathbb{D})$ (respectively, $\mathcal{S}_{q\times q}(\mathbb{D})$) is called a *left* (respectively, *right*) *denominator of* f if Bf (respectively, fB) belongs to $[\mathcal{N}_+(\mathbb{D})]^{p\times q}$. It is readily checked that every function $g \in [\mathcal{N}\mathcal{M}(\mathbb{D})]^{p\times q}$ has left and right denominators. The concept of left and right denominators was created by Arov [3] during his investigations on Darlington synthesis.

2 Left and Right Connected Pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions

In the following, we deal with pairs of matrix-valued Hardy functions. There is an inner connection between the $[H^2(\mathbb{D})]^{q\times q}$-functions which implies immediately, that this functions admit a pseudocontinuation.

Definition 1 *An ordered pair* $[\Phi, \Psi]$ *of functions which belong to* $[H^2(\mathbb{D})]^{q\times q}$ *is called left (respectively, right) connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions if there is an inner* $q \times q$ *Schur function* V *such that*

$$\underline{\Psi} = \underline{V}\ \underline{\Phi}^* \qquad (\textit{respectively, } \underline{\Psi} = \underline{\Phi}^* \underline{V}\ ) \tag{2}$$

holds true $\underline{\lambda}$*-a. e. on* $\mathbb{T}$*. Every such function* V *is said to be an inner function which realizes this left (respectively, right) connection of* $[\Phi, \Psi]$.

Remark 2 $[\Phi, \Psi]$ *is a left connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions if and only if* $[\Psi, \Phi]$ *is a right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions.*

Proposition 1 *Let* $[\Phi, \Psi]$ *be a left (respectively, right) connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions, and let* $V \in \mathcal{S}_{q\times q}(\mathbb{D})$ *be an inner function which realizes this left (respectively, right) connection of* $[\Phi, \Psi]$*. Then both functions* Φ *and* Ψ *admit pseudocontinuations* $\Phi^\#$ *and* $\Psi^\#$ *which satisfy*

$$\Psi = V\widehat{\Phi^\#} \quad \textit{and} \quad \Phi = \widehat{\Psi^\#}V$$
$$\left(\textit{respectively, } \Psi = \widehat{\Phi^\#}V \quad \textit{and} \quad \Phi = V\widehat{\Psi^\#}\ \right)\ . \tag{3}$$

In particular, V *is a left denominator of* $\widehat{\Phi^\#}$ *and a right denominator of* $\widehat{\Psi^\#}$ *(respectively, a right denominator of* $\widehat{\Phi^\#}$ *and a left denominator of* $\widehat{\Psi^\#}$*).*

Proof. From Remark 1 and (2) we see that $\Phi^\# = \widehat{\Psi}\widehat{V}^{-1}$ (respectively, $\Phi^\# = \widehat{V}^{-1}\widehat{\Psi}$) is a pseudocontinuation of Φ, and that $\Psi^\# = \widehat{V}^{-1}\widehat{\Phi}$ (respectively, $\Psi^\# = \widehat{\Phi}\widehat{V}^{-1}$) is a pseudocontinuation of Ψ. In view of $H^2(\mathbb{D}) \subseteq \mathcal{N}_+(\mathbb{D})$ the proof is complete. □

Remark 3 *Let* $[\Phi, \Psi]$ *be a left or right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions. Then* $\det \Phi$ *does not identically vanish if and only if* $\det \Psi$ *does not identically vanish.*

Lemma 1 *Let* $[\Phi, \Psi]$ *be a left (respectively, right) connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions. Suppose that the function* $\det \Phi$ *does not identically vanish. Then* $\det \Psi$ *does not identically vanish, and there is a unique inner function* $V \in \mathcal{S}_{q\times q}(\mathbb{D})$ *which realizes the left (respectively, right) connection of* $[\Phi, \Psi]$*. This function* V *admits the representations*

$$V = \Psi\left(\widehat{\Phi^{\#}}\right)^{-1} \quad \textit{and} \quad V = \left(\widehat{\Psi^{\#}}\right)^{-1}\Phi$$

$$\left(\textit{respectively, } V = \left(\widehat{\Phi^{\#}}\right)^{-1}\Psi \quad \textit{and} \quad V = \Phi\left(\widehat{\Psi^{\#}}\right)^{-1}\right). \tag{4}$$

Proof. From Proposition 1 we obtain that (3) holds. Thus the assumption that $\det \Phi$ does not identically vanish and Remark 3 yield the asserted statements. □

Proposition 2 *Let* Φ *be a function which belongs to* $[H^2(\mathbb{D})]^{q\times q}$ *and which admits a pseudocontinuation* $\Phi^{\#}$*. Let* V *be a left (respectively, right) denominator of* $\widehat{\Phi^{\#}}$*, and let* $\Psi := V\widehat{\Phi^{\#}}$ *(respectively,* $\Psi := \widehat{\Phi^{\#}}V$*). Then* Ψ *belongs to* $[H^2(\mathbb{D})]^{q\times q}$*, and* $[\Phi, \Psi]$ *is a left (respectively, right) connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions where* V *is an inner function which realizes this left (respectively, right) connection of* $[\Phi, \Psi]$*.*

Proof. Since V is a left (respectively, right) denominator of $\widehat{\Phi^{\#}}$ it follows that Ψ and $\underline{V}$ belong to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ and $[\mathcal{L}^\infty(\mathbb{T})]^{q\times q}$ respectively. On the other hand, we have $\underline{\Phi} \in [\mathcal{L}^2(\mathbb{T})]^{q\times q}$. In view of Remark 1, $\underline{\Psi} = \underline{V}\,\underline{\Phi}^*$ holds $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Hence $\underline{\Psi} \in [\mathcal{L}^2(\mathbb{T})]^{q\times q}$. Thus the maximum modulus principle for the Smirnov class (see, e. g., [9]) provides that Ψ even belongs to $[H^2(\mathbb{D})]^{q\times q}$. The rest of the assertion follows immediately. □

Proposition 3 *Let* $\Phi \in [H^2(\mathbb{D})]^{q\times q}$ *and* $\Psi \in [H^2(\mathbb{D})]^{q\times q}$ *be such that* $\det \Phi$ *and* $\det \Psi$ *do not identically vanish in* $\mathbb{D}$*. Then the following statements are equivalent:*

- *(i)* $[\Phi, \Psi]$ *is a left connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions.*
- *(ii)* Φ *admits a pseudocontinuation* $\Phi^{\#}$ *and* $V := \Psi(\widehat{\Phi^{\#}})^{-1}$ *is an inner* $q \times q$ *Schur function.*
- *(iii)* Ψ *admits a pseudocontinuation* $\Psi^{\#}$ *and* $W := (\widehat{\Psi^{\#}})^{-1}\Phi$ *is an inner* $q \times q$ *Schur function.*

Proof. $(i) \Rightarrow (ii)$, $(i) \Rightarrow (iii)$: Use Proposition 1 and Lemma 1.
$(ii) \Rightarrow (i)$: Because of (ii) we see $V\widehat{\Phi^{\#}} = \Psi \in [H^2(\mathbb{D})]^{q\times q} \subseteq [\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Therefore V is a left denominator of $\widehat{\Phi^{\#}}$. From Proposition 2 it follows (i).
$(iii) \Rightarrow (i)$: From (iii) we get $\widehat{\Psi^{\#}}W = \Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Hence W is a right denominator of $\widehat{\Psi^{\#}}$. Thus Proposition 2 shows that $[\Psi, \Phi]$ is a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions. Remark 2 then yields (i). □

Using Remark 2 the following analogous result can be immediately derived from Proposition 3.

Proposition 4 *Let* $\Phi \in [H^2(\mathbb{D})]^{q\times q}$ *and* $\Psi \in [H^2(\mathbb{D})]^{q\times q}$ *be such that* $\det \Phi$ *and* $\det \Psi$ *do not identically vanish in* $\mathbb{D}$*. Then the following statements are equivalent:*

(i) $[\Phi, \Psi]$ *is a right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions.*

(ii) Φ *admits a pseudocontinuation* $\Phi^{\#}$ *and* $V := (\widehat{\Phi^{\#}})^{-1}\Psi$ *is an inner* $q \times q$ *Schur function.*

(iii) Ψ *admits a pseudocontinuation* $\Psi^{\#}$ *and* $W := \Phi(\widehat{\Psi^{\#}})^{-1}$ *is an inner* $q \times q$ *Schur function.*

3 Some Particular Consideration on Matrix-valued Carathéodory Functions

A function $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is said to be a $q \times q$ *Carathéodory function (on* $\mathbb{D}$*)* if Ω is holomorphic in $\mathbb{D}$ and if Re $\Omega(z)$ is nonnegative Hermitian for all $z \in \mathbb{D}$. The set $\mathcal{C}_q(\mathbb{D})$ of all $q \times q$ Carathéodory functions is a subset of $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ (see [11, Corollary 2]). In particular, every $q \times q$ Carathéodory function Ω has radial boundary values $\underline{\Omega}$ with nonnegative Hermitian real part Re $\underline{\Omega}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. If $\Omega \in \mathcal{C}_q(\mathbb{D})$, then the matricial version of a famous theorem due to F. Riesz and Herglotz (see, e. g., [8, Theorem 2.2.2]) provides that there is a unique nonnegative Hermitian-valued Borel measure F on the unit circle $\mathbb{T}$ such that

$$\Omega(w) = \int_{\mathbb{T}} \frac{z+w}{z-w} F(\mathrm{d}z) + i \text{ Im } [\Omega(0)] \tag{5}$$

is satisfied for all $w \in \mathbb{D}$. This nonnegative Hermitian-valued measure F is called the *F. Riesz-Herglotz measure associated with* Ω. A $q \times q$ Carathéodory function Ω (on $\mathbb{D}$) is said to be *absolutely continuous* (respectively, *singular*) if the F. Riesz-Herglotz measure associated with Ω is absolutely continuous (respectively, singular) with respect to the linear Lebesgue-Borel measure $\underline{\lambda}$ on $\mathbb{T}$. In the following, we will use $\mathcal{C}_q^{(a)}(\mathbb{D})$ (respectively, $\mathcal{C}_q^{(s)}(\mathbb{D})$) in order to denote the set of all absolutely continuous (respectively, singular) $q\times q$ Carathéodory functions. A function $\Omega \in \mathcal{C}_q(\mathbb{D})$ is called *normalized* if Im $[\Omega(0)] = 0_{q\times q}$. We will write $\mathcal{C}_q^{\square}(\mathbb{D})$ for the set of all normalized $q \times q$ Carathéodory functions (on $\mathbb{D}$).

Lemma 2 *Every singular* $q \times q$ *Carathéodory function* Ω_s *admits a pseudocontinuation* $\Omega_s^{\#}$*, namely* $\Omega_s^{\#} = -\widehat{\Omega_s}$*, and fulfills* Re $\underline{\Omega_s} = 0_{q\times q}$ $\underline{\lambda}$*-almost everywhere on* $\mathbb{T}$.

Proof. By virtue of [11, Lemma 4], the function Re $\underline{\Omega_s}$ is a version of the Radon-Nikodym derivative of F_a with respect to $(1/2\pi)\underline{\lambda}$ where F_a is the absolutely continuous part in the Lebesgue decomposition of the F. Riesz-Herglotz measure associated with Ω_s with respect to $(1/2\pi)\underline{\lambda}$. Since Ω_s is singular thus it follows Re $\underline{\Omega_s} = 0_{q\times q}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Because of $\mathcal{C}_q(\mathbb{D}) \subseteq [\mathcal{NM}(\mathbb{D})]^{q\times q}$ we see that $g := -\widehat{\Omega_s}$ belongs to $[\mathcal{NM}(\mathbb{E})]^{q\times q}$. From Remark 1 we get finally

$$\underline{g} = -\underline{\Omega_s^*} = -2(\text{Re } \underline{\Omega_s}) + \underline{\Omega_s} = \underline{\Omega_s} \tag{6}$$

$\underline{\lambda}$-almost everywhere on $\mathbb{T}$. □

Lemma 3 *Let* $\Sigma \in [\mathcal{L}^1(\mathbb{T})]^{q\times q}$ *be such that* $\Sigma(z)$ *is nonnegative Hermitian for* $\underline{\lambda}$*-almost all* $z \in \mathbb{T}$*. Then* $\Omega_\Sigma : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ *given by*

$$\Omega_\Sigma(w) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{z+w}{z-w} \Sigma(z) \underline{\lambda}(\mathrm{d}z) \tag{7}$$

belongs to $\mathcal{C}_q^{(a)}(\mathbb{D}) \cap \mathcal{C}_q^{\square}(\mathbb{D})$ *and satisfies* $\mathrm{Re}\ \underline{\Omega_\Sigma} = \Sigma$ $\underline{\lambda}$*-almost everywhere on* $\mathbb{T}$*. The F. Riesz-Herglotz measure* F *associated with* Ω_Σ *admits the representation*

$$F(B) = \frac{1}{2\pi} \int_B \Sigma \ \mathrm{d}\underline{\lambda} \tag{8}$$

for every Borel subset B *of* $\mathbb{T}$*.*

Proof. Let $F : \mathfrak{B}_{\mathbb{T}} \longrightarrow \mathbb{C}^{q\times q}$ be given by (8). Then it is readily checked that F is a Hermitian-valued Borel measure on $\mathbb{T}$ which satisfies

$$\Omega_\Sigma(w) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{z+w}{z-w} F(\mathrm{d}z)$$

for all $w \in \mathbb{D}$. Thus one can easily see that Ω_Σ belongs to $\mathcal{C}_q^{(a)}(\mathbb{D}) \cap \mathcal{C}_q^{\square}(\mathbb{D})$. Lemma 4 in [11] yields finally that $\mathrm{Re}\ \underline{\Omega_\Sigma} = \Sigma$ holds $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. $\square$

If Σ and Ξ are functions which belong to $[\mathcal{L}^1(\mathbb{T})]^{q\times q}$, which have nonnegative Hermitian values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$ and which satisfy $\Sigma = \Xi$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, then the functions Ω_Σ and Ω_Ξ given by (7) coincide, i. e., the function Ω_Σ depends only on the equivalence class $\langle\Sigma\rangle$ of all functions $\Xi : \mathbb{T} \longrightarrow \mathbb{C}^{q\times q}$ which fulfill $\Xi = \Sigma$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. In the following, we will continue to use this notation $\langle\Sigma\rangle$. Moreover, we will write $\Omega_{\langle\Sigma\rangle}$ for the function which is given by $\Omega_{\langle\Sigma\rangle} := \Omega_\Sigma$ and (7).

Lemma 4 *Let* $\Sigma \in [\mathcal{L}^1(\mathbb{T})]^{q\times q}$ *be such that* $\Sigma(z)$ *is nonnegative Hermitian for* $\underline{\lambda}$*-almost all* $z \in \mathbb{T}$*. Then the set*

$$\mathcal{C}_{q,\langle\Sigma\rangle} := \{\Omega \in \mathcal{C}_q(\mathbb{D}) : \langle \mathrm{Re}\ \underline{\Omega}\rangle = \langle\Sigma\rangle\} \tag{9}$$

admits the representation

$$\mathcal{C}_{q,\langle\Sigma\rangle} = \{\Omega_{\langle\Sigma\rangle} + \Omega_s : \Omega_s \in \mathcal{C}_q^{(s)}(\mathbb{D})\}\ . \tag{10}$$

Proof. For each $\Omega \in \mathcal{C}_q(\mathbb{D})$, let F_Ω be the F. Riesz-Herglotz measure associated with Ω, and let $F_{\Omega,a}$ (respectively, $F_{\Omega,s}$) be the absolutely continuous part (respectively, the singular part) of F_Ω in the Lebesgue decomposition of F_Ω with respect to $(1/2\pi)\underline{\lambda}$. According to Lemma 2, we have $\mathrm{Re}\ \underline{\Omega_s} = 0_{q\times q}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$ for every singular $q \times q$ Carathéodory function Ω_s. Applying Lemma 3 we then see that, for each $\Omega_s \in \mathcal{C}_q^{(s)}(\mathbb{D})$, the function $\Omega_\blacksquare := \Omega_{\langle\Sigma\rangle} + \Omega_s$ belongs to $\mathcal{C}_{q,\langle\Sigma\rangle}$. Conversely, now assume that Ω is an arbitrary function which belongs to $\mathcal{C}_{q,\langle\Sigma\rangle}$. Using the arguments mentioned above then we see that $\Omega_\square := \Omega - \Omega_{\langle\Sigma\rangle}$ is a function which is holomorphic in $\mathbb{D}$ and which fulfills

$$\begin{aligned} \Omega_\square(w) &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{z+w}{z-w} \mathrm{Re}\ \underline{\Omega}(z)\underline{\lambda}(\mathrm{d}z) + \int_{\mathbb{T}} \frac{z+w}{z-w} F_{\Omega,s}(\mathrm{d}z) - \frac{1}{2\pi}\int_{\mathbb{T}} \frac{z+w}{z-w}\Sigma(z)\underline{\lambda}(\mathrm{d}z) \\ &= \int_{\mathbb{T}} \frac{z+w}{z-w} F_{\Omega,s}(\mathrm{d}z) \end{aligned}$$

for all $w \in \mathbb{D}$. Thus $\Omega_\square$ is a singular $q\times q$ Carathéodory function satisfying $\Omega = \Omega_{\langle\Sigma\rangle} + \Omega_s$. $\square$

If Φ belongs to $[H^2(\mathbb{D})]^{q\times q}$, then both functions $\underline{\Phi}\ \underline{\Phi}^*$ and $\underline{\Phi}^*\underline{\Phi}$ belong to $[\mathcal{L}^1(\mathbb{T})]^{q\times q}$. In view of Lemma 4 we then introduce the following notion.

Definition 2 *If* $\Phi \in [H^2(\mathbb{D})]^{q\times q}$*, then the set* $\mathcal{C}_{q,\langle \underline{\Phi}\,\underline{\Phi}^* \rangle}$ *(respectively,* $\mathcal{C}_{q,\langle \underline{\Phi}^*\underline{\Phi} \rangle}$*) defined by (9) is called the subclass of* $\mathcal{C}_q(\mathbb{D})$ *which is left (respectively, right) generated by* Φ.

Remark 4 *(a) Let* $[\Phi, \Psi]$ *be a left connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions. Then the subclass of* $\mathcal{C}_q(\mathbb{D})$ *which is left generated by* Φ *coincides with the subclass of* $\mathcal{C}_q(\mathbb{D})$ *which is right generated by* Ψ.
(b) Let $[\Phi, \Psi]$ *be a right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions. Then the subclass of* $\mathcal{C}_q(\mathbb{D})$ *which is right generated by* Φ *coincides with the subclass of* $\mathcal{C}_q(\mathbb{D})$ *which is left generated by* Ψ.

Lemma 5 *Let* Φ *be a function which belongs to* $[H^2(\mathbb{D})]^{q\times q}$ *and which admit a pseudocontinuation* $\Phi^{\#}$*. Then every function* Ω *that belongs to* $\mathcal{C}_{q,\langle \underline{\Phi}\,\underline{\Phi}^* \rangle}$ *admits a pseudocontinuation* $\Omega^{\#}$*, namely* $\Omega^{\#} = 2\Phi^{\#}\widehat{\Phi} - \widehat{\Omega}$.

Proof. Let $\Omega \in \mathcal{C}_{q,\langle \underline{\Phi}\,\underline{\Phi}^* \rangle}$, i. e., Ω belongs to $\mathcal{C}_q(\mathbb{D})$ and satisfies $\underline{\Omega} + \underline{\Omega}^* = 2\underline{\Phi}\,\underline{\Phi}^*$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Since $[H^2(\mathbb{D})]^{q\times q}$ and $\mathcal{C}_q(\mathbb{D})$ are subsets of $[\mathcal{NM}(\mathbb{D})]^{q\times q}$, the function $g := 2\Phi^{\#}\widehat{\Phi} - \widehat{\Omega}$ belongs to $[\mathcal{NM}(\mathbb{E})]^{q\times q}$. From Remark 1 we then see that $\underline{g} = 2\underline{\Phi}\,\underline{\Phi}^* - \underline{\Omega}^* = \underline{\Omega}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Hence g is a pseudocontinuation of Ω. $\square$

Analogously, the following result can be proved.

Lemma 6 *Let* Φ *be a function which belongs to* $[H^2(\mathbb{D})]^{q\times q}$ *and which admit a pseudocontinuation* $\Phi^{\#}$*. Then every function* Ω *that belongs to* $\mathcal{C}_{q,\langle \underline{\Phi}^*\underline{\Phi} \rangle}$ *admits a pseudocontinuation* $\Omega^{\#}$*, namely* $\Omega^{\#} = 2\widehat{\Phi}\Phi^{\#} - \widehat{\Omega}$.

4 Some Remarks on J-inner Functions

Throughout this section, let m be a positive integer, and let J be an $m \times m$ signature matrix, i. e., J belongs to $\mathbb{C}^{m\times m}$ and satisfies as well $J = J^*$ as $J^2 = I$. A matrix $A \in \mathbb{C}^{m\times m}$ is called *J-contractive* if $B := J - A^*JA$ is nonnegative Hermitian. If $A \in \mathbb{C}^{m\times m}$ even satisfies $A^*JA = J$, then A is said to be *J-unitary.* The *Potapov class* $\mathfrak{P}_J(\mathbb{D})$ consists of all $m \times m$ matrix-valued functions W which satisfy the following three conditions:
(i) W is meromorphic in $\mathbb{D}$.
(ii) The function $\det W$ does not identically vanish in $\mathbb{D}$.
(iii) For each z which belongs to the set $\mathbb{H}_W$ of all points of analyticity of W, the matrix $W(z)$ is J-contractive.

The Potapov class $\mathfrak{P}_J(\mathbb{D})$ is a subclass of $[\mathcal{NM}(\mathbb{D})]^{m\times m}$ (see, e. g., [10, Corollary 2]). In particular, every function W which belongs to $\mathfrak{P}_J(\mathbb{D})$ has radial boundary values $\underline{W}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. If $W \in \mathfrak{P}_J(\mathbb{D})$ satisfies $\underline{W}^*J\underline{W} = J$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, then W is said to be a *J-inner function.*

Remark 5 *Every J-inner function W admits a pseudocontinuation $W^{\#}$ (into $\mathbb{E}$). For each $z \in \mathbb{E}$ which fulfills $1/\overline{z} \in \mathbb{H}_W$ and* $\det W(1/\overline{z}) \neq 0$ *this pseudocontinuation $W^{\#}$ admits the representation* $W^{\#}(z) = J[W(1/\overline{z})]^{-*}J$.

Now we will focus our attention to the special $2q \times 2q$ signature matrix

$$j_{qq} := \operatorname{diag}\,(I_q, -I_q)\,. \tag{11}$$

In the sequel, when we will consider a $2q \times 2q$ complex matrix W or a $2q \times 2q$ matrix-valued function W, then we will often work with the $q \times q$ block partition

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{12}$$

of W. Furthermore, if a $2q \times 2q$ matrix-valued function W is given, then we will use $\mathbb{H}_W$ to denote the set of all points of analyticity of W.

A useful tool to treat problems which are formulated for functions which belong to $\mathfrak{P}_{j_{qq}}(\mathbb{D})$ is the transformation into the Schur class. The following result gives a summary of facts which are useful to do this.

Proposition 5 *Let $W \in \mathfrak{P}_{j_{qq}}(\mathbb{D})$. Then the following statements are fulfilled:*

(a) *For each $z \in \mathbb{H}_W$, the inequalities* $\det W_{22}(z) \neq 0$, $\det[W_{22}(z) + W_{21}(z)] \neq 0$ *and* $\det[W_{22}(z) + W_{12}(z)] \neq 0$ *hold true. Moreover, the functions* $\det(\widehat{W_{11}^{\#}} + \widehat{W_{12}^{\#}})$ *and* $\det(\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}})$ *do not identically vanish.*

(b) *The function*

$$S := \begin{bmatrix} W_{11} - W_{12}W_{22}^{-1}W_{21} & W_{12}W_{22}^{-1} \\ -W_{22}^{-1}W_{21} & W_{22}^{-1} \end{bmatrix} \tag{13}$$

belongs to the Schur class $\mathcal{S}_{2q \times 2q}(\mathbb{D})$. In particular, $S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21}$, $S_{12} := W_{12}W_{22}^{-1}$, $S_{21} := -W_{22}^{-1}W_{21}$ and $S_{22} := W_{22}^{-1}$ are matrix-valued Schur functions, whereby S_{12} and S_{21} are even strictly contractive.

(c) *The functions* $\det S_{11}$ *and* $\det S_{22}$ *do not identically vanish.*

(d) $\mathbb{H}_W = \{z \in \mathbb{D} : \det S_{22}(z) \neq 0\}$.

(e) *If W is a j_{qq}-inner function, then S is an inner function and the following identities are valid:*

$$S_{12} = (\widehat{W_{11}^{\#}})^{-1}\widehat{W_{21}^{\#}}\ ,\ \ S_{21} = -\widehat{W_{12}^{\#}}(\widehat{W_{11}^{\#}})^{-1} \text{ and } S_{11} = (\widehat{W_{11}^{\#}})^{-1}. \tag{14}$$

A proof of the results stated in Proposition 5 is given in [3], [13], [7] and [10]. Observe that the function S defined by (13) is called the *Potapov-Ginzburg transform of W (with respect to j_{qq}).* In some sense, the following result, which can be verified by straightforward calculation, is a converse statement to part (b) of Proposition 5.

Proposition 6 *Let W be an inner $2q \times 2q$ Schur function, and let (12) be the $q \times q$ block partition of W. Suppose that* $\det W_{22}$ *does not identically vanish. Then S given by (13) is a j_{qq}-inner function.*

5 An Analysis of the Block Structure of j_{qq}-inner Functions

In this section, we investigate the inner block structure of j_{qq}-inner functions. We will give a representation of such functions in terms of a pair of Hardy functions and a singular Carathéodory function.

Proposition 7 *Let W be a j_{qq}-inner function. Then:*

(a) The pair $[\Phi_{W,l}, \Psi_{W,l}]$ given by

$$\Phi_{W,l} := (W_{22} + W_{21})^{-1} \qquad \textit{and} \qquad \Psi_{W,l} := (\widehat{W^{\#}_{11}} + \widehat{W^{\#}_{12}})^{-1} \tag{15}$$

is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions, where

$$V_{W,l} := (W_{11} + W_{12})(W_{22} + W_{21})^{-1} \tag{16}$$

is the unique inner $q \times q$ Schur function which realizes this left connection.

(b) Both functions $\Phi_{W,l}$ and $\Psi_{W,l}$ admit pseudocontinuations $\Phi^{\#}_{W,l}$ and $\Psi^{\#}_{W,l}$, respectively, and satisfy the identities

$$V_{W,l}\widehat{\Phi^{\#}_{W,l}} = \Psi_{W,l} \qquad \textit{and} \qquad \widehat{\Psi^{\#}_{W,l}}V_{W,l} = \Phi_{W,l} \ . \tag{17}$$

(c) The function

$$\Omega_{W,l} := (W_{22} + W_{21})^{-1}(W_{22} - W_{21}) \tag{18}$$

belongs to the subclass $\mathcal{C}_{q,\langle\underline{\Phi_{W,l}}\,\underline{\Phi^*_{W,l}}\rangle}$ of $\mathcal{C}_q(\mathbb{D})$ which is left generated by $\Phi_{W,l}$ and which admits the representation

$$\Omega_{W,l} = (I + W_{22}^{-1}W_{21})^{-1}(I - W_{22}^{-1}W_{21}) \ . \tag{19}$$

Proof. From Proposition 5 we see that $S_{11} := (\widehat{W^{\#}_{11}})^{-1}$, $S_{22} := W_{22}^{-1}$, $S_{12} := (\widehat{W^{\#}_{11}})^{-1}\widehat{W^{\#}_{21}}$ and $S_{21} := -W_{22}^{-1}W_{21}$ are (well-defined) $q \times q$ Schur functions where both functions S_{12} and S_{21} are even strictly contractive. In view of a result due to Arov [3] we can conclude that $T_{21} := I - S_{21}$ and $T_{12} := I + S_{12}$ are outer functions in $[H^\infty(\mathbb{D})]^{q\times q}$. Thus $(I - S_{21})^{-1}$ and $(I + S_{12})^{-1}$ are outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Since $\mathcal{N}_+(\mathbb{D})$ is an algebra over $\mathbb{D}$, we get from the identities

$$\Phi_{W,l} = (I - S_{21})^{-1}S_{22} \qquad \text{and} \qquad \Psi_{W,l} = (I + S_{12})^{-1}S_{11} \tag{20}$$

that $\Phi_{W,l}$ and $\Psi_{W,l}$ belong to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Thus, according to the maximum modulus principle for the Smirnov class (see, e. g., [9]), it is sufficient to verify that the radial boundary functions $\underline{\Phi_{W,l}}$ and $\underline{\Psi_{W,l}}$ of $\Phi_{W,l}$ and $\Psi_{W,l}$, respectively, belong to $[\mathcal{L}^2(\mathbb{T})]^{q\times q}$ in order to prove that $\Phi_{W,l}$ and $\Psi_{W,l}$ belong to $[H^2(\mathbb{D})]^{q\times q}$. Since $\Omega_{W,l}$ admits the representation $\Omega_{W,l} = (I - S_{21})^{-1}(I + S_{21})$ the function $\Omega_{W,l}$ belongs to $\mathcal{C}_q(\mathbb{D})$ (see, e. g., [8, Propositions 2.1.2 and 2.1.3]). Since W has j_{qq}-unitary radial boundary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, we obtain from a result due to Potapov [16] (see also [8, Theorem 1.3.3]) that $\underline{W}^*$ has j_{qq}-unitary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Consequently, we have

$$\underline{W_{22}}\,\underline{W_{22}^*} - \underline{W_{21}}\,\underline{W_{21}^*} = I \tag{21}$$

$\underline{\lambda}$-almost everywhere on $\mathbb{T}$ and hence

$$\begin{aligned} \operatorname{Re}\,\underline{\Omega_{W,l}} &= \frac{1}{2}(\underline{\Omega_{W,l}} + \underline{\Omega_{W,l}^*}) = (\underline{W_{22}} + \underline{W_{21}})^{-1}(\underline{W_{22}}\,\underline{W_{22}^*} - \underline{W_{21}}\,\underline{W_{21}^*})(\underline{W_{22}} + \underline{W_{21}})^{-*} \\ &= \underline{\Phi_{W,l}}\,\underline{\Phi_{W,l}^*} \end{aligned} \tag{22}$$

$\underline{\lambda}$-almost everywhere on $\mathbb{T}$. From [11, Lemma 4] we know that $\operatorname{Re}\,\underline{\Omega_{W,l}}$ belongs to $[\mathcal{L}^1(\mathbb{T})]^{q\times q}$. Thus the identity (22) provides $\underline{\Phi_{W,l}} \in [\mathcal{L}^2(\mathbb{T})]^{q\times q}$. Therefore, $\Phi_{W,l} \in [H^2(\mathbb{D})]^{q\times q}$ is verified. Let $e := (I_q, I_q)$. Since W is a j_{qq}-inner function, the function $g := e(j_{qq} - W^* j_{qq} W)e^*$ has nonnegative Hermitian values the radial boundary values of which fulfill $\underline{g} = 0_{q\times q}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Using the block form of this inequality and this equality, one can easily verify that $V_{W,l}$ is an inner $q \times q$ Schur function. Since W has j_{qq}-unitary radial boundary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$ we get from Remark 1 that $\underline{V_{W,l}}\,\underline{\Phi_{W,l}^{-1}} = \underline{\Psi_{W,l}^{-*}}$ and hence $\underline{\Psi_{W,l}} = \underline{V_{W,l}}\,\underline{\Phi_{W,l}^*}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Taking into account that $V_{W,l}$ is an inner $q \times q$ Schur function and that $\underline{\Phi_{W,l}}$ belongs to $[\mathcal{L}^2(\mathbb{T})]^{q\times q}$ we thus see that $\underline{\Psi_{W,l}}$ also belongs to $[\mathcal{L}^2(\mathbb{T})]^{q\times q}$. Hence $\Psi_{W,l}$ belongs to $[H^2(\mathbb{D})]^{q\times q}$. Consequently, we get that $[\Phi_{W,l}, \Psi_{W,l}]$ is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions. In view of Lemma 1 it follows that $V_{W,l}$ is the unique inner function which realizes this left connection. Proposition 1 yields that $\Phi_{W,l}$ and $\Psi_{W,l}$ admit pseudocontinuations and that (17) holds true. Finally, since the $q \times q$ Carathéodory function $\Omega_{W,l}$ satisfies (22) we see that $\Omega_{W,l}$ belongs to $\mathcal{C}_{q,\langle \underline{\Phi_{W,l}}\,\underline{\Phi_{W,l}^*}\rangle}$. $\square$

Analogously to Proposition 7 the following result can be proved.

Proposition 8 *Let W be a j_{qq}-inner function. Then:*

(a) The pair $[\Phi_{W,r}, \Psi_{W,r}]$ given by

$$\Phi_{W,r} := (W_{22} + W_{12})^{-1} \quad \textit{and} \quad \Psi_{W,r} := (\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}})^{-1} \tag{23}$$

is a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions, where

$$V_{W,r} := (W_{22} + W_{12})^{-1}(W_{11} + W_{21}) \tag{24}$$

is the unique inner $q \times q$ Schur function which realizes this right connection.

(b) Both functions $\Phi_{W,r}$ and $\Psi_{W,r}$ admit pseudocontinuations $\Phi_{W,r}^{\#}$ and $\Psi_{W,r}^{\#}$, respectively, and satisfy the identities

$$\widehat{\Phi_{W,r}^{\#}} V_{W,r} = \Psi_{W,r} \quad \textit{and} \quad V_{W,r}\widehat{\Psi_{W,r}^{\#}} = \Phi_{W,r}\ . \tag{25}$$

(c) The function

$$\Omega_{W,r} := (W_{22} - W_{12})(W_{22} + W_{12})^{-1} \tag{26}$$

belongs to the subclass $\mathcal{C}_{q,\langle \underline{\Phi_{W,r}^*}\,\underline{\Phi_{W,r}}\rangle}$ of $\mathcal{C}_q(\mathbb{D})$ which is right generated by $\Phi_{W,r}$ and which admits the representation

$$\Omega_{W,r} = (I - W_{12}W_{22}^{-1})(I + W_{12}W_{22}^{-1})^{-1}\ . \tag{27}$$

Propositions 7 and 8 lead us to the following notions.

Definition 3 *Let W be a j_{qq}-inner function.*
(a) The pair $[\Phi_{W,l}, \Psi_{W,l}]$ given by (15) (respectively, $[\Phi_{W,r}, \Psi_{W,r}]$ given by (23)) is called the left (respectively, right) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W.
(b) The function $\Omega_{W,l}$ given by (18) (respectively, $\Omega_{W,r}$ given by (26)) is said to be the left (respectively, right) $q \times q$ Carathéodory function generated by W.

Proposition 9 *Let W be a j_{qq}-inner function, and let $[\Phi_{W,l}, \Psi_{W,l}]$ be the left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W. Then:*

(a) The left $q \times q$ Carathéodory function $\Omega_{W,l}$ generated by W can be represented via

$$\Omega_{W,l} = (\widehat{W_{11}^{\#}} - \widehat{W_{12}^{\#}})(\widehat{W_{11}^{\#}} + \widehat{W_{12}^{\#}})^{-1} . \tag{28}$$

Moreover, $\Omega_{W,l}$ admits a pseudocontinuation $\Omega_{W,l}^{\#}$ which satisfies the identities

$$\Omega_{W,l} + \widehat{\Omega_{W,l}^{\#}} = 2\Phi_{W,l}\widehat{\Phi_{W,l}^{\#}} \tag{29}$$

and

$$\widehat{\Omega_{W,l}^{\#}} = (W_{11} + W_{12})^{-1}(W_{11} - W_{12}) . \tag{30}$$

(b) The function

$$\Omega_{W,l,s} := \Omega_{W,l} - \Omega_{\langle \underline{\Phi_{W,l}}\, \underline{\Phi_{W,l}^*} \rangle} \tag{31}$$

is a singular $q \times q$ Carathéodory function.

Proof. From Remark 5 we know that all the $q \times q$ matrix-valued functions W_{11}, W_{12}, W_{21} and W_{22} admit pseudocontinuations. Hence $V := W_{22} - W_{21}$ admits a pseudocontinuation. On the other hand, we know from Proposition 1, that $\Phi_{W,l}$ admits a pseudocontinuation. As the product of the pseudocontinuable functions $\Phi_{W,l}$ and V the function $\Omega_{W,l}$ admits a pseudocontinuation $\Omega_{W,l}^{\#}$ as well. From Proposition 7 we see that $\Omega_{W,l}$ belongs to $\mathcal{C}_{q,\langle \underline{\Phi_{W,l}}\, \underline{\Phi_{W,l}^*} \rangle}$. In particular, we have $\operatorname{Re} \underline{\Omega_{W,l}} = \underline{\Phi_{W,l}}\ \underline{\Phi_{W,l}^*}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Thus Remark 1 implies (29). Moreover, since $\underline{W}$ is j_{qq}-unitary $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, we get

$$\underline{W_{21}}\, \underline{W_{11}^*} - \underline{W_{22}}\, \underline{W_{12}^*} = 0 \qquad \underline{\lambda}\text{-a. e. on } \mathbb{T} \tag{32}$$

and hence

$$(\underline{W_{22}} - \underline{W_{21}})(\underline{W_{11}} + \underline{W_{12}})^* = (\underline{W_{22}} + \underline{W_{21}})(\underline{W_{11}} - \underline{W_{12}})^* \quad \underline{\lambda}\text{-a. e. on } \mathbb{T}. \tag{33}$$

Using Remark 1 we thus obtain

$$(W_{22} - W_{21})(\widehat{W_{11}^{\#}} + \widehat{W_{12}^{\#}}) = (W_{22} + W_{21})(\widehat{W_{11}^{\#}} - \widehat{W_{12}^{\#}}) . \tag{34}$$

From Proposition 5 then it follows (28). Remark 1 implies

$$\underline{\Omega_{W,l}^*} = [(\underline{W_{11}^*} - \underline{W_{12}^*})(\underline{W_{11}^*} + \underline{W_{12}^*})^{-1}]^* \tag{35}$$

and consequently (30). Finally, the assertion stated in part (b) is an immediate consequence of $\Omega_{W,l} \in \mathcal{C}_{q,\langle \underline{\Phi_{W,l}}\, \underline{\Phi_{W,l}^*} \rangle}$ and Lemma 4. □

Similarly to Proposition 9, an analogous result for right connected pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions can be proved.

Proposition 10 *Let W be a j_{qq}-inner function, and let $[\Phi_{W,r}, \Psi_{W,r}]$ be the right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W. Then:*

(a) The right $q \times q$ Carathéodory function $\Omega_{W,r}$ generated by W can be represented via

$$\Omega_{W,r} = (\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}})^{-1}(\widehat{W_{11}^{\#}} - \widehat{W_{21}^{\#}}) \ . \tag{36}$$

Moreover, $\Omega_{W,r}$ admits a pseudocontinuation $\Omega^{\#}_{W,r}$ which satisfies the identities

$$\Omega_{W,r} + \widehat{\Omega^{\#}_{W,r}} = 2\widehat{\Phi^{\#}_{W,r}}\Phi_{W,r} \tag{37}$$

and

$$\widehat{\Omega^{\#}_{W,r}} = (W_{11} - W_{21})(W_{11} + W_{21})^{-1} \ . \tag{38}$$

(b) The function

$$\Omega_{W,r,s} := \Omega_{W,r} - \Omega_{\langle \underline{\Phi^{*}_{W,r}}\, \underline{\Phi_{W,r}} \rangle} \tag{39}$$

is a singular $q \times q$ Carathéodory function.

Definition 4 *Let W be a j_{qq}-inner function. Then $\Omega_{W,l,s}$ defined by (31) (respectively, $\Omega_{W,r,s}$ defined by (39)) is called the left (respectively, right) singular $q \times q$ Carathéodory function generated by W.*

Theorem 1 *Let W be a j_{qq}-inner function, let $[\Phi_{W,l}, \Psi_{W,l}]$ and $\Omega_{W,l}$ be the left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions and the left $q\times q$ Carathéodory function, respectively, generated by W. Then W admits the representation*

$$W = \frac{1}{2} \cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}_{W,l}})^{-1}, \Phi^{-1}_{W,l}\right] \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}_{W,l}} & I - \widehat{\Omega^{\#}_{W,l}} \\ I - \Omega_{W,l} & I + \Omega_{W,l} \end{bmatrix} \ . \tag{40}$$

If $[\Phi, \Psi]$ is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions such that the function $\det \Phi$ *does not identically vanish in $\mathbb{D}$ and if Ω is a $q \times q$ Carathéodory function which admits a pseudocontinuation $\Omega^{\#}$ such that the representation*

$$W = \frac{1}{2} \cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}})^{-1}, \Phi^{-1}\right] \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}} & I - \widehat{\Omega^{\#}} \\ I - \Omega & I + \Omega \end{bmatrix} \tag{41}$$

of W is satisfied, then $\Phi = \Phi_{W,l}$, $\Psi = \Psi_{W,l}$ and $\Omega = \Omega_{W,l}$.

Proof. In view of Remark 1, it is sufficient to verify

$$\underline{W} = \frac{1}{2} \cdot \operatorname{diag}\left[\underline{\Psi^{-*}_{W,l}}, \underline{\Phi^{-1}_{W,l}}\right] \cdot \begin{bmatrix} I + \underline{\Omega^{*}_{W,l}} & I - \underline{\Omega^{*}_{W,l}} \\ I - \underline{\Omega_{W,l}} & I + \underline{\Omega_{W,l}} \end{bmatrix} \qquad \underline{\lambda}\text{-a. e. on } \mathbb{T} \tag{42}$$

in order to prove (40). However, according to Remark 1 and equation (30) we get

$$\begin{aligned} \frac{1}{2}\underline{\Psi^{-*}_{W,l}}(I + \underline{\Omega^{*}_{W,l}}) &= \frac{1}{2}(\underline{W_{11}} + \underline{W_{12}})[I + (\underline{W_{11}} + \underline{W_{12}})^{-1}(\underline{W_{11}} - \underline{W_{12}})] \\ &= \underline{W_{11}} \end{aligned}$$

and analogously

$$\frac{1}{2}\underline{\Psi^{-*}_{W,l}}(I - \underline{\Omega^{*}_{W,l}}) = \underline{W_{12}} \quad , \quad \frac{1}{2}\underline{\Phi^{-1}_{W,l}}(I - \underline{\Omega_{W,l}}) = \underline{W_{21}} \quad , \quad \frac{1}{2}\underline{\Phi^{-1}_{W,l}}(I + \underline{\Omega_{W,l}}) = \underline{W_{22}}$$

$\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Thus (42) and hence (40) are checked. Now assume that $[\Phi, \Psi]$ is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions such that $\det\Phi$ does not identically vanish in $\mathbb{D}$. Then Proposition 1 shows that $\det\widehat{\Psi^{\#}}$ does not identically vanish in $\mathbb{D}$. Further assume that Ω is a pseudocontinuable $q\times q$ Carathéodory function such that (41) is fulfilled. Then

$$W_{21} + W_{22} = \frac{1}{2}\Phi^{-1}[(I - \Omega) + (I + \Omega)] = \Phi^{-1} \tag{43}$$

and therefore $\Phi = \Phi_{W,l}$. Using (43) and

$$W_{22} - W_{21} = \frac{1}{2}\Phi^{-1}[(I + \Omega) - (I - \Omega)] = \Phi^{-1}\Omega \ ,$$

we can conclude that $\Omega = (W_{22} + W_{21})^{-1}W_{22} - W_{21} = \Omega_{W,l}$. Moreover, we see from (41) that

$$W_{11} + W_{12} = \frac{1}{2}(\widehat{\Psi^{\#}})^{-1}[(I + \widehat{\Omega^{\#}}) + (I - \widehat{\Omega^{\#}})] = (\widehat{\Psi^{\#}})^{-1}$$

and consequently $\widehat{\Psi^{\#}} = (W_{11} + W_{12})^{-1}$. This implies finally $\Psi = \Psi_{W,l}$. □

Corollary 1 *Let W be a j_{qq}-inner function. Then there exist a unique left connected pair $[\Phi, \Psi]$ of $[H^2(\mathbb{D})]^{q\times q}$-functions such that the functions* $\det\Phi$ *and* $\det\Psi$ *do not identically vanish in $\mathbb{D}$ and a unique singular $q\times q$ Carathéodory function Ω_s such that*

$$W = \frac{1}{2}\cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}})^{-1}, \Phi^{-1}\right]\cdot \begin{bmatrix} I + \widehat{\Omega^{\#}_{\langle\underline{\Phi}\,\underline{\Phi}^*\rangle}} + \widehat{\Omega^{\#}_{s}} & I - \widehat{\Omega^{\#}_{\langle\underline{\Phi}\,\underline{\Phi}^*\rangle}} - \widehat{\Omega^{\#}_{s}} \\ I - \Omega_{\langle\underline{\Phi}\,\underline{\Phi}^*\rangle} - \Omega_s & I + \Omega_{\langle\underline{\Phi}\,\underline{\Phi}^*\rangle} + \Omega_s \end{bmatrix} \ , \tag{44}$$

namely the left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions and the left singular $q\times q$ Carathéodory function generated by W.

Proof. From Lemma 2 we know that every singular $q\times q$ Carathéodory function admits a pseudocontinuation. Further we can conclude from Proposition 1 and Lemma 5 that $\Omega_{\langle\underline{\Phi}\,\underline{\Phi}^*\rangle}$ also admits a pseudocontinuation. Thus the application of Theorem 1 and Lemma 4 provides the assertion. □

Analogously to Theorem 1 the following result can be verified.

Theorem 2 *Let W be a j_{qq}-inner function, let $[\Phi_{W,r}, \Psi_{W,r}]$ and $\Omega_{W,r}$ be the right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions and the right $q\times q$ Carathéodory function, respectively, generated by W. Then W admits the representation*

$$W = \frac{1}{2}\cdot \begin{bmatrix} I + \widehat{\Omega^{\#}_{W,r}} & I - \Omega_{W,r} \\ I - \widehat{\Omega^{\#}_{W,r}} & I + \Omega_{W,r} \end{bmatrix}\cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}_{W,r}})^{-1}, \Phi^{-1}_{W,r}\right] \ . \tag{45}$$

If $[\Phi, \Psi]$ *is a right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions such that the function* $\det \Phi$ *does not identically vanish in* $\mathbb{D}$ *and if* Ω *is a* $q \times q$ *Carathéodory function which admits a pseudocontinuation* $\Omega^{\#}$ *such that the representation*

$$W = \frac{1}{2} \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}} & I - \Omega \\ I - \widehat{\Omega^{\#}} & I + \Omega \end{bmatrix} \cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}})^{-1}, \Phi^{-1}\right] \tag{46}$$

of W *is satisfied, then* $\Phi = \Phi_{W,r}$*,* $\Psi = \Psi_{W,r}$ *and* $\Omega = \Omega_{W,r}$*.*

Corollary 2 *Let* W *be a* j_{qq}*-inner function. Then there exist a unique right connected pair* $[\Phi, \Psi]$ *of* $[H^2(\mathbb{D})]^{q\times q}$*-functions such that the functions* $\det \Phi$ *and* $\det \Psi$ *do not identically vanish in* $\mathbb{D}$ *and a unique singular* $q \times q$ *Carathéodory function* Ω_s *such that*

$$W = \frac{1}{2} \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}_{\langle \underline{\Phi}^* \underline{\Phi}\rangle}} + \widehat{\Omega^{\#}_s} & I - \Omega_{\langle \underline{\Phi}^* \underline{\Phi}\rangle} - \Omega_s \\ I - \widehat{\Omega^{\#}_{\langle \underline{\Phi}^* \underline{\Phi}\rangle}} - \widehat{\Omega^{\#}_s} & I + \Omega_{\langle \underline{\Phi}^* \underline{\Phi}\rangle} + \Omega_s \end{bmatrix} \cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}})^{-1}, \Phi^{-1}\right], \tag{47}$$

namely the right connected pair of $[H^2(\mathbb{D})]^{q\times q}$*-functions and the right singular* $q \times q$ *Carathéodory function generated by* W*.*

Proof. In view of Lemmas 2 and 6 and Proposition 1, the assertion follows easily from Theorem 2 and Lemma 4. We omit the details. □

Corollaries 1 and 2 lead us to the following notions.

Definition 5 *Let* W *be a* j_{qq}*-inner function.*
(a) The triple $[\Phi_l, \Psi_l, \Omega_{l,s}]$ *where* $[\Phi_l, \Psi_l]$ *is the left connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions generated by* W *and where* $\Omega_{l,s}$ *is the left singular* $q \times q$ *Carathéodory function generated by* W *is called the left ADD-parametrization of* W*.*
(b) The triple $[\Phi_r, \Psi_r, \Omega_{r,s}]$ *where* $[\Phi_r, \Psi_r]$ *is the right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions generated by* W *and where* $\Omega_{r,s}$ *is the right singular* $q \times q$ *Carathéodory function generated by* W *is said to be the right ADD-parametrization of* W*.*

Theorem 3 *Let* $[\Phi, \Psi]$ *be a left connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions such that the function* $\det \Phi$ *does not identically vanish. Further, let* $\Omega \in \mathcal{C}_{q,\langle \underline{\Phi}\,\underline{\Phi}^*\rangle}$*. Then* Ω *admits a pseudocontinuation* $\Omega^{\#}$ *and*

$$W := \frac{1}{2} \cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}})^{-1}, \Phi^{-1}\right] \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}} & I - \widehat{\Omega^{\#}} \\ I - \Omega & I + \Omega \end{bmatrix} \tag{48}$$

is a j_{qq}*-inner function. Moreover,* $[\Phi, \Psi]$ *and* Ω *are the left connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions and the left* $q \times q$ *Carathéodory function, respectively, generated by* W*. If* V *is the (unique) inner* $q \times q$ *Schur function which realizes the left connection of* $[\Phi, \Psi]$*, then the Potapov-Ginzburg transform* S *of* W *admits the representation*

$$S = \begin{bmatrix} 2\Psi(I+\Omega)^{-1} & V - 2\Psi(I+\Omega)^{-1}\Phi \\ -(I-\Omega)(I+\Omega)^{-1} & 2(I+\Omega)^{-1}\Phi \end{bmatrix}. \tag{49}$$

Proof. Since Ω belongs to $\mathcal{C}_{q,\langle \underline{\Phi}\,\underline{\Phi}^*\rangle}$ and the function $\det \Phi$ does not identically vanish in $\mathbb{D}$, the functions $\det(\Omega + \Omega^*)$ and $\det \Psi$ do not identically vanish in $\mathbb{D}$. Using Remark 4 and the identities $\underline{\Omega}(\underline{\Omega} + \underline{\Omega}^*)^{-1} + \underline{\Omega}^*(\underline{\Omega} + \underline{\Omega}^*)^{-1} = I$ and

$$\begin{aligned}
&\underline{\Omega}(\underline{\Omega} + \underline{\Omega}^*)^{-1}\underline{\Omega}^* - \underline{\Omega}^*(\underline{\Omega} + \underline{\Omega}^*)^{-1}\underline{\Omega} \\
&\quad = \ [\underline{\Omega}(\underline{\Omega} + \underline{\Omega}^*)^{-1}\underline{\Omega}^* + \underline{\Omega}^*(\underline{\Omega} + \underline{\Omega}^*)^{-1}\underline{\Omega}^*] - [\underline{\Omega}^*(\underline{\Omega} + \underline{\Omega}^*)^{-1}\underline{\Omega}^* + \underline{\Omega}^*(\underline{\Omega} + \underline{\Omega}^*)^{-1}\underline{\Omega}] \\
&\quad = \ \underline{\Omega}^* - \underline{\Omega}^* \ = \ 0 \ ,
\end{aligned}$$

which hold true $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, we obtain that

$$\begin{aligned}
&\frac{1}{4}(I + \underline{\Omega})(\underline{\Psi}^*\underline{\Psi})^{-1}(I + \underline{\Omega}^*) - (I - \underline{\Omega}^*)(\underline{\Phi}\,\underline{\Phi}^*)^{-1}(I - \underline{\Omega}) \\
&\quad = \frac{1}{2}[(I + \underline{\Omega})(\underline{\Omega} + \underline{\Omega}^*)^{-1}(I + \underline{\Omega}^*) - (I - \underline{\Omega}^*)(\underline{\Omega} + \underline{\Omega}^*)^{-1}(I - \underline{\Omega})] = I
\end{aligned} \tag{50}$$

and

$$\frac{1}{4}(I - \underline{\Omega})(\underline{\Psi}^*\underline{\Psi})^{-1}(I - \underline{\Omega}^*) - (I + \underline{\Omega}^*)(\underline{\Phi}\,\underline{\Phi}^*)^{-1}(I + \underline{\Omega}) \ = \ -I \tag{51}$$

are valid $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Applying the same arguments we also get the equations

$$(I + \underline{\Omega})(\underline{\Psi}^*\underline{\Psi})^{-1}(I - \underline{\Omega}^*) - (I - \underline{\Omega}^*)(\underline{\Phi}\,\underline{\Phi}^*)^{-1}(I + \underline{\Omega}) \ = \ 0 \tag{52}$$

and

$$(I - \underline{\Omega})(\underline{\Psi}^*\underline{\Psi})^{-1}(I + \underline{\Omega}^*) - (I + \underline{\Omega}^*)(\underline{\Phi}\,\underline{\Phi}^*)^{-1}(I - \underline{\Omega}) \ = \ 0 \tag{53}$$

are satisfied $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. In view of (50), (51), (52), (53) and Remark 1, then it follows

$$\begin{aligned}
\underline{W}^* j_{qq} \underline{W} \ &= \ \frac{1}{4}\begin{bmatrix} I + \underline{\Omega} & I - \underline{\Omega}^* \\ I - \underline{\Omega} & I + \underline{\Omega}^* \end{bmatrix} \operatorname{diag}\left[\underline{\Psi}^{-1}\underline{\Psi}^{-*}, -\underline{\Phi}^{-*}\underline{\Phi}^{-1}\right] \begin{bmatrix} I + \underline{\Omega}^* & I - \underline{\Omega}^* \\ I - \underline{\Omega} & I + \underline{\Omega} \end{bmatrix} \\
&= \ j_{qq} \qquad \underline{\lambda}\text{-a. e. on } \mathbb{T} \ .
\end{aligned} \tag{54}$$

Let W be partitioned into $q \times q$ blocks via (12). Our following considerations are aimed to show that

$$S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21} \quad , \quad S_{12} := W_{12}W_{22}^{-1} \tag{55}$$

and

$$S_{21} := -W_{22}^{-1}W_{21} \quad , \quad S_{22} := W_{22}^{-1} \tag{56}$$

are well-defined functions which belong to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Obviously, $W_{22} = \frac{1}{2}\Phi^{-1}(I + \Omega)$. Since Ω belongs to $\mathcal{C}_q(\mathbb{D})$, we can conclude that $\det W_{22}$ does not identically vanish in $\mathbb{D}$ (see, e. g., [8, Part (a) of Proposition 2.1.3]) and that S_{22} is a well-defined function which admits the representation

$$S_{22} = 2(I + \Omega)^{-1}\Phi \ . \tag{57}$$

From [11, Proposition 3] we know that $(I + \Omega)^{-1}$ is an outer function in $\mathcal{S}_{q\times q}(\mathbb{D})$. Since $\mathcal{S}_{q\times q}(\mathbb{D})$ and $[H^2(\mathbb{D})]^{q\times q}$ are subsets of $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ and because $\mathcal{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$, we see then that S_{22} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. An easy calculation shows

$$S_{21} = -(I - \Omega)(I + \Omega)^{-1} \ . \tag{58}$$

Since $\Omega \in \mathcal{C}_q(\mathbb{D})$ thus S_{21} belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$ and therefore to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ (see, e. g., [8, Part (b) of Proposition 2.1.3]). From Proposition 1 we know that

$$(\widehat{\Psi^{\#}})^{-1}\Phi = V \ . \tag{59}$$

Because of $\Omega \in \mathcal{C}_{q,\langle \underline{\Phi}\,\underline{\Phi}^*\rangle}$ and Remark 4 we have $\Omega \in \mathcal{C}_{q,\langle \underline{\Psi}^*\underline{\Psi}\rangle}$. Hence $\underline{\Psi}^* = \frac{1}{2}(\underline{\Omega} + \underline{\Omega}^*)\underline{\Psi}^{-1}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Thus Lemma 5 and Remark 1 imply that Ω admits a pseudocontinuation $\Omega^{\#}$ for which the identity $\widehat{\Psi^{\#}} = \frac{1}{2}(\Omega + \widehat{\Omega^{\#}})\Psi^{-1}$ and consequently

$$(\widehat{\Psi^{\#}})^{-1} = 2\Psi(\Omega + \widehat{\Omega^{\#}})^{-1} \tag{60}$$

are valid. Using (48), (59) and (60) we obtain

$$\begin{aligned} S_{12} &= (\widehat{\Psi^{\#}})^{-1}(I - \widehat{\Omega^{\#}})(I+\Omega)^{-1}\Phi \\ &= (\widehat{\Psi^{\#}})^{-1}[(I+\Omega) - (\Omega + \widehat{\Omega^{\#}})](I+\Omega)^{-1}\Phi \\ &= (\widehat{\Psi^{\#}})^{-1}\Phi - (\widehat{\Psi^{\#}})^{-1}(\Omega + \widehat{\Omega^{\#}})(I+\Omega)^{-1}\Phi \\ &= V - 2\Psi(I+\Omega)^{-1}\Phi \ . \end{aligned} \tag{61}$$

Since V, Ψ, $(I+\Omega)^{-1}$ and Φ belong to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ we thus see that S_{12} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ as well. Since W has j_{qq}-unitary radial boundary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, we obtain that $\det \underline{W_{11}} \neq 0$, $\det \underline{W_{22}} \neq 0$ and $\underline{W_{11}}^{-*} = \underline{W_{11}} - \underline{W_{12}}\,\underline{W_{22}^{-1}}\,\underline{W_{21}} = \underline{S_{11}}$ hold $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. In view of Remark 1 this implies $(\widehat{W_{11}^{\#}})^{-1} = S_{11}$. On the other hand, formula (48) provides

$$\widehat{W_{11}^{\#}} = \frac{1}{2}(I+\Omega)\Psi^{-1} \ . \tag{62}$$

Consequently
$$S_{11} = 2\Psi(I+\Omega)^{-1} \ . \tag{63}$$

The same arguments as above yield then $S_{11} \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Therefore all the functions S_{11}, S_{12}, S_{21} and S_{22} belong to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. In view of (54) then from Arov's fundamental result (see [3]) it follows that W is a j_{qq}-inner function. According to (48) we have $W_{12} = \frac{1}{2}(\widehat{\Psi^{\#}})^{-1}(I - \widehat{\Omega^{\#}})$ and hence $\widehat{W_{12}^{\#}} = \frac{1}{2}(I - \Omega)\Psi^{-1}$. Using (62) we thus obtain $\widehat{W_{11}^{\#}} + \widehat{W_{12}^{\#}} = \Psi^{-1}$. Because the identity $W_{21} + W_{22} = \Phi^{-1}$ is also valid, we can see that $[\Phi, \Psi]$ is the left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions associated with W. Lemma 1 and (59) show that V is the inner function which realizes this left connection. Obviously, $W_{22} - W_{21} = \Phi^{-1}\Omega$ and therefore $(W_{22} + W_{21})^{-1}(W_{22} - W_{21}) = \Omega$, i. e., Ω is the $q \times q$ Carathéodory function which is left generated by W. Finally, we observe that (55), (56), (57), (58), (61) and (63) provide immediately the representation (49) of the Potapov-Ginzburg transform S of W. □

Theorem 4 *Let* $[\Phi, \Psi]$ *be a right connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions such that the functions* $\det \Phi$ *does not identically vanish. Further, let* $\Omega \in \mathcal{C}_{q,\langle \underline{\Phi}^*\underline{\Phi}\rangle}$. *Then* Ω *admits a pseudocontinuation* $\Omega^{\#}$ *and*

$$W := \frac{1}{2} \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}} & I - \Omega \\ I - \widehat{\Omega^{\#}} & I + \Omega \end{bmatrix} \cdot \operatorname{diag}\left[(\widehat{\Psi^{\#}})^{-1}, \Phi^{-1}\right] \tag{64}$$

is a j_{qq}-inner function. Moreover, $[\Phi, \Psi]$ and Ω are the right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions and the right $q \times q$ Carathéodory function, respectively, generated by W. If V is the (unique) inner $q \times q$ Schur function which realizes the right connection of $[\Phi, \Psi]$, then the Potapov-Ginzburg transform S of W admits the representation

$$S = \begin{bmatrix} 2(I+\Omega)^{-1}\Psi & (I+\Omega)^{-1}(I-\Omega) \\ -V+2\Phi(I+\Omega)^{-1}\Psi & 2\Phi(I+\Omega)^{-1} \end{bmatrix} . \tag{65}$$

Theorem 5 *Let $[\Phi, \Psi]$ be a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions such that the function $\det \Phi$ does not identically vanish, and let Ω_s be a singular $q \times q$ Carathéodory function. Then there is a unique j_{qq}-inner function W such that $[\Phi, \Psi, \Omega_s]$ is the left ADD-parametrization of W.*

Proof. Lemma 4 shows that $\Omega := \Omega_{\langle \underline{\Phi}\, \underline{\Phi}^* \rangle} + \Omega_s$ belongs to $\mathcal{C}_{q,\langle \underline{\Phi}\, \underline{\Phi}^* \rangle}$. Theorem 3 yields then that W given by (44) is a j_{qq}-inner function, that $[\Phi, \Psi]$ and Ω are the left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions and the left $q \times q$ Carathéodory function, respectively, generated by W. Thus $[\Phi, \Psi, \Omega_s]$ is the left ADD-parametrization of W. On the other hand, Corollary 1 provides that there is at most one j_{qq}-inner function the left ADD-parametrization of which is exactly $[\Phi, \Psi, \Omega_s]$. □

Theorem 6 *Let $[\Phi, \Psi]$ be a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions such that the function $\det \Phi$ does not identically vanish, and let Ω_s be a singular $q \times q$ Carathéodory function. Then there is a unique j_{qq}-inner function W such that $[\Phi, \Psi, \Omega_s]$ is the right ADD-parametrization of W.*

Proof. Using Lemma 4, Theorem 4 and Corollary 2, one can easily prove Theorem 6 analogously to the proof of Theorem 5. □

6 Construction of j_{qq}-inner Functions by Given Left and Right Carathéodory Functions

Let $\Delta : \mathbb{T} \longrightarrow \mathbb{C}^{q\times q}$ be a function which is integrable with respect to the linear Legesgue-Borel measure $\underline{\lambda}$ on $\mathbb{T}$ and which satisfies $\Delta(z) \in \mathbb{C}^{q\times q}_{\geq}$ for $\underline{\lambda}$-almost all $z \in \mathbb{T}$. Then a function $\Theta \in [H^2(\mathbb{D})]^{q\times q}$ is called a *left* (respectively, *right*) *spectral factor of* $\langle \Delta \rangle$ if $\underline{\Theta}\, \underline{\Theta}^* = \Delta$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. (respectively, $\underline{\Theta}^* \underline{\Theta} = \Delta$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$). It is said to be *normalized* if $\Theta(0) \in \mathbb{C}^{q\times q}_{\geq}$. If

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log(\det \Delta) \mathrm{d}\underline{\lambda} \;>\; -\infty \;, \tag{66}$$

then Masani [14] proved that there are a unique normalized left spectral factor Φ_0 of Δ and a unique normalized right spectral factor Ψ_0 of Δ, and, moreover, that every left spectral factor Φ of Δ and every right spectral factor Ψ of Δ are outer functions in $[H^2(\mathbb{D})]^{q\times q}$.

Remark 6 *Let ρ be a function which belongs to $[\mathcal{N}\mathcal{M}(\mathbb{D})]^{q\times q}$. Then $\det \rho$ does not identically vanish if and only if*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log(\det |\underline{\rho}|) \mathrm{d}\underline{\lambda} \;>\; -\infty$$

(see, e. g., [9]).

We know from Lemma 4 in [11] that, for each $\Omega \in \mathcal{C}_q(\mathbb{D})$, the function Re $\underline{\Omega}$ belongs to $[\mathcal{L}^1(\mathbb{T})]^{q\times q}$. Thus a $q \times q$ Carathéodory function is called a $q \times q$ *Carathéodory function of finite entropy* if

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log[\det(\mathrm{Re}\ \underline{\Omega})]\mathrm{d}\underline{\lambda}\ >\ -\infty\ . \tag{67}$$

If $\Omega \in \mathcal{C}_q(\mathbb{D})$ admits a pseudocontinuation $\Omega^{\#}$, then we see from Remarks 1 and 6 that Ω is a $q \times q$ Carathéodory function of finite entropy if and only if $\det(\Omega + \widehat{\Omega^{\#}})$ does not identically vanish.

In some sense, the following lemma can be considered as converse statement to Lemmas 5 and 6.

Lemma 7 *Let Ω be a $q \times q$ Carathéodory function of finite entropy. Suppose that Ω admits a pseudocontinuation $\Omega^{\#}$. Then:*

(a) Every left spectral factor Φ of $\langle \mathrm{Re}\ \underline{\Omega}\rangle$ admits a pseudoconitnuation $\Phi^{\#}$, namely $\Phi^{\#} = \frac{1}{2}(\Omega^{\#} + \widehat{\Omega})\widehat{\Phi^{-1}}$.

(b) Every right spectral factor Ψ of $\langle \mathrm{Re}\ \underline{\Omega}\rangle$ admits a pseudoconitnuation $\Psi^{\#}$, namely $\Psi^{\#} = \frac{1}{2}\widehat{\Psi^{-1}}(\Omega^{\#} + \widehat{\Omega})$.

Proof. (a) Let Φ be a left spectral factor of $\langle \mathrm{Re}\ \underline{\Omega}\rangle$. Then Φ is an outer function which belongs to $[H^2(\mathbb{D})]^{q\times q}$ and which fulfills $\underline{\Omega} + \underline{\Omega}^* = 2\underline{\Phi}\,\underline{\Phi}^*$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Remark 1 shows that the function $g := \frac{1}{2}(\Omega^{\#}+\widehat{\Omega})\widehat{\Phi^{-1}}$ belongs to $[\mathcal{NM}(\mathbb{E})]^{q\times q}$ and satisfies $\underline{g} = \frac{1}{2}(\underline{\Omega} + \underline{\Omega}^*)\underline{\Phi}^{-*} = \underline{\Phi}$ $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Hence g is a pseudocontinuation of Φ.
(b) This part can be analogously proved. We omit the details. □.

Theorem 7 *Let Ω be a $q \times q$ Carathéodory function of finite entropy. Suppose that Ω admits a pseudocontinuation $\Omega^{\#}$. Let Φ_l be a left spectral factor of $\langle \mathrm{Re}\ \underline{\Omega}\rangle$, and let V_l be a left denominator of $\widehat{\Phi_l^{\#}}$. Then*

$$W := \frac{1}{2}\mathrm{diag}\left[V_l\Phi_l^{-1}, \Phi_l^{-1}\right] \cdot \begin{bmatrix} I + \widehat{\Omega^{\#}} & I - \widehat{\Omega^{\#}} \\ I - \Omega & I + \Omega \end{bmatrix} \tag{68}$$

is a j_{qq}-inner function the left $q \times q$ Carathéodory function generated of which is Ω. Moreover, $[\Phi_l, \Psi_l]$ where $\Psi_l := V_l\widehat{\Phi_l^{\#}}$ is the left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W.

Proof. Since Φ_l is a left spectral factor of $\langle \mathrm{Re}\ \underline{\Omega}\rangle$ we have $\Omega \in \mathcal{C}_{\langle \underline{\Phi_l}\,\underline{\Phi_l}^*\rangle}$. Because of the fact that Φ_l is an outer function in $[H^2(\mathbb{D})]^{q\times q}$, the function $\det \Phi_l$ does not identically vanish. From Proposition 2 we see that $[\Phi_l, \Psi_l]$ is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions where V_l is an inner function which realizes this left connection. According to Proposition 1, then we infer that Ψ_l admits a pseudocontinuation which satisfies $(\widehat{\Psi_l^{\#}})^{-1} = V_l\Phi_l^{-1}$. The application of Theorem 3 completes the proof. □

Using Theorem 4 the following result can be analogously proved.

Theorem 8 *Let Ω be a $q \times q$ Carathéodory function of finite entropy. Suppose that Ω admits a pseudocontinuation $\Omega^{\#}$. Let Φ_r be a right spectral factor of $\langle \mathrm{Re}\ \underline{\Omega} \rangle$, and let V_r be a right denominator of $\widehat{\Phi_r^{\#}}$. Then*

$$W := \frac{1}{2} \begin{bmatrix} I + \widehat{\Omega^{\#}} & I - \Omega \\ I - \widehat{\Omega^{\#}} & I + \Omega \end{bmatrix} \cdot \mathrm{diag}\left[\Phi_r^{-1} V_r, \Phi_r^{-1}\right] \tag{69}$$

is a j_{qq}-inner function the right $q \times q$ Carathéodory function generated of which is Ω. Moreover, $[\Phi_r, \Psi_r]$ where $\Psi_r := \widehat{\Phi_r^{\#}} V_r$ is the right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W.

Obviously, for each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, the functions $I - \underline{f}\,\underline{f}^*$ and $I - \underline{f}^*\underline{f}$ are bounded $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Hence both functions belong to $[\mathcal{L}^1(\mathbb{T})]^{p\times p}$ and $[\mathcal{L}^1(\mathbb{T})]^{q\times q}$, respectively. In view of this fact, a function $f \in \mathcal{S}_{p\times q}(\mathbb{D})$ is called a *$p\times q$ Schur function of finite entropy* if

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log[\det(I - \underline{f}\,\underline{f}^*)] \mathrm{d}\underline{\lambda} \ > \ -\infty \ . \tag{70}$$

Because

$$\det(I - K\,K^*) = \det \begin{bmatrix} I_p & K \\ K^* & I_q \end{bmatrix} = \det(I - K^* K)$$

holds true for every contractive $p \times q$ complex matrix K, a function $f \in \mathcal{S}_{p\times q}(\mathbb{D})$ is a $p \times q$ Schur function of finite entropy if and only if

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log[\det(I - \underline{f}^*\underline{f})] \mathrm{d}\underline{\lambda} \ > \ -\infty \ .$$

Remark 7 *If f is a $p \times q$ Schur function which admits a pseudocontinuation $f^{\#}$, then the following statements are equivalent:*
(i) $\det(I - f\,\widehat{f^{\#}})$ *does not identically vanish.*
(ii) $\det(I - \widehat{f^{\#}}\,f)$ *does not identically vanish.*
(iii) f is a $p \times q$ Schur function of finite entropy.

Lemma 8 *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$. Then* $\det(I + \Omega)$ *nowhere vanishes in $\mathbb{D}$ and the function $f := (I - \Omega)(I + \Omega)^{-1}$ belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$. The function Ω admits a pseudocontinuation if and only if f admits a pseudocontinuation. Moreover, Ω has finite entropy if and only if f has finite entropy.*

A proof of Lemma 8 is given in [12, Lemmas 1 – 3].

Proposition 11 *Let $f \in \mathcal{S}_{p\times q}(\mathbb{D})$. Suppose that f admits a pseudocontinuation $f^{\#}$. Let $\rho := I - f\widehat{f^{\#}}$ and $\sigma := I - \widehat{f^{\#}}f$. Assume that* $\det \rho$ *does not identically vanish. Let ϕ be the unique normalized left spectral factor of $\langle \rho \rangle$, and let ψ be the unique normalized right spectral factor of $\langle \sigma \rangle$. Further, let $b \in \mathcal{S}_{p\times p}(\mathbb{D})$ and $c \in \mathcal{S}_{q\times q}(\mathbb{D})$ be such that*

$$c\psi\widehat{f^{\#}}\rho^{-1}\phi b \in [\mathcal{N}_+(\mathbb{D})]^{q\times p} \ . \tag{71}$$

Then

$$U := \mathrm{diag}\,[I_p, c] \cdot \begin{bmatrix} \phi & f \\ -\psi\widehat{f^{\#}}\rho^{-1}\phi & \psi \end{bmatrix} \cdot \mathrm{diag}\,[b, I_q]$$

is an inner $(p+q)\times(p+q)$ Schur function.

Proof. In view of $\mathcal{S}_{p\times q}(\mathbb{D}) \subseteq [\mathcal{N}_+(\mathbb{D})]^{p\times q}$, $[H^2(\mathbb{D})]^{p\times p} \subseteq [\mathcal{N}_+(\mathbb{D})]^{p\times p}$ and the fact that the Smirnov class is an algebra over $\mathbb{C}$, we get that $\phi b \in [\mathcal{N}_+(\mathbb{D})]^{p\times p}$, $f \in [\mathcal{N}_+(\mathbb{D})]^{p\times q}$ and $c\psi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Thus, in view of (71), we have

$$U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)} \ . \tag{72}$$

Using Remark 1 it is readily checked that

$$\underline{U}\,\underline{U}^* = \left[\begin{array}{cc} \underline{\phi}\,\underline{\phi}^* + \underline{f}\,\underline{f}^* & (I - \underline{\phi}\,\underline{\phi}^*\underline{\rho}^{-1})\underline{f}\,\underline{\psi}^*\underline{c} \\ \underline{c}\,\underline{\psi}\,\underline{f}^*(I - \underline{\rho}^{-1}\underline{\phi}\,\underline{\phi}^*) & \underline{c}\,\underline{\psi}(I + \underline{f}^*\underline{\rho}^{-1}\underline{\phi}\,\underline{\phi}^*\underline{\rho}^{-1}\underline{f})\underline{\psi}^*\underline{c}^* \end{array}\right] \tag{73}$$

holds true $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Obviously, Remark 1 also provides that

$$\underline{\phi}\,\underline{\phi}^* + \underline{f}\,\underline{f}^* = I \qquad , \qquad I - \underline{\rho}^{-1}\underline{\phi}\,\underline{\phi}^* = 0 \tag{74}$$

and

$$I + \underline{f}^*\underline{\rho}^{-1}\underline{\phi}\,\underline{\phi}^*\underline{\rho}^{-1}\underline{f} = I + \underline{f}^*(I - \underline{f}\,\underline{f}^*)^{-1}\underline{f} = (I - \underline{f}^*\underline{f})^{-1} = \underline{\psi}^{-1}\underline{\psi}^{-*} \tag{75}$$

is valid $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Hence we infer from (73), (74) and (75) that

$$\underline{U}\,\underline{U}^* = I \qquad\qquad\qquad \underline{\lambda}\text{-a. e. on } \mathbb{T}. \tag{76}$$

Applying the maximum modulus principle for the Smirnov class (see, e. g., [9]), we get from (72) and (76) that U belongs to $\mathcal{S}_{(p+q)\times(p+q)}(\mathbb{D})$. $\square$

Theorem 9 *Let Ω_1 and Ω_2 be $q \times q$ Carathéodory functions of finite entropy. Suppose that both functions Ω_1 and Ω_2 admit pseudocontinuations. Then $f_1 := (I - \Omega_1)(I + \Omega_1)^{-1}$ and $f_2 := (I - \Omega_2)(I + \Omega_2)^{-1}$ are pseudocontinuable $q \times q$ Schur functions of finite entropy. In particular, $\rho_1 := I - f_1\widehat{f_1^{\#}}$, $\rho_2 := I - f_2\widehat{f_2^{\#}}$, $\sigma_1 := I - \widehat{f_1^{\#}}f_1$ and $\sigma_2 := I - \widehat{f_2^{\#}}f_2$ are functions whose determinants do not identically vanish. Let ϕ_1 and ϕ_2 be the unique normalized left spectral factors of $\langle\underline{\rho_1}\rangle$ and $\langle\underline{\rho_2}\rangle$, respectively, and let ψ_1 and ψ_2 be the unique normalized right spectral factors of $\langle\underline{\sigma_1}\rangle$ and $\langle\underline{\sigma_2}\rangle$, respectively. Then the following statements are equivalent:*

(i) There exists a j_{qq}-inner function W such that Ω_1 and Ω_2 are the left $q\times q$ Carathéodory function generated by W and the right $q \times q$ Carathéodory function generated by W, respectively.

(ii) There are inner $q \times q$ Schur functions b_1 and c_2 such that $f_1 = c_2\psi_2\widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2$.

(iii) There are inner $q \times q$ Schur functions c_1 and b_2 such that $f_2 = c_1\psi_1\widehat{f_1^{\#}}\rho_1^{-1}\phi_1 b_1$.

Proof. From Lemma 8 we see that f_1 and f_2 are pseudocontinuable $q \times q$ Schur functions of finite entropy. Thus we obtain from Remark 7 that the functions $\det\rho_1$, $\det\sigma_1$, $\det\rho_2$ and $\det\sigma_2$ do not identically vanish.
$(i) \Rightarrow (ii)$: Let (i) be satisfied. From Propositions 5, 7 and 8 we obtain

$$\Omega_1 = (I - S_{21})(I + S_{21})^{-1} \quad \text{and} \quad \Omega_2 = (I - S_{12})(I + S_{12})^{-1} \tag{77}$$

where $S_{21} := -W_{22}^{-1}W_{21}$ and $S_{12} := W_{12}W_{22}^{-1}$ are strictly contractive $q\times q$ Schur functions. Using a property of the Cayley transform (see, e. g., [8, Lemma 1.3.12]) we thus infer

$$S_{21} = -(I-\Omega_1)(I+\Omega_1)^{-1} = -f_1 \quad \text{and} \quad S_{12} = (I-\Omega_2)(I+\Omega_2)^{-1} = f_2\ . \tag{78}$$

Parts (b) and (e) of Proposition 5 show that the Potapov-Ginzburg transfrom S of W is an inner $2q\times 2q$ Schur function. Setting $S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21}$ and $S_{22} := W_{22}^{-1}$ then we see in particular that $\det S_{11}$ does not identically vanish and that

$$I - \underline{S_{12}}\,\underline{S_{12}^*} = \underline{S_{11}}\,\underline{S_{11}^*} \quad , \quad \underline{S_{21}} = -\underline{S_{22}}\,\underline{S_{12}^*}\,\underline{S_{11}^{-*}} \tag{79}$$

and

$$I - \underline{S_{12}^*}\,\underline{S_{12}} = \underline{S_{22}^*}\,\underline{S_{22}} \tag{80}$$

hold $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. In view of Remark 1 and (78) thus it follows

$$\begin{aligned}\underline{S_{22}}\,\underline{f_2^*}\,\underline{\rho_2}^{-1}\underline{S_{11}} &= \underline{S_{22}}\,\underline{S_{12}^*}(I-\underline{S_{12}}\,\underline{S_{12}^*})^{-1}\underline{S_{11}}\\ &= \underline{S_{22}}\,\underline{S_{12}^*}\,\underline{S_{11}^{-*}} = -\underline{S_{21}} = \underline{f_1} \quad \underline{\lambda}\text{-a. e. on } \mathbb{T}.\end{aligned} \tag{81}$$

Hence we can conclude from Remark 1 that

$$S_{22}\widehat{f_2^{\#}}\rho_2^{-1}S_{11} = f_1\ . \tag{82}$$

From part (b) of Proposition 5 we know that both functions S_{11} and S_{22} belong to $[H^2(\mathbb{D})]^{q\times q}$. According to Masani's factorization theorem [14], there are unique inner $q\times q$ Schur functions b_2 and c_2, and there are unique normalized outer functions $\phi_2^{\square}$ and $\psi_2^{\square}$ which belong to $[H^2(\mathbb{D})]^{q\times q}$ such that $S_{11} = \phi_2^{\square}b_2$ and $S_{22} = c_2\psi_2^{\square}$. Therefore the equation (82) can be written as

$$f_1 = c_2\psi_2^{\square}\widehat{f_2^{\#}}\rho_2^{-1}\phi_2^{\square}b_2\ . \tag{83}$$

Using (79), (80) and Remark 1 we infer that

$$\underline{\phi_2^{\square}}(\underline{\phi_2^{\square}})^* = \underline{S_{11}}\,\underline{S_{11}^*} = I - \underline{S_{12}}\,\underline{S_{12}^*} = \underline{\rho_2}$$

and

$$(\underline{\psi_2^{\square}})^*\underline{\psi_2^{\square}} = \underline{S_{22}^*}\,\underline{S_{22}} = I - \underline{S_{12}^*}\,\underline{S_{12}} = \underline{\sigma_2}$$

is valid $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. Hence $\phi_2^{\square} = \phi_2$ and $\psi_2^{\square} = \psi_2$. Thus the identity (83) yields (ii).

(ii) ⇒ *(i)*: Suppose (ii). Then Proposition 11 shows that

$$U := \begin{bmatrix} \phi_2 b_2 & f_2 \\ -f_1 & c_2\psi_2 \end{bmatrix} \tag{84}$$

is an inner $2q\times 2q$ Schur function, where $\det(c_2\psi_2)$ does not identically vanish. In view of Proposition 6 we see that

$$W := \begin{bmatrix} \phi_2 b_2 + f_2(c_2\psi_2)^{-1}f_1 & f_2(c_2\psi_2)^{-1} \\ (c_2\psi_2)^{-1}f_1 & (c_2\psi_2)^{-1} \end{bmatrix} \tag{85}$$

is a j_{qq}-inner function. Let Ω_l and Ω_r be the left $q \times q$ Carathéodory function and the right $q \times q$ Carathéodory function, respectively, generated by W. Since Ω_1 and Ω_2 admit the representations

$$\Omega_1 = (I + f_1)^{-1}(I - f_1) \quad \text{and} \quad \Omega_2 = (I - f_2)(I + f_2)^{-1}$$

(see, e. g., [8, Lemma 1.3.12]), we have

$$\begin{aligned} \Omega_l &= [(c_2\psi_2)^{-1} + (c_2\psi_2)^{-1}f_1]^{-1}[(c_2\psi_2)^{-1} - (c_2\psi_2)^{-1}f_1] \\ &= (I + f_1)^{-1}(I - f_1) \ = \ \Omega_1 \end{aligned} \tag{86}$$

and

$$\begin{aligned} \Omega_r &= [(c_2\psi_2)^{-1} - f_2(c_2\psi_2)^{-1}][(c_2\psi_2)^{-1} + f_2(c_2\psi_2)^{-1}]^{-1} \\ &= (I - f_2)(I + f_2)^{-1} \ = \ \Omega_2 \ , \end{aligned} \tag{87}$$

i. e., Ω_1 and Ω_2 are exactly the left $q \times q$ Carathéodory function and the right $q \times q$ Carathéodory function, respectively, generated by the j_{qq}-inner function W. Hence (i) holds.
$(i) \Leftrightarrow (iii)$: This equivalence can be analogously verified as the fact that (ii) is necessary and sufficient for (i). □

Theorem 10 *Let Ω_1 and Ω_2 be $q \times q$ Carathéodory functions of finite entropy. Suppose that both functions Ω_1 and Ω_2 admit pseudocontinuations. Let $f_1 := (I - \Omega_1)(I + \Omega_1)^{-1}$, $f_2 := (I - \Omega_2)(I + \Omega_2)^{-1}$, $\rho_2 := I - f_2\widehat{f_2^{\#}}$ and $\sigma_2 := I - \widehat{f_2^{\#}}f_2$. Let ϕ_2 be the unique normalized left spectral factors of $\langle \underline{\rho_2} \rangle$, and let ψ_2 be the unique normalized right spectral factors of $\langle \underline{\sigma_2} \rangle$. Further, let c_2 and b_2 be inner $q \times q$ Schur functions such that*

$$f_1 = c_2\psi_2\widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2 \ . \tag{88}$$

Then

$$W := \begin{bmatrix} I & f_2 \\ \widehat{f_2^{\#}} & I \end{bmatrix} \cdot \operatorname{diag}\left[\rho_2^{-1}\phi_2, \psi_2^{-1}\right] \cdot \operatorname{diag}\left[b_2, c_2^{-1}\right] \tag{89}$$

is a j_{qq}-inner function such that Ω_1 is the left $q \times q$ Carathéodory function and Ω_2 is the right $q \times q$ Carathéodory function generated by W, and

$$V := \operatorname{diag}\left[c_2, b_2^{-1}\right] \cdot \operatorname{diag}\left[\psi_2\sigma_2^{-1}, \phi_2^{-1}\right] \cdot \begin{bmatrix} I & \widehat{f_2^{\#}} \\ f_2 & I \end{bmatrix} \tag{90}$$

is a j_{qq}-inner function such that Ω_1 is the right $q \times q$ Carathéodory function and Ω_2 is the left $q \times q$ Carathéodory function generated by W.

Proof. Lemma 8 shows that f_2 admits a pseudocontinuation $f_2^{\#}$ and that f_2 has finite entropy. In particular, both functions $\det \rho_2$ and $\det \sigma_2$ does not identically vanish. Let (12) be the $q \times q$ block partition of W. Then we have

$$W_{22}^{-1} = c_2\psi_2 \quad , \quad W_{12}W_{22}^{-1} = f_2 \ , \tag{91}$$

$$W_{11} - W_{12}W_{22}^{-1}W_{21} = \rho_2^{-1}\phi_2 b_2 - f_2\widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2 = \phi_2 b_2 \tag{92}$$

and, in view of (88),

$$-W_{22}^{-1}W_{21} = -c_2\psi_2\widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2 = -f_1\ . \tag{93}$$

Thus the Potapov-Ginzburg transform S of W has the shape

$$S = \operatorname{diag}\left[I_q, c_2\right] \cdot \left[\begin{array}{cc} \phi_2 & f_2 \\ -c_2\psi_2\widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2 & \psi_2 \end{array}\right] \cdot \operatorname{diag}\left[b_2, I_q\right]\ . \tag{94}$$

Since $\mathcal{S}_{q\times q}(\mathbb{D})$ is a subset of $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ we obtain from (88), (94) and Proposition 11 that S is an inner $2q\times 2q$ Schur function. Using Proposition 6 then we can conclude that the Potapov-Ginzburg transform $W^{\square}$ of S is a j_{qq}-inner function. On the other hand, since S is the Potapov-Ginzburg transform of W, we have $W^{\square} = W$, i. e., W is a j_{qq}-inner function. Let Ω_l (respectively, Ω_r) denote the left (respectively, right) $q\times q$ Carathéodory function generated by W. Then we get from (91) and (93) that (86) and (87) hold true. Let

$$\Delta := \left[\begin{array}{cc} 0 & I_q \\ I_q & 0 \end{array}\right]\ .$$

Using the identity $\sigma_2^{-1}\widehat{f_2^{\#}} = \widehat{f_2^{\#}}\rho_2^{-1}$ and the fact that a $2q\times 2q$ Schur function T is inner if and only if $U := \Delta T\Delta$ is an inner $2q\times 2q$ Schur function, the other part of the assertion can be verified analogously. $\square$

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