Latin Tableaux: Solutions, Embeddings, and Redundant Disequalities

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Abstract
The Latin Tableau (LT) Constraint Satisfaction Problem has as a special case the Latin Square (LS) Problem. Questions about LS are reformulated and partially solved in the LT context, in particular questions related to the existence of solutions, to the existence of embeddings in an LS, and to the classification of its redundant disequalities.
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1 Introduction

The Latin Square problem of size $N$ requires filling out an $N \times N$ square with numbers from 1 to $N$ in such a way that each row and column contains every number exactly once. A common formulation of Latin Square as a constraint satisfaction problem (CSP) [8] uses (a) $N \times N$ variables $x_{ij}, i, j \in [1..N]$, representing the value assigned to the cell in row $i$ and column $j$ of the square, (b) $N \times N$ domain constraints indicating that the domain of each variable $x_{ij}$ is $[1..N]$, and (c) $2 \times N$ all_different constraints [11] of $N$ variables each, one for the variables in each column and in each row. We refer to this CSP, that is, to the set of $N \times N$ domain constraints and $2 \times N$ all_different constraints, as $\text{LatinSquare}(N)$.

An all_different constraint for $N$ variables can equivalently be specified as the conjunction of the $N \times (N - 1)/2$ pairwise binary disequality constraints on its input variables. For example, all_different($\{x_1, x_2, x_3\}$) is logically equivalent to the conjunction of the constraints $x_1 \neq x_2$, $x_1 \neq x_3$, and $x_2 \neq x_3$.

Loosely speaking, a set of constraints is redundant if it does not restrict the set of solutions. For instance in $\{x < y, y < z, x < z\}$, the set $\{x < z\}$ is redundant, as adding it to $\{x < y, y < z\}$ does not change the set of solutions.

For the Latin Square problem, [3] defines two sets, $R_1$ and $R_2$, of disequalities:

\[
R_1(N) = \{x_{1i} \neq x_{1j}|1 \leq i < j \leq N\}
\]

\[
R_2(N) = \{x_{1i} \neq x_{1j}|1 \leq j \leq N\} \cup \{x_{i1} \neq x_{j1}|1 < i < j \leq N\}
\]

*Part of this research was performed in partial fullfilment of the requirements for obtaining the Bachelor in Informatics of the first author.
then proves that both $R_1$ and $R_2$ are redundant sets of disequalities, and that every set of redundant disequalities is (up to symmetry) contained in either $R_1$ or $R_2$.

We seek here to generalize that statement to Latin Tableaux. We learned about Latin Tableaux from [http://www-math.mit.edu/~tchow/g4g7.pdf](http://www-math.mit.edu/~tchow/g4g7.pdf). A Latin Tableau can be seen as part of a Latin Square, in which the domain of every cell $(i,j)$ is restricted to $1...\min(size(row_i), size(col_j))$, and whose form is a Young diagram. As an example, Figure 1 shows a solved Latin Tableau of size 8.

![Figure 1: A solved Latin Tableau](image)

An (empty) Latin Tableau is characterized by the sizes of the subsequent rows, so we denote the Tableau in Figure 1 by $LT[8,7,7,5,5,3,3,1]$. We are only interested in studying square $LT$’s, so the length of the list of sizes must be equal to the first number in that list. We name this list the defining list of the $LT$. We use $LT(N)$ to denote any Latin Tableau with $N$ the length of the first row. Note that for convenience, we write the numbers of the defining list in descending order, but the order is really immaterial.

An alternative way to characterize the shape of an $LT(N)$ is to give a list of the lengths of the columns in it: that list could be obtained by first rotating the picture around the main diagonal, and then constructing the list of row sizes. We name the list so obtained the conjugate list of the $LT(N)$.

The problems we study here are

- characterize the Latin Tableaux that have a solution
- the embedding of a solution of an $LT(N)$ in a solution of $LS(M)$, possibly with $M > N$
- characterize the $LT(N)$ that have an embedding in an $LS(N)$
- the redundant disequalities in an $LT(N)$ with a solution

While our proofs are formal, some of the intuition behind the proofs resulted often from running a constraint logic program for small values of $N$. We used B-Prolog [12] for this, but any Prolog with constraints would have been adequate.
Note: At the time of doing this research, we had not yet fully understood the Wide Partition Conjecture and other results mentioned in [1] and we basically ignored it. We refer to it once later. An accompanying report [4] uses WPC to derive other (conditional) results about LTs.

2 Some necessary conditions for solving an LT

Every LS(N) has a solution, so it is worth contemplating whether the same holds for every LT(N). The following theorem shows that this is not the case.

Theorem 2.1. If an LT has a solution, its defining and conjugate list must be equal. Otherwise said: the LT shape is symmetric around the main diagonal.

Proof. Let \([n_1, \ldots, n_N]\) and \([m_1, \ldots, m_N]\) be the defining list and the conjugate list of the LT(N). In a solved LT, one can see that

\[
\forall i, 1 \leq i \leq N: \sum_{i \geq n_k} 1 = \sum_{i \geq m_k} 1
\]

because both sums indicate how often the number \(i\) occurs in the solved LT. From this it follows that \([n_1, \ldots, n_N]\) equals \([m_1, \ldots, m_N]\).

Note that if the LT were not square, it has no solution, because the longest row (or column) could not be filled out.

Figure 2 shows that the symmetry property is not enough: it shows LT\([5,5,3,2,2]\) and LT\([5,5,2,2,2]\): both are symmetric but have no solution.

Figure 2: Two symmetric LT’s with no solution

Some numbers have been filled out: up to symmetry, that is the only possibility for them in the first two rows. Clearly, the third row cannot be completed. Since an LT(N) needs to be symmetric w.r.t. the main diagonal for it to have a solution, one could think that every solvable LT(N) has a solution that is symmetric w.r.t. the main diagonal: this is true for LS(N), but it is not true for all LTs. For instance: LT\([5,5,5,3,3]\), LT\([6,5,5,3,3,1]\) and LT\([7,7,7,7,7,5,5]\) are solvable (see example solutions below), but they have no solution that is symmetric w.r.t. the main diagonal.
Figure 2 visualizes with the dashed line what we mean by a slice of an LT: it is the set of cells to the right of a vertical cut through the LT. The same figure shows for each row its number of cells, and the upper line is the index of the row, in reverse order. More formally, define a k-slice $S$ of an LT(N) as the cells from column $k$ up to 1. Denote by $rows^S$ the rows in $S$.

**Theorem 2.2.** In a solvable LT(N), the following holds for each $1 \leq k < N$: let $S$ be a k-slice, then

$$\text{len}(\text{col}_{k+1}) - \text{len}(\text{col}_k) \geq \# \{ r | r \in rows^S, \text{len}(r) = \text{len}(\text{col}_k) \}$$

Proof. Take a solution $Sol$ of the LT(N). Let $S$ be the k-slice and let $f = \# \{ r | r \in rows^S, \text{len}(r) = \text{len}(\text{col}_k) \}$ (see Figure 3).

The cells in these $f$ rows have a column domain constraint imposed on which constraints its values to 1..len(col$_k$) (or smaller, but len(col$_k$) is good enough as a bound). So, in $Sol$, these $f$ slice rows contain exactly the values 1..len(col$_k$), and these values cannot occur in the first 1..f cells of the (k+1)$^{\text{th}}$ column: these values must occur in the rest of that column. So that column needs enough spare cells to put those values in, meaning $\text{len}(\text{col}_{k+1}) - f \geq \text{len}(\text{col}_k)$.

Figure 3: $k = 3$, $\text{len}_{k+1} - \text{len}_k = 1$; the slice has 2 rows of length 3 (= $\text{len}_k$), so this LT has no solution.

One consequence of Theorem 2.2 is that a solvable LT(N) can’t miss cells above the anti-diagonal, i.e. $i \leq \text{size}(\text{col}_i)$. The even stronger condition $i < \text{size}(\text{col}_i)$ is not sufficient$^1$ as the unsolvable counterexample [9,9,9,6,5,4,4,4] shows. The same example shows that Theorem 2.2 does not give a sufficient condition.

$^1$It is certainly not necessary: LT[4,3,2,1] is solvable.
3 Redundant domain constraints

**Theorem 3.1.** For a solvable LT, the domain constraints on the rows are redundant, i.e. the LT(N)-CSP (Cells, cell(i, j) → 1...size(col_j), All_different) is equivalent with the LT(N)-CSP (Cells, cell(i, j) → 1...min(size(row_i), size(col_j)), All_different).

**Proof.** Consider a solution S and take a number \(i\) from \([1..N]\). Because of the column domain and all_different constraints, the number of occurrences of \(i\) (in S), denoted \(occ(i)\) equals the number of columns (and thus rows because of symmetry) with length \(\geq i\). We prove that \(i\) occurs in every row with length \(\geq i\): this will allow us to conclude that each row in S contains exactly the numbers 1 up to the length of that row.

So, assume there is a row \(j\) with length \(l \geq i\) that does not contain \(i\). Because of the row all_different constraints, now row can contain \(i\) twice, so there must be a row with \(j'\) with length \(l' < i\) containing \(i\). This prevents row \(j'\) to contain all numbers in \(1..l'\), meaning that for each row missing a number smaller than its length, there exists a strictly smaller row with the same property. That is clearly impossible.

**Relation to WPC** In Conjecture 1 of [1], the constraints on the tableau do not state the column domain restrictions. This is in line with Theorem 3.1 since for an LT, rows and columns are interchangeable.

4 Embedding of an LT in an LS

As LT(N) is part of an \(N \times N\)-square, the question arises naturally whether (a solution of) an LT(N) can be part of (a solution to) an LS(N). This is not always the case, e.g. LT[4,3,3,1] has no solution that is part of a solution to LS(4). However, it is part of LS(6) as the following solution shows.

![Figure 4: An embedding of LT[4,3,3,1] in LS(6)](image)

Figure 4 also shows (at the right) that because of the row and column interchange symmetry of the LS, it does not matter where exactly we draw the LT in the LS. However, it is convenient to imagine that the embedding is upper-left. Using a theorem from [10] it is easy to see that for every solvable LT(N) there exists an \(M\) such that it can be embedded in LS(M):
**Theorem by B. Smetaniuk:** Any $n \times n$ partial Latin Square with at most $n - 1$ entries can be completed to a Latin square.

Since an LT(N) has at most $N^2$ entries, one can take $n$ above equal to $N^2 + 1$, and conclude that every solvable LT(N) has an embedding in LS($N^2 + 1$).

The above indicates that an LT(N) can be seen as a CSP whose variables are also in an LS(M) with $M \geq N$. It is convenient to have a way to talk about the other variables and their domain and constraints:

**Definition 1.** The $i$-complement (with $i \geq 0$) of an LT(N) $A$, denoted by $\overline{A}^i$, is an LT(M) that consists of the cells configured as LS($N + i$) without the cells of the LT(N). Clearly, an $i$-complement is itself an LT: the domains and constraints of an $\overline{A}^i$ follow naturally.

Figure 5 shows an example of an LT(7) and its 0-complement, and an LT(5) with its 2-complement.

Figure 5: An LT(7) with its 0-complement, and an LT(5) with its 2-complement

Note that in Figure 5, the LT(7) has a solution, but its 0-complement has not; the LT(5) in Figure 5 and its 2-complement both have a solution.

**Lemma 1.** Let the LT(N) $A$ be solvable, then $\overline{A}^i$ has a solution if and only if a solution of $A$ can be embedded in a solution of LS($N + i$).

**Proof.** Figure 6 shows how this works in one direction on a small example.

Figure 6: An LT with its 0-complement, both having a solution

The idea is to construct a solution for $\overline{A}^i$, replace the value $v$ in each of its cells by $N + i + 1 - v$, and then glue the two together.

The other direction of the claim can be derived from that.
We have indicated above already that every solvable LT(N) has an embedding in an LS(M) for some M, but M was quite large. Theorem 4.1 tightens that result, and Theorem 4.2 shows that in some sense Theorem 4.1 is optimal.

In Appendix I, we give our chronologically first construction of an LS-embedding for a general solvable LT(N): it uses a theorem from [6]. Our current result is stronger and relies on the more powerful Ryser’s theorem [9] which states:

**Theorem by H. Ryser:** Let \( N_R(o_i) \) denote the number of occurrences of \( o_i \) in an \( r \times s \) latin rectangle \( R \) on the symbols \( o_1, \ldots, o_n \), then \( R \) may be completed to form a latin square of side \( n \) if and only if \( \forall i : 1 \leq i \leq n \rightarrow N_R(o_i) \geq r + s - n \)

**Theorem 4.1.** Every solvable LT(N) \( A \) has an \( N \)-embedding, i.e. an embedding in LS(2*N).

**Proof.** In Ryser’s theorem take \( r = s = N \) for some solvable LT(N), and \( n = 2 \ast N \). Then \( r + s - n = 0 \) and since the 0-complement of an LT(N) can be filled with \( N \) new symbols so that each row and each column has only different elements, and each symbol occurs at least zero times, Ryser’s theorem implies that any solvable LT(N) has an embedding in LS(2*N).

The following is also a consequence of the proof of Theorem 4.1.

**Corollary 1.** \( \forall N, M \geq 2N \), every solvable LT(N) has an M-embedding.

Note that Theorem 4.1 is also a direct consequence of Theorem 2 in [5].

This result is clearly not optimal for each LT. For instance LT[N,N-1,N-2,...,1] has a solvable 0-complement and thus can be embedded in LS(N). Indeed, in general, every solvable LT(N) with a solvable 0-complement can be embedded in LS(N). However, \( 2 \ast N \) is a tight bound, as Theorem 4.2 shows. We first need the construction of a particular LT(N) for \( N \) large enough.

**Lemma 2.** LT[N,...,N,N-1,N-1,N-3] has a solution for \( N \geq 6 \).

**Proof.** First consider LT[N,...,N,N-3,N-3,N-3]: it is the (N-3)-complement of LS(3), which obviously has a solution. Since \( N \geq 6 = 2 \ast 3 \), this LS(3) (seen as LT[3,3,3]) and its (N-3)-complement can be embedded in LS(N), so LT[N,...,N,N-3,N-3,N-3] has a solution. To a solution of this LT(N), glue a solution of LS(2), in which to every element \( N - 3 \) is added: this is shown in Figure 7.

**Theorem 4.2.** For \( N \geq 7 \) and \( M < 2N \), LT[N,...,N,N-1,N-1,N-3] has no embedding in LS(M).
Figure 7: Turning an LT[7,7,7,7,4,4,4] into an LT[7,7,7,7,6,6,3]

Proof. Denote by $A$ the LT[N,...,N,N-1,N-1,N-3]. As $A^0$ is an LT[3,1,1], it has no solution, so at least one new number is needed to complete $A$ into a latin rectangle $R$ with size $N \times N$. Let LS(M) be an embedding Latin Square of $A$, then the amount of new numbers to be used for this LS(M) is $M - N$. Ryser’s theorem indicates that $M - N \geq N - \min_{i=1}^{M} N_R(i)$ and since $N \geq 7$ we derive $M - N \geq 7 - \min_{i=1}^{M} N_R(i)$. The new numbers $i$ have at most $N_R(i) = 2$, so $M - N \geq 5$. This means that at least 5 new numbers are needed, and then they can have at most an $N_R(i) = 1$, so we get $M - N \geq 7 - 1 = 6$. This means that least 6 new numbers are needed. One of these numbers cannot occur in $R$, so $\min_{i=1}^{M} N_R(i) = 0$ and we derive $M \geq 2N$.

Note that Theorem 4.2 is not a direct consequence of [5], because we established here that the bound $2N$ is tight for a strictly smaller class of partial LS, namely the solvable LT.

5 Embeddings and redundant disequalities

Our initial hope was that once an LT(N) is embedded in an LS(M), then the results of [3] applied to the LS(M) can be lifted to the LT(N). This can be done to some extent.

The situation is as follows:

- there are two disjoint sets of variables $Vars_1$ and $Vars_2$ - their union is $Vars$
- there are domain constraints $D_i$ and disequalities $C_i$ on $Vars_i$
- there are domain constraints $D$ and disequalities $C$ on $Vars$
- we know about a subset $R$ of $C$ that it is redundant, i.e. $D + C \setminus R \rightarrow R$

For a set of (domain) constraints $X$, denote by $Proj_i(X)$ the constraints from $X$ that only involve variables from $Vars_i$. We would like to prove that

$$D_1 + C_1 \setminus Proj_1(R) \rightarrow Proj_1(R)$$
Denote by $\text{Int}(X) = X \setminus (\text{Proj}_1(X) + \text{Proj}_2(X))$.

We know that

- $D_i \rightarrow \text{Proj}_i(D)$
- $C_i = \text{Proj}_i(C)$
- $D_1 + D_2 \rightarrow \text{Int}(C)$
- $\text{Proj}_1(D) + \text{Proj}_2(D) = D$

So, now we have the following chain:

$$(D_1 + C_1 \setminus \text{Proj}_1(R)) + (D_2 + C_2 \setminus \text{Proj}_2(R)) =$$
$$(D_1 + D_2) + (C_1 \setminus \text{Proj}_1(R) + C_2 \setminus \text{Proj}_2(R)) \rightarrow$$
$$\text{Proj}_1(D) + \text{Int}(C) + \text{Proj}_2(D) + (C_1 + C_2) \setminus (\text{Proj}_1(R) + \text{Proj}_2(R)) =$$
$$\text{Proj}_1(D) + \text{Proj}_2(D) + (C_1 + C_2 + \text{Int}(C)) \setminus (\text{Proj}_1(R) + \text{Proj}_2(R)) \rightarrow$$
$$D + (C \setminus R) \rightarrow$$
$$R \rightarrow \text{Proj}_1(R)$$

So now we are in the situation that we have two sets of constraints with a solution $S_i = (D_1 + C_i \setminus \text{Proj}_i(R))$ over two disjoint sets $\text{Vars}_i$ and $S_1 + S_2 \rightarrow \text{Proj}_i(R)$ which contains only variables from $\text{Vars}_1$. In such a case, one can conclude that $S_1 \rightarrow \text{Proj}_1(R)$, or $D_1 + C_1 \setminus \text{Proj}_1(R) \rightarrow \text{Proj}_1(R)$ what we wanted to prove.

Using the above for a solvable LT consists in realising that one can take $LT(N)$ as $(\text{Vars}_1, D_1, C_1)$. The large enough i-complement of the $LT(N)$ has the subscripts 2. $R$ is either $R_1$ or $R_2$ from [3]. So we arrive at the statement: $\text{proj}_1(R_1)$ and $\text{proj}_1(R_2)$ are redundant. We just say $R_1$ and $R_2$ are redundant. Figure 8 shows two of these projected redundant sets for $LT[4,3,3,1]$.

![Figure 8: Left: the projection of a $R_1$; right: the projection of a $R_2$](image)

Although the above proves that $R_1$ or $R_2$ are redundant for any solvable LT(N), it gives no insight in their maximality. Moreover, [7] contains an infinite sequence of $LT(N)$s with redundant sets of disequalities that are strictly larger than $R_1$ and $R_2$. However, the next section shows that for a non-trivial and infinite class of LTs, $R_1$ and $R_2$ are the only maximal redundant sets, just as for Latin Squares.
6 The maximal redundant sets for some LT(N)

In an LS, pairs of disequalities belong to one of the 8 classes shown in Figure 9 taken from [3].

![Figure 9: All possible pairs of disequalities in an LS(N) with \( N \geq 4 \), up to symmetry](image)

The number of configurations is so small thanks to the spatial symmetries of LS(N). These symmetries do not hold for a general LT(N), so we must expect restrictions on the general results, and more labour in proving them. We can still distinguish these 8 patterns, but we must interpret them as:

1. the disequalities are in different rows and have endpoints in two columns
2. the disequalities are in different rows and have endpoints in three columns
3. the disequalities are in different rows and have endpoints in four columns
4. one disequality is in a row, the other in a column, and they share one endpoint
5. one disequality is in a row, the other in a column, and they share no column or row
6. one disequality is in a row, the other in a column, and they share a column, but no endpoint
7. the two disequalities share a row and an endpoint
8. the two disequalities share a row and no endpoint

We name those the extended patterns of Figure 9. In Theorem 1 [3], it was proven that patterns 1..5 form bad pairs, i.e. non-redundant pairs of disequalities: we seek to prove a close statement. We do this by stating a series of five lemmas that for each of the extended patterns 1..5 proves a sufficient condition for it to be a bad pair. The first lemma is detailed, the four others are variants and only the difference with the first is highlighted.

Note that we do not need to prove separately that patterns 6,7,8 are redundant: this is implied by the result in Section 5.
Lemma 3. Let $A$ be an LT($N$) with a solvable 0-complement $\gamma$. Let $K > 0$ be the size of $\gamma$. Let $D = (d_1, d_2)$ be a pair of disequalities in $A$ of the first sort of Figure 9. If $N \geq 2 \times K + 3$, then $D$ forms a bad pair.

Proof. Consider Figure 10. $A$ consists of the regions $\alpha$ and $\beta$. $A^0$ is region $\gamma$: the dashed line represents the border between $\beta$ and $\gamma$. The exact form of this border is not important, but we know it can not be outside the $K \times K$-square (that is is not indicated in the figure). $\beta \cup \gamma$ equals a $(K + e) \times (K + e)$-square: for the current lemma $e = 2$. Note that $A$ has solution symmetries that allow to reorder the first $N - K$ columns, and likewise for the rows.

The proof takes a number of steps:

1. take a pair of disequalities of the first extended pattern of Figure 9 but within $A$; by using the symmetries, we can assume that the disequalities are completely inside $\beta$ if $e \geq 2$. Two is enough because pattern 1 when squeezed together completely in 2 rows and 2 columns, meaning that all the endpoints that were not yet in $\beta$ can be moved there by the symmetries Figure 11 shows the situation at the left: $\alpha$ is not important at this moment and shown; now fix values for the endpoints of the disequalities, as in the middle part of Figure 11.

2. Ryser’s theorem guarantees that there is an LS($K+e$) solution $S$ for $\beta \cup \gamma$ in which the indicated cells have exactly those values: restrict $S$ to $\beta$. $S$ uses every number from 1..($K + e$) at least once.

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2In the later lemmas, it is better to use Theorems 1.42, 1.78 or 1.81 from [2].
3. since $\gamma$ is a solvable LT(K), we can take any solution of it and shift if over $(K + e)$ - i.e. add $(K + e)$ to each entry - and together with the $S$ restricted to $\beta$, we get a latin rectangle with values in $1..(2 \times K + e)$ and in which each value appears at least once: Ryser’s theorem now guarantees that this rectangle can be extended to a complete Latin Square of size $N$ iff $N \geq 2 \times (K + e) - 1$

4. restricting this solution to $A$ (up to some surgery on the domains) gives a solution of the initial LT(N), in which the endpoints of the disequalities have exactly the values indicated; now change those values as in the right of Figure 11 we obtain a non-solution to $A$ in which only the two disequalities are violated, so we can conclude that the original two disequalities were a bad pair

As $e = 2$ the bound $N \geq 2 \times K + 3$ follows.

Note that the condition that $A^0$ has a solution, is necessary in the proof: without that condition, one can fill $\gamma$ with $K$ new symbols such that the disequalities are respected, but that would not necessarily make the completion of $\alpha$ together with $\beta$ into a valid solution for the LT(N).

The next four lemmas are variants of Lemma 3: the pattern number is different, the equivalent of Figure 11 and possibly the value for $e$.

**Lemma 4.** Let $A$ be an LT(N) with a solvable 0-complement $\gamma$. Let $K > 0$ be the size of $\gamma$. Let $D = (d_1, d_2)$ be a pair of disequalities in $A$ of the fourth sort of Figure 9. If $N \geq 2 \times K + 3$, then $D$ forms a bad pair.

**Proof.** Take $e = 2$ and then Figure 12 shows how to do it.

![Figure 12: Extended pattern 4 in $\beta$](image)

Note that actually one must take into account all 4 orientations of this pattern, but that poses no extra problem.

**Lemma 5.** Let $A$ be an LT(N) with a solvable 0-complement $\gamma$. Let $K > 0$ be the size of $\gamma$. Let $D = (d_1, d_2)$ be a pair of disequalities in $A$ of the second sort of Figure 9. If $N \geq 2 \times K + 5$, then $D$ forms a bad pair.

**Proof.** Take $e = 3$ and Figure 13.
Lemma 6. Let $A$ be an LT($N$) with a solvable 0-complement $\gamma$. Let $K > 0$ be the size of $\gamma$. Let $D = (d_1, d_2)$ be a pair of disequalities in $A$ of the third sort of Figure 9. If $N \geq 2 \times K + 5$, then $D$ forms a bad pair.

Proof. Consider Figure 14.

Now we need a bit more reasoning: the pattern uses at most 4 columns plus 2 rows outside the $K \times K$-square. That means that instead of extending $\gamma$ to a square of size $(K + e) \times (K + e)$ for some $e$, we can extend it to a rectangle of size $(K + 4) \times (K + 2)$ and restrict an LS(K+4) solution to it, so that the values we need are in this $((K + 4) \times (K + 2))$-rectangle. Now superimpose the (shifted) solution in $\gamma$. In the latin $((K + 4) \times (K + 2))$-rectangle, there are now $(2 \times K + 4)$ values, each occurring at least once, so it can be extended to an LS(N) solution iff $1 \geq (K + 4) + (K + 2) + N$, or $N \geq 2 \times K + 5$.

Lemma 7. Let $A$ be an LT($N$) with a solvable 0-complement $\gamma$. Let $K > 0$ be the size of $\gamma$. Let $D = (d_1, d_2)$ be a pair of disequalities in $A$ of the fifth sort of Figure 9. If $N \geq 2 \times K + 5$, then $D$ forms a bad pair.

Proof. Take $e = 3$ and Figure 15.
Summarizing Lemmas 3 up to 7 we can state without proof:

**Theorem 6.1.** Let $A$ be an $LT(N)$ with a solvable 0-complement $\gamma$. Let $K > 0$ be the size of $\gamma$. Let $D = (d_1, d_2)$ be any pair of disequalities in $A$ from Figure 4. If $N \geq 2 \times K + 5$, then $D$ forms a bad pair.

Note that the set of $LT(N)$s satisfying the conditions of Theorem 6.1 is not trivial: take any solvable $LT(K)$ and construct its $p$-complements with $p \geq (K + 5)$ and take this complement as the $A$ in the theorem.

We can now prove the final result of this section:

**Corollary 2.** Let $A$ be an $LT(N)$ with a solvable 0-complement $B$, and let $2 \times (N - \text{size}(\text{col}_N)) + 5 \leq N$. Let $S$ be a set of disequalities of the $LT(N)$, then $S$ contains a bad pair, or is part of $R_1$ or $R_2$. Moreover, the maximal redundant sets of disequalities for $A$ are $R_1$ or $R_2$ up to symmetry.

### 7 Discussion and Future Work

The important achievements of our research are the embedding result of Theorem 4.1 and its tightness (see Theorem 4.2), and the redundancy characterization of Theorem 6.1: the latter is in some sense the best we can hope for, because in [7], a family of $LT(N)$s is established that has strictly larger redundant sets of disequalities than the ones inherited from the Latin Square problem: this family consists of $LT(N)$s that differ from $LS(N)$ as much as possible. Our Theorem 6.1 indicates that there is a cross-over point at which an $LT(N)$ behaves as an $LS(N)$ as far as it concerns the redundant sets of disequalities. That this cross-over point can be characterized at all is remarkable. It is also interesting that the relative size of a solvable complement characterizes the property. We would like to find out whether our Theorem 6.1 also holds when 5 is replaced by a smaller number. Another variant of Theorem 6.1 could require that the $LT(N)$ has a solution, but not necessarily its 0-complement. Finally, an efficient algorithm returning the smallest $LS$-embedding of a solvable $LT(N)$ would be welcome.

### References


A Constructing an embedding

The sequence of pictures in Figure 16 shows how row by row, new cells are added to an LT(N): it starts by adding one cell to the top row. Now consider row 2, 3, ... in sequence. When the i-th row is considered, let \( p = \text{length}(\text{row}_i) - \text{length}(\text{row}_{i-1}) \) (note that at this moment, all rows with index strictly smaller than \( i \) have the same length. Now add 2\( p \) cells to the i-th row and \( p \) cells to the rows above it. Fill the first \( p \) rows of \( \text{row}_i \) with the numbers ...

At the end, we have a rectangle in which every row has the same length \( M \), each row contains every number from 1 to \( M \) exactly once, and where each column contains no number twice.

We can now use the result from [6] which says that

**Theorem by M. Hall:** Given a rectangle of \( n-r \) rows and \( n \) columns such that each of the numbers 1, 2, ..., \( n \) occurs once in every row and no number occurs twice in any column, then there exist \( r \) rows which may be added to the given rectangle to form a latin square.

and complete our rectangle to a latin square. This method adds up to \( 2^N - 1 \) columns to an LT(N) and is far from optimal.