

**Termination analysis of logic programs  
using acceptability with general term  
orders**

*Alexander Serebrenik  
Danny De Schreye*

*Report CW 291, May 2000*



**Katholieke Universiteit Leuven**  
Department of Computer Science  
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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The advantage of the approach over standard acceptability is that in some cases, where complex level mappings are needed, fairly simple term-orderings may be easily generated. The advantage over transformational approaches is that it avoids the transformation step all together.

**Keywords :** termination analysis, acceptability, term-orderings.

# Termination analysis of logic programs using acceptability with general term orders

Alexander Serebrenik, Danny De Schreye

Department of Computer Science, K.U. Leuven  
Celestijnenlaan 200A, B-3001, Heverlee, Belgium  
Email: {Alexander.Serebrenik, Danny.DeSchreye}@cs.kuleuven.ac.be

Technical report CW 291

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**Keywords:** termination analysis, acceptability, term-orderings.

## 1 Introduction

There are many different approaches to termination analysis of logic programs. One particular distinction is between *transformational* approaches and “*direct*” ones. A transformational approach first transforms the logic program into an “equivalent” term-rewrite system (or, in some cases, into an equivalent functional program). Here, equivalence means that, at the very least, the termination of the term-rewrite system should imply the termination of the logic program, for some predefined collection of queries<sup>1</sup>. Direct approaches do not include such a transformation, but prove the termination directly on the basis of the logic program.

Besides the transformation step itself, there is one other technical difference between these approaches. Direct approaches usually prove termination on the basis of a well-founded ordering over the natural numbers. More specifically, they use a *level mapping*, which maps atoms to natural numbers, and, they verify

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<sup>1</sup> The approach of Arts [4] is exceptional in the sense that the termination of the logic program is concluded from a weaker property of *single-redex normalisation* of the term-rewrite system.

appropriate decreases of this level mapping on the atoms occurring in the clauses. On the other hand, transformational approaches make use of more general well-founded orderings over terms, such as reduction orders, or more specifically a simplification order, or others (see [11]).

At least for the direct approaches the systematic choice for level mappings and norms, instead of general term orders, seems arbitrary and ad hoc. This has been the main motivation for this paper. We present an initial study on the use of general well-founded term-orderings as a means of directly proving the termination of logic programs—without intermediate transformation. In particular,

- we study whether the theoretical results on acceptability can be reformulated on the basis of general term orders,
- we evaluate to what extent the use of the general term orderings (instead of level mappings) either improves or deteriorates the direct approaches.

To illustrate the latter point, consider the following program, that formulates some of the rules for computing the repeated derivative of a linear function in one variable  $u$  (see also [13]) :

*Example 1.*

$$\begin{aligned}
d(\text{der}(u), 1). \\
d(\text{der}(A), 0) \leftarrow \text{number}(A). \\
d(\text{der}(X + Y), DX + DY) \leftarrow d(\text{der}(X), DX), d(\text{der}(Y), DY). \\
d(\text{der}(X * Y), X * DY + Y * DX) \leftarrow d(\text{der}(X), DX), d(\text{der}(Y), DY). \\
d(\text{der}(\text{der}(X)), DD X) \leftarrow d(\text{der}(X), DX), d(\text{der}(DX), DD X).
\end{aligned}$$

Proving termination of this program on the basis of a level-mapping is hard. For this example, the required level-mapping is a non-linear function. In particular, a level mapping, such that:  $|d(X, Y)| = \|X\|$ ,  $|\text{number}(X)| = 0$ ,  $\|\text{der}(X)\| = 2^{\|X\|}$ ,  $\|X + Y\| = \max(\|X\|, \|Y\|) + 1$ ,  $\|X * Y\| = \max(\|X\|, \|Y\|) + 1$ ,  $\|u\| = 2$ ,  $\|n\| = 2$ , if  $n$  is a number, would be needed. No automatic system for proving termination on the basis of level mappings is able to generate such mappings. Moreover, we believe, that it would be very difficult to extend existing systems to support generation of appropriate non-linear mappings.  $\square$

Although we have not yet presented our general-well-founded term ordering approach, it should be intuitively clear, that we can capture the decrease in order between the  $\text{der}(X)$  and  $DX$  by using an ordering on terms that gives the highest “priority” to the functor  $\text{der}$ .

As an example of the fact that moving to general ordering can also introduce deterioration, consider the following program from [7, 10].

*Example 2.*

$$\begin{aligned}
\text{conf}(X) \leftarrow \text{delete}_2(X, Z), \text{delete}(U, Y, Z), \text{conf}(Y). \\
\text{delete}_2(X, Y) \leftarrow \text{delete}(U, X, Z), \text{delete}(V, Z, Y). \\
\text{delete}(X, [X|T], T). \\
\text{delete}(X, [H|T], [H|T1]) \leftarrow \text{delete}(X, T, T1).
\end{aligned}$$

Note that by reasoning in terms of sizes of terms, we can infer that the size decreases by 2 after the call to  $delete_2$  predicate in the first clause and then increases by 1 in the subsequent call to the  $delete$  predicate. In total, sizes allow to conclude a decrease. Reasoning in terms of order relations only, however, does not allow to conclude the overall decrease from the inequalities  $arg3 < arg2$  for the  $delete$  predicate and  $arg1 > arg2$  for the  $delete_2$  predicate.  $\square$

As can be expected, theoretically both approaches are essentially equivalent. We will introduce a variant of the notion of acceptability, based on general term orders, which is again equivalent to termination in a similar way as in the level mapping based approach. On the more practical level, as illustrated in the two examples above, neither of the approaches is strictly better: the general term orders provide a larger set of orders to select from (in particular, note that orders based on level mappings and norms are a term order), the level mapping approach provides arithmetic, on top of mere ordering.

In the remainder of this paper, we will start off from a variant of the notion of *acceptability with respect to a set*, as introduced in [8], obtained by replacing level mappings by term orderings. We show how this variant of acceptability remains equivalent to termination under the left-to-right selection rule, for certain goals. Then, we illustrate how this result can be used to prove termination with some examples. We also provide a variant of the *acceptability* condition, as introduced in [3], and discuss advantages and disadvantages of each approach. Next, we discuss automation of the approach. We elaborate on a demand-driven method to set-up and verify sufficient preconditions for termination. In this method, the aim is to derive—in, as much as possible, a constructive way—a well-founded ordering over the set of all atoms and terms of the language underlying the program, that satisfies the termination condition.

## 2 Preliminaries

### 2.1 Term ordering

An *order* over a set  $S$  is an irreflexive, asymmetric and transitive relation  $>$  defined on elements of  $S$ . As usual,  $s \geq t$  denotes that either  $s > t$  or  $s =_> t$ .  $s =_> t$  denotes that  $s$  and  $t$  are equal under the order, but not necessarily identical, and  $s \parallel t$  denotes that  $s$  and  $t$  are incomparable. An ordered set  $S$  is said to be *well-founded* if there are no infinite descending sequences  $s_1 > s_2 > \dots$  of elements of  $S$ . If the set  $S$  is clear from the context we will say that the order, defined on it, is well-founded.

**Definition 1.** *Let  $>$  be an order on a set  $T$ . An order  $\succ$  defined on a set  $S \supseteq T$  is called an extension of  $>$  if for any  $t_1, t_2 \in T$   $t_1 > t_2$  implies  $t_1 \succ t_2$ .*

The study of termination of term-rewriting systems caused intensive study of term orderings. A number of useful properties of term orderings were established.

**Definition 2.** *Let  $>$  be an ordering on terms.*

- If  $s_1 > s_2$  implies  $f(\bar{t}_1, s_1, \bar{t}_2) > f(\bar{t}_1, s_2, \bar{t}_2)$  and  $p(\bar{t}_1, s_1, \bar{t}_2) > p(\bar{t}_1, s_2, \bar{t}_2)$  for any sequences of terms  $\bar{t}_1$  and  $\bar{t}_2$ , function symbol  $f$  and predicate  $p$ , then  $>$  is called *monotone*.
- If for any term  $f(\bar{t}_1, s, \bar{t}_2)$  holds that  $f(\bar{t}_1, s, \bar{t}_2) > s$ , then  $>$  is said to have the *subterm property*.

The following are examples of order relations:  $>$  on the set of numbers, lexicographic order on the set of strings (this is a way the entries are ordered in dictionaries), multiset ordering and recursive path ordering [11].

For our purposes monotonicity and subterm properties are too restrictive. Thus, we assign to each predicate or functor a subset of argument positions, such that for the argument positions in this subset the specified properties hold. We will say that a predicate  $p$  (a functor  $f$ ) is *monotone* (has a *subterm property*) on a specified subset of argument positions. The formal study of these weaker notions may be found in Section 3.

*Example 3.* Let  $f$  be a functor of arity two, and  $a, b$  two terms, such that  $a > b$ . Let  $f$  be monotone in the first argument position. Then,  $f(a, c) > f(b, c)$  holds for any term  $c$ , but there might be some term  $c$ , such that  $f(c, a) \not> f(c, b)$ .

## 2.2 Logic Programs

We follow the standard notation for terms and atoms. A *query* is a finite sequence of atoms. Given an atom  $A$ ,  $rel(A)$  denotes the predicate occurring in  $A$ .  $Term_P$  and  $Atom_P$  denote, respectively, sets of all terms and atoms that can be constructed from the language underlying  $P$ . The extended Herbrand Universe  $U_P^E$  (the extended Herbrand base  $B_P^E$ ) is a quotient set of  $Term_P$  ( $Atom_P$ ) modulo the variant relation.

We refer to an SLD-tree constructed using the left-to-right selection rule of Prolog, as an LD-tree. We will say that a goal  $G$  *LD-terminates* for a program  $P$ , if the LD-tree for  $(P, G)$  is finite.

The following definition is borrowed from [1].

**Definition 3.** Let  $P$  be a program and  $p, q$  be predicates occurring in it.

- We say that  $p$  *refers to*  $q$  in  $P$  if there is a clause in  $P$  that uses  $p$  in its head and  $q$  in its body.
- We say that  $p$  *depends to*  $q$  in  $P$  and write  $p \sqsupseteq q$ , if  $(p, q)$  is in the transitive, reflexive closure of the relation *refers to*.
- We say that  $p$  and  $q$  are *mutually recursive* and write  $p \simeq q$ , if  $p \sqsupseteq q$  and  $q \sqsupseteq p$ .

If  $p \sqsupseteq q$ , but  $q \not\sqsupseteq p$  we also write  $p \sqsupset q$ .

### 3 Revising properties of orders

Unfortunately, monotonicity and subterm properties even being common for the orders, useful for the termination proofs do not hold for orders like list-size norm based order. Indeed,  $\|[[1, 2, 3], 4]\| = 2 < 3 = \|[1, 2, 3]\|$ . Moreover, as we will see if those properties are assumed the notion of rigidity deteriorates to groundedness.

Thus, we extend the notions above further, to incorporate these orders as well. We start with some preliminary remarks.

#### 3.1 Preliminaries

We start by recalling the classical definition of the characteristic function of a set.

**Definition 4.** *Let  $I$  be a set. The characteristic function  $\chi_I$  is defined as following:*

$$\chi_I(x) = \begin{cases} 1 & x \in I \\ 0 & x \notin I \end{cases}$$

We associate with each functor  $f/n$  some set  $I_{f/n} \subseteq \{1, \dots, n\}$ . Then, for term  $s$  we can extend the definition above in the following way:

**Definition 5.** *Let  $L$  be a language. Let  $\mathcal{I}$  be the family of sets, associated with functors, such that  $I_{f/n} \in \mathcal{I}$  exists for every  $f/n \in L$ . Let  $s$  be a term and  $\vec{v}$  a vector of integers. Then the characteristic function  $\chi_s$  is defined as following:*

$$\chi_s^{\mathcal{I}}(\vec{v}) = \begin{cases} 1 & \vec{v} \text{ is an empty vector} \\ 1 & s \text{ is a variable or a constant} \\ \chi_{I_{f/n}}(i_1) * \chi_{I_{t_1}}^{\mathcal{I}}(\overrightarrow{i_2, \dots, i_k}) & \text{if } s = f(t_1, \dots, t_n) \end{cases}$$

We illustrate this definition by the following example.

*Example 4.* Let  $s$  be  $f(a, g(b, h(c, X)))$  and let  $I_{f/2}$  be  $\{1, 2\}$ ,  $I_{g/2}$  be  $\{2\}$  and  $I_{h/2}$  be  $\{1\}$ . Then, the following holds  $\chi_s^{\mathcal{I}}(\vec{1}) = 1$ ,  $\chi_s^{\mathcal{I}}(\vec{2}, \vec{1}) = 0$ ,  $\chi_s^{\mathcal{I}}(\vec{2}, \vec{2}) = 1$ ,  $\chi_s^{\mathcal{I}}(\vec{2}, \vec{2}, \vec{1}) = 1$ ,  $\chi_s^{\mathcal{I}}(\vec{2}, \vec{2}, \vec{2}) = 0$ .

The definition above also suggests that the vector notation denotes a branch in the tree representation of term. When no confusion can be caused we will also talk about the value of the characteristic function for the subterm, that is denoted by a vector and not for the vector itself.

*Example 5.* Continuing Example 4 we can rewrite the values of the characteristic function as following:  $\chi_s^{\mathcal{I}}(a) = 1$ ,  $\chi_s^{\mathcal{I}}(b) = 0$ ,  $\chi_s^{\mathcal{I}}(h(c, d)) = 1$ ,  $\chi_s^{\mathcal{I}}(c) = 1$ ,  $\chi_s^{\mathcal{I}}(X) = 0$ .

**Proposition 1.** *Let  $s$  be a term and  $t$  be a subterm of it. Then, if  $\chi_s^{\mathcal{I}}(t) = 0$ , for any subterm  $t'$  of  $t$  holds that  $\chi_s^{\mathcal{I}}(t') = 0$ .*

*Proof.* Immediately, from the fact that the vector defining  $t$  is a prefix of the vector defining  $t'$ .

### 3.2 Monotonicity - revised

As we've mentioned already the properties of monotonicity and subterm might be seen as very restrictive. Thus, we introduce the following properties, that relax those notions.

**Definition 6.** Let  $S$  be a set of terms over the language  $L$ , and let  $>$  be an order defined on  $S$ . We say that  $>$  is partially monotone if for every functor  $f/n$  in  $L$  exists  $\mathcal{M}_{f/n} \subseteq \{1, \dots, n\}$ , such that if  $t_1 > t_2$  and  $i \in \mathcal{M}_{f/n}$  then for any sequences of terms  $\bar{s}$ , (occupying positions  $1 \dots i-1$ ) and  $\bar{u}$  (occupying positions  $i+1 \dots n$ ) holds  $f(\bar{s}, t_1, \bar{u}) > f(\bar{s}, t_2, \bar{u})$ . The sets  $\mathcal{M}_{f/n}$  are called defined by  $>$ .

*Example 6.* The order defined by the list-size norm is not monotone, but is partially monotone. It defines  $\mathcal{M}_{./2} = \{2\}$  and for other functors  $f/n$  holds that  $\mathcal{M}_{f/n} = \emptyset$ .

**Proposition 2.** Let  $S$  be a set. If  $>$ , defined on  $S$  is monotone, than  $>$  is partially monotone.

*Proof.* Immediate from the definitions.  $\mathcal{M}_{f/n} = \{1, \dots, n\}$ .

We denote the collection of all  $\mathcal{M}_{f/n}$  by  $\mathcal{M}$ .

**Lemma 1.** Let  $s$  be a term, and  $X_{(i)}$  be a variable occurrence in it, such that  $\chi_s^{\mathcal{M}}(X_{(i)}) = 1$ . Then,  $t_1 > t_2$  implies  $s\{t_1 \rightarrow X_{(i)}\} > s\{t_2 \rightarrow X_{(i)}\}$ .

*Proof.* Let  $\vec{i}_1, \dots, \vec{i}_k$  be a vector that denotes  $X_{(i)}$ . Then, by definition of the characteristic function  $\chi_{\mathcal{M}_{f_1/n_1}}(\vec{i}_1) * \dots * \chi_{\mathcal{M}_{f_k/n_k}}(\vec{i}_k) = 1$ . That is for any  $j$  holds  $\chi_{\mathcal{M}_{f_j/n_j}}(\vec{i}_j) = 1$ .

Since  $\chi_{\mathcal{M}_{f_k/n_k}}(\vec{i}_k) = 1$ ,  $i_k \in \mathcal{M}_{f_k/n_k}$ . Thus,  $t_1 > t_2$  implies  $f(\vec{u}_1, t_1, \vec{u}_2) > f(\vec{u}_1, t_2, \vec{u}_2)$  for any sequences  $\vec{u}_1$  and  $\vec{u}_2$ . In particular, this holds for the sequences appearing in  $s$ . Repetitive application of this argument proves the lemma.

The following example shows, that if  $\chi_s^{\mathcal{M}}(X_{(i)}) = 0$  the lemma does not necessary hold.

*Example 7.* Let  $>$  be order based on the list-size norm, and let  $s$  be  $[X, 5]$ . The only variable in  $s$  is  $X$  and  $\chi_s^{\mathcal{M}}(X) = 0$ . On the other hand, let  $t_1$  be  $[1, 2]$  and  $t_2$  be  $[3]$ . According to the definition of  $>$ ,  $t_1 > t_2$  holds. However,  $s\{t_1 \rightarrow X\} = [[1, 2], 5]$ ,  $s\{t_2 \rightarrow X\} = [[2], 5]$  and  $[[1, 2], 5] =_> [[2], 5]$ .

### 3.3 The subterm property - revised

Now we are going to discuss the extension of notion of the subterm property.



**Definition 7.** Let  $S$  be a set of terms over the language  $L$ , and let  $>$  be an order defined on  $S$ . We say that  $>$  has a partially subterm property if for every functor  $f/n$  in  $L$  exists  $\mathcal{S}_{f/n} \subseteq \{1, \dots, n\}$ , such that if  $i \in \mathcal{S}_{f/n}$  then for any sequences of terms  $\bar{s}$ , (occupying positions  $1 \dots i-1$ ) and  $\bar{u}$  (occupying positions  $i+1 \dots n$ ) holds  $f(\bar{s}, t, \bar{u}) > t$ . The sets  $\mathcal{S}_{f/n}$  are called defined by  $>$ .

*Example 8.* The order defined by the list-size norm does not have a subterm property, but it has a partial subterm property. It defines  $\mathcal{S}_{./2} = \{2\}$  and for other functors  $f/n$  holds that  $\mathcal{S}_{f/n} = \emptyset$ .

**Proposition 3.** Let  $S$  be a set of atoms or terms. If  $>$ , defined on  $S$  has a subterm property, than  $>$  has a partially subterm property.

*Proof.* Immediate from the definitions.  $\mathcal{S}_{f/n} = \{1, \dots, n\}$ .

We denote the collection of all  $\mathcal{S}_{f/n}$  by  $\mathcal{S}$ .

**Lemma 2.** Let  $s$  be a term, and  $t$  be a subterm occurring in it, such that  $\chi_s^{\mathcal{S}}(t) = 1$ . Then,  $s > t$ .

*Proof.* Let  $\overrightarrow{i_1, \dots, i_k}$  be a vector that denotes  $t$ . Then, by definition of the characteristic function  $\chi_{\mathcal{S}_{f_1/n_1}}(i_1) * \dots * \chi_{\mathcal{S}_{f_k/n_k}}(i_k) = 1$ . That is for any  $j$  holds  $\chi_{\mathcal{S}_{f_j/n_j}}(i_j) = 1$ .

Since  $\chi_{\mathcal{S}_{f_k/n_k}}(i_k) = 1$ ,  $i_k \in \mathcal{S}_{f_k/n_k}$ . Thus,  $f(\bar{u}_1, t, \bar{u}_2) > t$  for any sequences  $\bar{u}_1$  and  $\bar{u}_2$ . In particular, this holds for the sequences appearing in  $s$ . Repetitive application of this argument and the transitivity of  $>$  proves the lemma.

The following example shows, that if  $\chi_s^{\mathcal{S}}(t) = 0$  the lemma does not necessary hold.

*Example 9.* Let  $>$  be order based on the list-size norm, and let  $s$  be  $[[1, 2, 3], 5]$ . Observe, that  $\chi_s^{\mathcal{S}}([1, 2, 3]) = 0$ . On the other hand,  $[[1, 2, 3], 5] < [1, 2, 3]$ .

## 4 Term-acceptability with respect to a set

In this section we present and discuss some of the theory we developed to extend acceptability to general term orders. In the literature, there are different variants of acceptability. The most well-known of these is the acceptability as introduced by Apt and Pedreschi [3]. This version is defined and verified on the level of ground instances of clauses, but draws its practical power mostly from the fact that termination is proved for *any bounded* goal. Here, boundedness is a notion related to the selected level mapping and requires that the set  $\{|G\theta| \mid \theta \text{ is a grounding substitution for goal } G\}$  is bounded in the natural numbers, where  $|\cdot| : B_P \rightarrow \mathcal{N}$  denotes the level mapping.

Another notion of acceptability is the “acceptability with respect to a set of goals”, introduced by De Schreye et. al. in [8]. This notion allows to prove termination with respect to any set of goals of interest. However, it relies on

procedural concepts, such as calls and computed answer substitution. It was designed to be verified through global analysis, for instance through abstract interpretation.

A variant of acceptability w.r.t. a set that avoids the drawbacks of using procedural notions and that can be verified on a local level was designed in [10]. This variant required that the goals of interest are *rigid* under the given level mapping. Here, rigidity means that  $|G\theta| = |G|$ , for any substitution  $\theta$ , where  $|\cdot| : B_P^E \rightarrow \mathcal{N}$  now denotes a generalised level mapping, defined on the extended Herbrand base.

Comparing the notions of boundedness and rigidity in the context of a level mapping based approach, it is clear that boundedness is more general than rigidity. If the level mapping of a goal is invariant under substitution, then the level mapping is bounded on the set of instances of the goal, but not conversely.

Given the latter observation and given that acceptability of [3] is a more generally known and accepted notion, we started our work by generalising this variant.

However, it turned out that generalising the concept of boundedness to general term orders proved to be very difficult. We postpone the discussion on this issue until after we formulated the results, but because of these complications, we only arrived at generalised acceptability conditions that are useful in the context of well-mode, simply-mode programs and goals.

Because of this, we then turned our attention to acceptability with respect to a set. Here, the generalisation of rigidity was less complicated, so that in the end we obtained the strongest results for this variant of acceptability. Therefore, we first present term-acceptability with respect to a set of goals. We need the following notion.

**Definition 8.** [9] *Let  $P$  be a definite program and  $S$  be a set of atomic queries. The call set,  $Call(P, S)$ , is the set of all atoms  $A$ , such that a variant of  $A$  is a selected atom in some derivation for  $P \cup \{\leftarrow Q\}$ , for some  $Q \in S$  and under the left-to-right selection rule.*

To illustrate this definition recall the following example [1, 10].

*Example 10.*

$$\begin{aligned} &permute([], []). \\ &permute(L, [El|T]) \leftarrow delete(El, L, L1), permute(L1, T). \\ &delete(X, [X|T], T). \\ &delete(X, [H|T], [H|T1]) \leftarrow delete(X, T, T1). \end{aligned}$$

Let  $S$  be  $\{permute(t_1, t_2) \mid t_1 \text{ is a nil-terminated list and } t_2 \text{ is a free variable}\}$ . Then,  $Call(P, S) =$

$$S \cup \{delete(t_1, t_2, t_3) \mid t_1, t_3 \text{ are free variables and } t_2 \text{ is a nil-terminated list}\}.$$

Such information about  $S$  could for instance be expressed in terms of the rigid types of [15] and  $Call(P, S)$  could be computed using the type inference of [15].  $\square$

The following definition generalises the notion of acceptability w.r.t. a set [9] in two ways: 1) it generalises it to general term orders, 2) it generalises it to mutual recursion, using the standard notation of mutual recursion [1].

**Definition 9.** Let  $S$  be a set of atomic queries and  $P$  a definite program.  $P$  is term-acceptable w.r.t.  $S$  if there exists a well-founded order  $>$ , such that

- for any  $A \in \text{Call}(P, S)$
- for any clause  $A' \leftarrow B_1, \dots, B_n$  in  $P$ , such that  $\text{mgu}(A, A') = \theta$  exists,
- for any atom  $B_i$ , such that  $\text{rel}(B_i) \simeq \text{rel}(A)$
- for any computed answer substitution  $\sigma$  for  $\leftarrow (B_1, \dots, B_{i-1})\theta$ :

$$A > B_i\theta\sigma$$

In order to establish the connection between term-acceptability w.r.t. a set  $S$  and LD-termination for queries in  $S$  we recall the notion of *directed sequence* and related results, introduced by Verschaetse in [20].

**Definition 10.** Let  $G_0, G_1, G_2, \dots, \theta_1, \theta_2, \dots$  be a derivation with selected atoms  $A_0, A_1, A_2, \dots$  and applied renamed clauses  $H^i \leftarrow B_1^i, \dots, B_{n_i}^i$  ( $i = 1, 2, \dots$ ). We say that  $A_k$  is a direct descendant of  $A_i$ , if  $k > i$  and  $A_k$  is the atom  $B_j^{i+1}\theta_{i+1} \dots \theta_k$ , ( $1 \leq j \leq n_{i+1}$ ).

**Definition 11.** Let  $G_0, G_1, G_2, \dots, \theta_1, \theta_2, \dots$  be a derivation with selected atoms  $A_0, A_1, A_2, \dots$ . A subsequence of derivation steps,  $G_{i(0)}, G_{i(1)}, \dots, \theta_{i(0)+1}, \dots$  with selected atoms  $A_{i(0)}, A_{i(1)}, A_{i(2)}, \dots$  is directed, if for each  $k$  ( $k > 1$ ),  $A_{i(k)}$  is a direct descendant of  $A_{i(k-1)}$  in the given derivation.

**Definition 12.** A derivation  $G_0, G_1, G_2, \dots, \theta_1, \theta_2, \dots$  is directed if it is its own directed subsequence.

Verschaetse [20] proved also the following lemma:

**Lemma 3.** Let  $P$  be a definite program and  $A$  an atom. If  $(P, \leftarrow A)$  has an infinite derivation, then it has an infinite directed derivation.

Based on this results we prove the characterisation of the LD-termination in terms of term-acceptability w.r.t. a set.

**Theorem 1.** Let  $P$  be a program.  $P$  is term-acceptable w.r.t. a set of atomic queries  $S$  if and only if  $P$  is LD-terminating for all queries in  $S$ .

*Proof.*  $\Rightarrow$  Let  $A \in S$ , and assume that  $P \cup \{\leftarrow A\}$  has an infinite LD-derivation.

This derivation contains an infinite directed subsequence, that is a subsequence of goals  $G_{i(0)}, G_{i(1)}, \dots$  such that the selected atom of  $G_{i(k+1)}$ ,  $A_{k+1}$ , is a direct descendant of the selected atom of  $G_{i(k)}$ ,  $A_k$ . There is some  $k_0$ , such that for any  $m, n > k_0$  holds  $\text{rel}(A_m) \simeq \text{rel}(A_n)$ . Let  $k$  be greater than  $k_0$ .

Then there is a clause  $H \leftarrow B_1, \dots, B_n$ , such that the  $\text{mgu}(A_k, H) = \theta$  exists and for some  $j$  holds that  $A_{k+1} = B_j\theta\sigma$ , where  $\sigma$  is a computed answer

substitution for  $\leftarrow B_1, \dots, B_{j-1}$ . Observe that the choice of  $k$  implies that  $\text{rel}(B_j) \simeq \text{rel}(H)$ .

Since  $A_k$  is one of the selected atoms in  $\text{Call}(P, S)$  the condition of the term-acceptability w.r.t.  $S$  is applicable. Thus,  $A_k > B_j\theta\sigma$ , i.e.,  $A_k > A_{k+1}$ . By proceeding in this way we construct an infinitely decreasing chain of atoms, contradicting the well-foundedness of  $>$ .

$\Leftarrow$  Let  $G$  be in  $\text{Call}(P, S)$ , and let be  $H \leftarrow B_1, \dots, B_n$  be a clause in  $P$ , such that the  $\text{mgu}(H, G) = \theta$  exists. We define a relation  $\succ$ , such that  $G \succ B_i\theta\sigma$  for any  $1 \leq i \leq n$ , where  $\sigma$  is the computed answer substitution for  $\leftarrow (B_1, \dots, B_{i-1})\theta$ . Let  $>$  be the transitive closure of  $\succ$ .

We start from the following observation. If  $A \succ B$  there is a derivation, started by  $A$ , and having a goal with selected atom  $B$  in it. Similarly, if  $A \succ B$  and  $B \succ C$  there is a derivation, started by  $A$ , and having a goal with selected atom  $B$  in it, followed (not necessary immediately) by a goal with the selected atom  $C$ . We extend this observation to  $>$ , and claim that if  $A > B$  then there is a derivation for  $A$ , having a goal with the selected atom  $B$  in it.

An additional observation is that if  $X > Y$  is defined for some  $X$  and  $Y$ , then  $X$  and  $Y$  are in  $\text{Call}(P, S)$ .

In particular, if  $A > A$  there is a derivation for  $A$  that has a goal  $G$  with the selected atom  $A$  in it. Thus, we can continue the derivation from  $G$  in the same way we have constructed a derivation from  $A$  to  $G$ . This process can be done forever, contradicting that  $P$  terminates for  $A$ .

If  $A > B$ ,  $B > C$  from the observation above we construct a derivation from  $A$  that has a goal with the selected atom  $B$ , and then continue by mimicking the derivation from  $B$ , that has a goal with the selected goal  $C$ . Thus,  $A > C$ . This also proves the asymmetry (if  $A > B$  and  $B > A$ , then  $A > A$ , by transitivity, and this causes contradiction by irreflexivity).

The well-foundedness follows from the finiteness of all the derivations. The term-acceptability w.r.t.  $S$  follows immediately from the definition of  $>$ .

We postpone applying the Theorem 1 to Example 10 until a more syntactic way of verifying term-acceptability w.r.t. a set is developed.

To do this, we extend the sufficient condition of [8], that impose the additional requirement of rigidity of the level mapping on the call set, to the case of general term orders and study the notion of *rigidity*.

## 5 Rigidity

First we adapt the notion of rigidity to general term orders.

**Definition 13.** (see also [6]) *The term or atom  $A \in U_P^E \cup B_P^E$  is called rigid w.r.t.  $>$  if for any substitution  $\theta$ ,  $A =_> A\theta$ . In this case  $>$  is said to be rigid on  $A$ .*

The notion of the rigidity on a term (an atom) is naturally extended to the notion of rigidity on a set of atoms (terms). In particular, we will be interested

in orders that are rigid on  $\text{Call}(P, S)$  for some  $P$  and  $S$ . Intuitively, the rigid order ignores argument positions that might be occupied by free variables in  $\text{Call}(P, S)$ .

**Proposition 4.** *The following simple properties of rigid terms and atoms hold:*

1. *If  $A$  is ground, then  $A$  is rigid.*
2. *If  $A$  is rigid w.r.t.  $>$ , then  $A\theta$  is rigid w.r.t.  $>$  for any substitution  $\theta$ .*

Also, observe, that similarly to the notion of rigidity w.r.t. a norm,  $A =_> B$  and  $A$  is rigid does not imply that  $B$  is rigid.

Let  $>$  be an order and  $s$  be a term, that is not rigid. Then there must occur some variables in  $s$  whose substitution causes the disequality to hold. We want to identify each of their occurrences.

**Definition 14.** (see [6]) *Let  $>$  be an order and  $s$  be a term. The  $i$ th occurrence  $X_{(i)}$ , of a variable  $X$  in the term  $s$  is relevant w.r.t.  $>$  whenever there exists a replacement  $\{t \rightarrow X_{(i)}\}$  of the term  $t$  for the  $i$ th occurrence of  $X$  in  $s$ , such that  $s\{t \rightarrow X_{(i)}\} \neq_> s$ .  $\text{VREL}_{>}(s)$  is the set of all the relevant occurrences of variables in  $s$ .*

Bossi, Cocco and Fabris [6] proved that for the semi-linear norm  $\|\cdot\|$  the term  $s$  is rigid w.r.t.  $\|\cdot\|$  if and only if  $\text{VREL}_{\|\cdot\|}(s) = \emptyset$ . This allowed to reduce a check of the rigidity w.r.t. the semi-linear norm to a syntactical condition. As the following example demonstrates, the assumption of semi-linearity is essential.

*Example 11.* ([6]) Let  $\|\cdot\|_{\text{bal}}$  be the following norm:

$$\|t\|_{\text{bal}} = \begin{cases} 0 & t \text{ is void or is not a tree} \\ 0 & t \text{ is a } \text{tree}(a, l, r) \text{ and } l = r \\ 1 & t \text{ is a } \text{tree}(a, l, r) \text{ and } l \neq r, \end{cases}$$

where  $=$ , as usual, means syntactical equality.

This norm is not semi-linear, and indeed the term  $\text{tree}(a, X, X)$  is rigid w.r.t. it since for any substitution of the variable  $X$ , the two subtrees remain syntactically equal. Nevertheless,  $\text{VREL}(\text{tree}(a, X, X)) = \{X_{(1)}, X_{(2)}\}$  since by replacing one single occurrence of the variable  $X$  we can change the norm of the term.

Our goal is similar to one stated above, i.e., to reduce the check of rigidity w.r.t. an order to some syntactic condition. Similarly, for the most general case the emptiness of  $\text{VREL}_{>}(s)$  is not related to rigidity. The first example demonstrates that  $\text{VREL}_{>}(s) = \emptyset$  is not necessary for the rigidity, while the second one - that it is not sufficient for it.

*Example 12.* Let  $f/1$  be the only functor and  $a$  the only constant of the language. Let  $>$  be defined as follows:  $f(X, X) = f(t, t) < f(a, X) = f(X, a)$ , for any term  $t$  constructed from the functors and constants of the underlying language and a set of fresh variables. In this case  $f(X, X)$  is rigid, but  $\text{VREL}_{>}(f(X, X)) \neq \emptyset$ .

Observe that  $>$  in this example is not monotone (otherwise,  $f(a, X) > f(a, a)$  implies  $f(f(a, X), f(a, X)) > f(f(a, a), f(a, a))$ , that contradicts the definition above. Additionally,  $>$  does not have a subterm property either, since  $f(a, X) > f(f(a, X), f(a, X))$ .

*Example 13.* Let  $>$  be defined as  $f(X, Y) = f(a, Y) = f(f(\dots), Y) = f(X, a) = f(X, f(\dots)) = f(X, X)$ . Then  $\text{VREL}_{>}(f(X, Y)) = \emptyset$ . Observe, that this does not imply, for example, that  $\text{VREL}_{>}(f(a, Y)) = \emptyset$ . This allows us to define  $f(X, Y) < f(a, a)$ , thus, making  $f(X, Y)$  non-rigid.

Similarly to the example above,  $>$  is not monotone and does not have a subterm properties. Indeed, if  $>$  was monotone  $f(X, f(X, Y)) < f(X, f(a, a))$  was obtained, while both of the terms can be obtained by one step replacement from  $f(X, Y)$ , thus providing a contradiction to monotonicity. In the same way  $f(X, f(a, a)) < f(a, a)$  contradicts the subterm property.

We start by studying the rigidity under monotonicity and subterm property assumptions of  $>$ . Later on these assumptions will be relaxed.

**Lemma 4.** *Let  $>$  be a monotone order, having a subterm property, on the set of terms  $S$ . Then, if  $>$  is rigid on  $s \in S$  then  $\text{VREL}_{>}(s) = \emptyset$ .*

*Proof.* Assume that  $\text{VREL}_{>}(s) \neq \emptyset$ . This means that there exists an occurrence  $X_{(i)}$  of a variable  $X$  and a replacement  $\{t \rightarrow X_{(i)}\}$ , such that  $s\{t \rightarrow X_{(i)}\} \neq s$ . On the other hand,  $>$  is rigid on  $s$ . Thus, the replacement cannot be extended to a substitution  $\theta$ , such that  $s\theta = s\{t \rightarrow X_{(i)}\}$ . This means, that  $s$  is non-linear in its variables, i.e.,  $X$  appears among the variables of  $s$  at least twice. Let  $X_{(n_1)}, \dots, X_{(n_m)}$  be all occurrences of  $X$  in  $s$ .  $X_{(i)}$  is one of them.

We distinguish the following cases:

1.  $s < s\{t \rightarrow X_{(i)}\}$ . Let  $s'$  be a term, obtained from  $s$  by a simultaneous replacement of  $X_{(n_1)}, \dots, X_{(n_m)}$  in  $s$  by  $s$ . Let  $t'$  be a term, obtained from  $s$  by a simultaneous replacement of  $X_{(n_1)}, \dots, X_{(n_m)}$  in  $s$  by  $s\{t \rightarrow X_{(i)}\}$ . Then, by the monotonicity of  $>$  holds  $s' < t'$ . However, since  $X$  does not appear in  $s$ , except for  $X_{(n_1)}, \dots, X_{(n_m)}$  and all of those, and only them, have been replaced by a new terms the compositions of the replacements above are substitutions. This means that there are two substitutions  $\theta_1$  and  $\theta_2$ , such that  $s\theta_1 < s\theta_2$ , contradicting the rigidity of  $s$ .
2.  $s\{t \rightarrow X_{(i)}\} < s$ . Similarly to the previous case.
3.  $s\{t \rightarrow X_{(i)}\} \parallel s$ . Let  $t'$  be a term, obtained from  $s$  by a simultaneous replacement of  $X_{(n_1)}, \dots, X_{(n_m)}$  in  $s$  by  $s\{t \rightarrow X_{(i)}\}$ . Then, by the subterm property of  $>$  holds that  $s\{t \rightarrow X_{(i)}\} < t'$ . By the same reasoning as above  $t'$  is an instance of  $s$ . Thus,  $s$  is equal to it w.r.t.  $>$  (rigidity), i.e.,  $s\{t \rightarrow X_{(i)}\} < s$  providing a contradiction to the incomparability.

**Lemma 5.** *Let  $>$  be an order, having a subterm property, on the set of terms  $S$ . Then, if for some  $s \in S$  holds that  $\text{VREL}_{>}(s) = \emptyset$ ,  $s$  is ground.*

*Proof.* Assume for the sake of contradiction, that  $s$  is not ground. Thus,  $s$  has at least one variable occurrence say  $X_{(i)}$ . Then,  $s\{s \rightarrow X_{(i)}\} > s$ , contradicting that  $\text{VREL}_{>}(s) = \emptyset$ .

As we have already pointed out above every ground term is rigid, allowing to conclude.

**Corollary 1.** *Let  $>$  be a monotone order, having a subterm property on the set of terms  $s$ . Then, the following statements are equivalent:*

1.  $>$  is rigid on some  $s \in S$
2.  $\text{VREL}_{>}(s) = \emptyset$
3.  $s$  is ground

Thus, if  $>$  has monotonicity and subterm properties the notion of rigidity deteriorates to groundness. As we are going to see this is not necessary the case if the restrictions on  $>$  are relaxed.

In more general setting, presented in Section 3 some correspondence between argument positions possessing monotonicity and subterm properties and  $\text{VREL}_{>}$  should be established.

**Definition 15.** *Let  $s$  be a term, and let  $>$  be a partially monotone order, having a partial subterm property. Let  $\text{VarInst}(s)$  be the set of all variable instances of  $s$ . Then, we denote by  $M_{>}(s)$  and by  $S_{>}(s)$  the following sets:*

$$M_{>}(s) = \{X_{(j)} \mid X_{(j)} \in \text{VarInst}(s), \forall i \chi_s^M(X_{(i)}) = 1\}$$

$$S_{>}(s) = \{X_{(j)} \mid X_{(j)} \in \text{VarInst}(s), \forall i \chi_s^S(X_{(i)}) = 1\}$$

**Lemma 6.** 1.  $S_{>}(s) \subseteq \text{VREL}_{>}(s)$   
 2. If there exist  $t_1$  and  $t_2$ , such that  $t_1 > t_2$ , then  $M_{>}(s) \cup S_{>}(s) \subseteq \text{VREL}_{>}(s)$

*Proof.* 1. Let  $X_{(i)} \notin \text{VREL}_{>}(s)$ . This means that for any term  $t$ ,  $s\{t \rightarrow X_{(i)}\} = s$ . In particular,  $s\{s \rightarrow X_{(i)}\} = s$ . Clearly,  $s$  is a subterm of  $s\{s \rightarrow X_{(i)}\}$ . Thus, by Lemma 2  $\chi_{s\{s \rightarrow X_{(i)}\}}^S(s) = 0$ , i.e.,  $\chi_s^S(X_{(i)}) = 0$ . Thus,  $S_{>}(s) \subseteq \text{VREL}_{>}(s)$ .  
 2. Let  $t_1 > t_2$ . If  $X_{(i)} \in M_{>}(s)$ , then  $s\{t_1 \rightarrow X_{(i)}\} > s\{t_2 \rightarrow X_{(i)}\}$  holds. Thus, at least one of those terms is not equal to  $s$  and  $X_{(i)} \in \text{VREL}_{>}(s)$ . Hence,  $M_{>}(s) \subseteq \text{VREL}_{>}(s)$ , proving the second statement of the lemma as well.

Summarising the definitions introduced so far, an order  $>$  maps each functor  $f/n$  (predicate  $p/n$ ) to a pair of sets  $\mathcal{M}_{f/n}$  and  $\mathcal{S}_{f/n}$  ( $\mathcal{M}_{p/n}$  and  $\mathcal{S}_{p/n}$ ) that are defined by  $>$ .

Now we are able to reformulate the results, stated previously only for orders, that have a monotonicity and subterm properties state them for orders, that are partially monotone and have a partial subterm property.

**Lemma 7.** *Let  $>$  be a partially monotone order, having a partial subterm property, on the set of terms  $S$ . Then, if  $>$  is rigid on  $s \in S$  then  $M_{>}(s) \cap S_{>}(s) = \emptyset$ .*

*Proof.* (Sketch) Let  $M_{>}(s) \cap S_{>}(s)$  be non-empty and let  $X_{(i)} \in M_{>}(s) \cap S_{>}(s)$ . Then, Lemma 1 and Lemma 2 allow to conclude monotonicity and subterm properties for the argument positions occupied by the instances of  $X$  and to mimic the proof of Lemma 4.

If comparing this lemma with Lemma 4 we see that the assumptions of Lemma 4 are *more restrictive* and the conclusion is *stronger* than of the recent lemma.

Unfortunately, the second direction of the claim, analogous to Lemma 5 does not hold. Example 13 illustrates this (under assumption that  $\mathcal{M}_{f/2} = \mathcal{S}_{f/2} = \emptyset$ ).

**Definition 16.** Let  $>$  be a partially monotone order, having a partial subterm property. Then, the term  $s$  is called pseudo-rigid w.r.t.  $>$  if

- for every substitution  $\theta$ ,
- and for every  $X \in \text{Dom}(\theta)$ , such that
- for every occurrence  $X_{(i)}$  of  $X$  holds  $\chi_s^{\mathcal{M}}(X_{(i)}) = 0$  and  $\chi_s^{\mathcal{S}}(X_{(i)}) = 0$

$$s\theta = s$$

*Example 14.* Term  $[X, Y]$  is pseudo-rigid w.r.t. the list-size based norm.

**Lemma 8.** Let  $>$  be a partially monotone order, having a partial subterm property, such that there exist  $t_1$  and  $t_2$  for which  $t_1 > t_2$  holds. If  $\text{VREL}_{>}(s) = \emptyset$  and  $s$  is pseudo-rigid w.r.t.  $>$ , then  $s$  is rigid w.r.t.  $>$ .

*Proof.* Let  $\theta$  be a substitution. If  $\theta = \varepsilon$ ,  $s\theta = s$  and the proof is done. Otherwise, exists some  $X$ , such that  $X \in \text{Dom}(\theta)$ . By Lemma 6  $\text{M}_{>}(s) \cup \text{S}_{>}(s) = \emptyset$ . Thus,  $\text{M}_{>}(s) = \text{S}_{>}(s) = \emptyset$ . This means, that for any  $X \in \text{Dom}(\theta)$ , for all occurrences  $X_{(i)}$  holds that  $\chi_s^{\mathcal{S}}(X_{(i)}) = \chi_s^{\mathcal{M}}(X_{(i)}) = 0$ . But now, the pseudo-rigid condition is applicable, and  $s\theta = s$ .

For all the examples to be considered further we assume  $>$  to be pseudo-rigid on  $\text{Call}(P, S)$  as well as the existence of  $t_1$  and  $t_2$ , such that  $t_1 > t_2$ . Under these assumptions Lemma 8 allows us to reduce verifying rigidity to verifying the emptiness of  $\text{VREL}_{>}$ . That is, we impose that the ordering is invariant on predicate argument positions and functor argument positions that may occur with a free variable in  $\text{Call}(P, S)$ .

## 6 Sufficient condition for termination

Recall that our goal is to extend the sufficient condition of [8], that impose the additional requirement of rigidity of the level mapping on the call set, to the case of general term orders. Except for the notion of rigidity, we also need interargument relations based on general term orders.

**Definition 17.** Let  $P$  be a definite program,  $p/n$  a predicate in  $P$  and  $>$  an order on  $U_P^E$ . An interargument relation is a relation  $R_p = \{(t_1, \dots, t_n) \mid \varphi_p(t_1, \dots, t_n)\}$ , where:

- $\varphi_p(t_1, \dots, t_n)$  is a formula in a disjunctive normal form
- each conjunct in  $\varphi_p$  is either  $s_1 > s_2$ ,  $s_1 \Rightarrow s_2$  or  $s_1 \parallel s_2$ , where  $s_i$  are constructed from  $t_1, \dots, t_n$  by applying functors of  $P$ .



$R_p$  is a valid interargument relation for  $p/n$  w.r.t. an order  $>$  if and only if for every  $p(t_1, \dots, t_n) \in \text{Atom}_P$ : if  $P \models p(t_1, \dots, t_n)$  then  $(t_1, \dots, t_n) \in R_p$ .

*Example 15.* Consider the following program.

$$\begin{aligned} p(0, []) \\ p(f(X), [X|T]) \leftarrow p(X, T) \end{aligned}$$

The following interargument relations can be considered for  $p$ :  $\{(t_1, t_2) \mid t_2 > t_1 \vee t_1 =_> t_2\}$ , valid w.r.t. an ordering imposed by a list-length norm. Recall, that for lists  $\|[t_1|t_2]\|_l = 1 + \|t_2\|_l$ , while the list-length of other terms is considered to be 0. On the other hand,  $\{(t_1, t_2) \mid t_1 > t_2 \vee t_1 =_> t_2\}$ , valid w.r.t. an ordering imposed by a term-size norm.  $\square$

Using the notion of rigidity we present a sufficient condition for term-acceptability w.r.t. a set.

**Theorem 2.** (*rigid term-acceptability w.r.t. S*) Let  $S$  be a set of atomic queries and  $P$  be a definite program. Let  $>$  be an order on  $U_P^E$  and for each predicate  $p$  in  $P$ , let  $R_p$  be a valid interargument relation for  $p$  w.r.t.  $>$ . If there exists a well-founded extension  $\succ$  of  $>$  to  $U_P^E \cup B_P^E$ , which is rigid on  $\text{Call}(P, S)$  such that

- for any clause  $H \leftarrow B_1, \dots, B_n \in P$ , and
- for any atom  $B_i$  in its body, such that  $\text{rel}(B_i) \simeq \text{rel}(H)$ ,
- for substitution  $\theta$ , such that the arguments of the atoms in  $(B_1, \dots, B_{i-1})\theta$  all satisfy their associated relations  $R_{\text{rel}(B_1)}, \dots, R_{\text{rel}(B_{i-1})}$

$$H\theta \succ B_i\theta$$

then  $P$  is term-acceptable w.r.t.  $S$

*Proof.* Suppose the above condition is satisfied for  $P$ . Take any  $A \in \text{Call}(P, S)$  and any clause  $A' \leftarrow B_1, \dots, B_n$  such that  $\text{mgu}(A, A') = \theta$  exists. Suppose, that  $B_i$  is a body atom, such that  $\text{rel}(B_i) \simeq \text{rel}(A)$  and that  $\sigma$  is a computed answer substitution for  $\leftarrow (B_1, \dots, B_{i-1})\theta$ .

Then  $A\theta$  is identical to  $A'\theta$ , and thus,  $A\theta\sigma$  is identical to  $A'\theta\sigma$ . Since  $\succ$  is rigid on  $\text{Call}(P, S)$  and  $A \in \text{Call}(P, S)$ ,  $A'\theta\sigma =_{\succ} A$ .

Finally, since  $\sigma$  is a computed answer substitution  $P \models B_j\theta\sigma$ , for all  $j < i$ . Thus, by definition of valid interargument relation the arguments of  $B_j\theta\sigma$  satisfy  $R_{\text{rel}(B_j)}$ , for all  $j < i$ . Thus, by the rigid term-acceptability assumption  $A'\theta\sigma \succ B_i\theta\sigma$ . Combined with  $A'\theta\sigma =_{\succ} A$ , we get  $A > B_i\theta\sigma$ .

We continue the analysis of Example 10.

*Example 16.* Let  $>$  be a well-founded ordering on  $B_P^E \cup U_P^E$ , such that:

- for all terms  $t_1, t_{21}$  and  $t_{22}$ :  $\text{permute}(t_1, t_{21}) =_> \text{permute}(t_1, t_{22})$ .

- for all terms  $t_{11}, t_{12}, t_2, t_{31}, t_{32}$ :  $delete(t_{11}, t_2, t_{31}) =_> delete(t_{12}, t_2, t_{32})$ .
- for all terms  $t_{11}, t_{12}$  and  $t_2$ :  $[t_{11}|t_2] =_> [t_{12}|t_2]$ .

That is, we impose that the ordering is invariant on predicate argument positions and functor argument positions that may occur with a free variable in  $Call(P, S)$ . Furthermore, we impose that  $>$  has the subterm and monotonicity properties at all remaining predicate or functor argument positions.

First we investigate the rigidity of  $>$  on  $Call(P, S)$ , namely:  $G\theta =_> G$  for any  $G \in Call(P, S)$  and any  $\theta$ . Now any effect that the application of  $\theta$  to  $G$  may have on  $G$  needs to be through the occurrence of some variable in  $G$ . However, because we imposed that  $>$  is invariant on all predicate and functor argument positions that may possibly contain a variable in some call,  $G\theta =_> G$ .

Associate with *delete* the interargument relation  $R_{delete} = \{(t_1, t_2, t_3) \mid t_2 > t_3\}$ . First, we verify that this interargument relationship is valid. Note, that an interargument relationship is valid whenever it is a model for its predicate. Thus, to check whether  $R_{delete}$  is valid,  $T_P(R_{delete}) \subseteq R_{delete}$  is checked. For the non-recursive clause of *delete* the inclusion follows from the subset property of  $>$ , while for the recursive one, from the monotonicity of it.

Then, consider the recursive clauses of the program.

- *permute*. If  $delete(El, L, L1)\theta$  satisfies  $R_{delete}$ , then  $L\theta > L1\theta$ . By the monotonicity,  $permute(L, T)\theta > permute(L1, T)\theta$ . By the property stated above,  $permute(L, [El|T])\theta =_> permute(L, T)\theta$ . Thus, the desired decrease  $permute(L, [El|T])\theta > permute(L1, T)\theta$  holds.
- *delete*. By the properties of  $>$  stated above:  $delete(X, [H|T], [H|T1]) > delete(X, T, [H|T1])$  and  $delete(X, T, [H|T1]) =_> delete(X, T, T1)$ . Thus,  $delete(X, [H|T], [H|T1]) > delete(X, T, T1)$ .

We have shown that all the conditions of Theorem 2 are satisfied, and thus,  $P$  is term-acceptable w.r.t.  $S$ . By Theorem 1,  $P$  terminates for all queries in  $S$ .

Observe, that we do not need to construct the actual order, but only to prove that there is some, that meets all the requirements posed. In this specific case, the requirement of subterm and monotonicity on the remaining argument positions is satisfiable.  $\square$

## 7 The results for standard acceptability

In this section we briefly discuss some of the results we obtained in generalising the acceptability notion of [3]. Since these results are weaker than those presented in the previous section, we do not elaborate on them in full detail. In particular, we do not recall the definitions of well-moded programs and goals, nor those of simply moded programs and goals, that we use below, but instead refer to [1], respectively [2]. Below, we assume that in-output modes for the program and goal are given. For any atom  $A$  and a mode  $m_A$  for  $A$ , we denote by  $A^{inP}$  the atom obtained from  $A$  by removing all output arguments. E.g., let  $A = p(f(2), 3, X)$  and  $m_A = p(in, in, out)$ , then  $A^{inP} = p(f(2), 3)$ .

**Definition 18.** Let  $>$  be an order relation on  $B_P^E$ . We say that  $>$  is output-independent if for any two moded atoms  $A$  and  $B$ :  $A^{inP} = B^{inP}$  implies  $A => B$ .

For well-moded programs, term-acceptability in the style of [3] can now be defined as follows.

**Definition 19.** Let  $P$  be a well-moded program,  $>$  an output-independent well-founded order and  $I$  a model for  $P$ . The program  $P$  is called term-acceptable w.r.t.  $>$  and  $I$  if for all  $A \leftarrow B_1, \dots, B_n$  in  $P$  and all substitutions  $\theta$ , such that  $(A\theta)^{inP}$  and  $B_1\theta, \dots, B_{i-1}\theta$  are ground and  $I \models B_1\theta \wedge \dots \wedge B_{i-1}\theta$  holds:  $A\theta > B_i\theta$ .

$P$  is called *term-acceptable* if it is term-acceptable w.r.t. some output-independent well-founded order and some model. The following theorem states that term-acceptability of a well-moded program is sufficient for termination of well-moded goals w.r.t. this program.

**Theorem 3.** Let  $P$  be a well-moded program, that is term-acceptable w.r.t. an output-independent well-founded order  $>$  and a model  $I$ . Let  $G$  be a well-moded goal, then  $G$  LD-terminates.

*Proof.* We base our proof on the notion of directed sequence. Let  $G$  be non-terminating, i.e.,  $P \cup \{G\}$  has an infinite derivation. By Lemma 3 it has an infinite directed derivation as well. Let  $G_0, G_1, \dots$  be this infinite directed derivation. We denote  $G_i = \leftarrow A_1^i, \dots, A_{n_i}^i$  and  $G_{i+1} = \leftarrow A_1^{i+1}, \dots, A_{n_{i+1}}^{i+1}$ . There is a clause  $H \leftarrow B_1, \dots, B_n$ , and substitutions  $\sigma$  and  $\theta$  such that  $A_1^i = H\sigma$ , for some  $1 \leq j \leq n_{i+1}$ ,  $A_1^{i+1} = B_j\sigma\theta$ ,  $I \models B_1\sigma\theta \wedge \dots \wedge B_{j-1}\sigma\theta$  and  $B_1\sigma\theta \wedge \dots \wedge B_{j-1}\sigma\theta$  is ground. Note, that  $(H\sigma)^{inP}$  is ground due to the well-modedness. The term-acceptability condition implies that  $H\sigma\theta > B_j\sigma\theta$ , that is  $A_1^i\theta > A_1^{i+1}$ . Since  $P$  and  $G$  are well-founded  $(A_1^i)^{inP}$  is ground. Thus,  $(A_1^i)^{inP} = (A_1^i\theta)^{inP}$  and, by the output-independence of  $>$ ,  $A_1^i => A_1^i\theta$ . By transitivity,  $A_1^i > A_1^{i+1}$ . Thus, selected atoms of the goals in the infinite directed derivation form an infinite decreasing chain w.r.t.  $>$ , contradicting the well-foundedness of the order.

Note that if the requirement of well-modedness is relaxed the theorem does not hold anymore.

*Example 17.*

$$p(a) \leftarrow q(X) \quad q(f(X)) \leftarrow q(X)$$

We assume the modes  $p(in)$  and  $q(in)$  to be given. This program is not well-moded w.r.t. the given modes, but it satisfies the remaining conditions of term-acceptability with respect to the following order  $>$  on terms

$$p(a) > \dots > q(f(f(f(a)))) > q(f(f(a))) > q(f(a)) > q(a)$$

and a model  $I = \{p(a), q(a), q(f(a)), q(f(f(a))), \dots\}$ . However, note that the well-founded goal  $p(a)$  is non-terminating.  $\square$

Unfortunately, well-modedness is not sufficient to make the converse to hold. That is, there is a well-moded program  $P$  and a well-moded goal  $G$ , such that  $G$  is LD-terminating w.r.t.  $P$ , but  $P$  is not term-acceptable.

*Example 18.* Consider the following program

$$p(f(X)) \leftarrow p(g(X))$$

with the mode  $p(out)$ . This program is well-moded, the well-moded goal  $p(X)$  terminates w.r.t. this program, but it is not term-acceptable, since the required decrease  $p(f(X)) > p(g(X))$  violates output-independence of  $>$ .  $\square$

Intuitively, the problem in the example occurred, since some information has been passed via the output positions, i.e.  $P$  is not simply moded. Indeed, if  $P$  is simply-moded,[2], the second direction of the theorem holds as well.

We start the presentation by a number of useful lemmas.

**Lemma 9.** *Let  $P_0, \dots, P_m$  be directed sequence, such that  $P_i = \leftarrow A_1^i, \dots, A_{n_i}^i$ . Then, for any suffix  $\mathbf{S}$ , exist a sequence of substitutions  $\theta_1, \dots, \theta_m$  and a sequence of suffices  $\mathbf{R}_1, \dots, \mathbf{R}_m$  such that*

$$\leftarrow A_1^0, \mathbf{S}; \leftarrow A_1^1, \mathbf{R}_1 \theta_1, \mathbf{S} \theta_1; \dots; \leftarrow A_{n_m}^m, \mathbf{R}_m \theta_1 \dots \theta_m, \mathbf{S} \theta_1 \dots \theta_m$$

*is directed.*

**Lemma 10.** *Let  $P_0, \dots, P_m$  and  $Q_0, \dots, Q_k$  be two directed sequences, such that  $P_m = \leftarrow A_1, \dots, A_s$  and  $Q_0 = \leftarrow B_1, \dots, B_t$  and  $A_1 = B_1$ . Then, exists a directed sequence  $R_0, \dots, R_{m+k}$ , such that the selected atom of  $R_0$  is the selected atom of  $P_0$ , and the selected atom of  $R_{m+k}$  is the selected atom of  $Q_k$ .*

*Proof.* We define  $R_0, \dots, R_{m+k}$  as following:

$$R_i = \begin{cases} P_i & \text{if } 0 \leq i \leq m \\ T_{i-m} & \text{if } m \leq i \leq m+k \end{cases}$$

where  $T_j$  is the  $j$ -th element in the sequence, generated by Lemma 9 for the directed sequence  $Q_0, \dots, Q_k$  with  $\mathbf{S} = A_2, \dots, A_s$ .

The sequence  $R_0, \dots, R_{m+k}$  is well-defined: if  $i = m$ , on one hand we get that  $R_m = P_m$ , and on the other hand,  $R_m = B_1, \mathbf{S}$ , that is  $P_m$ . For  $i \neq m$  only one of those definitions is applicable.

The requirement of the lemma are clearly fulfilled.

**Corollary 2.** *Let  $P$  be a simply moded program and  $Q, Q'$  - simply moded goals. Let  $P_0, \dots, P_m$  be a directed sequence obtained from one of the derivations for  $P \cup \{Q\}$  and  $Q_0, \dots, Q_k$  be a directed sequence obtained from one of the derivations for  $P \cup \{Q'\}$ . Let also  $P_m$  be  $\leftarrow A_1, \dots, A_s$ ,  $Q_0$  be  $\leftarrow B_1, \dots, B_t$  and  $A_1^{\text{in}P} = B_1^{\text{in}P}$ . Then, exists a directed sequence  $R_0, \dots, R_{m+k}$ , such that the selected atom of  $R_0$  is the selected atom of  $P_0$ , and the selected atom of  $R_{m+k}$  is the selected atom of  $Q_k$ .*

*Proof.* Since  $P_m$  and  $Q_0$  are queries in some of the LD-derivations of the simply moded queries and of the simply moded program, they are simply moded [2]. Thus, the output positions, both of  $A_1$  and of  $B_1$ , are occupied by distinct variables. Since  $A_1^{inp} = B_1^{inp}$  we can claim that  $A_1 = B_1$ , up to variables renaming. Thus, Lemma 10 becomes applicable (note that we never required in the lemma that both directed sequences shall originate from the derivations of the same LD-tree), and we can obtain a new directed sequence as required.

**Theorem 4.** *Let  $P$  be a well-moded and simply moded program, LD-terminating for any well-moded and simply-moded goal. Then there exists a model  $I$  and an output-independent well-founded order  $>$ , such that  $P$  is term-acceptable w.r.t.  $I$  and  $>$ .*

*Proof.* We base the choice of  $>$  on the LD-trees. More precisely, we define  $A > B$  if there is a well-moded and simply moded goal  $G$  and there is a directed sequence  $P_0, \dots, P_m$  in the LD-tree for  $P \cup \{G\}$ , such that the selected atom of  $P_0$  is  $A_1$  and the selected atom of  $P_m$  is  $B_1$  and  $A^{inp} = A_1^{inp}$ ,  $B^{inp} = B_1^{inp}$ . Let  $I$  be a least Herbrand model of  $P$ . We have to prove that:

1.  $>$  is an order relationship, i.e., is irreflexive, asymmetric and transitive;
  2.  $>$  is output-independent;
  3.  $>$  is well-founded;
  4.  $P$  is term-acceptable w.r.t.  $>$  and  $I$
1.  $>$  is an order relationship, that is  $>$  is irreflexive, asymmetric and transitive.
    - (a) Irreflexivity. If  $A > A$  holds, then exists a directed sequence  $P_0, \dots, P_k$ , such that the selected atom of  $P_0$  is  $A_1$ , the selected atom of  $P_k$  is  $A_2$ ,  $A_1^{inp} = A^{inp}$  and  $A_2^{inp} = A^{inp}$ . By repetitive application of Corollary 2 an infinite branch is build and the contradiction to the finiteness of the LD-tree is obtained.
    - (b) Asymmetry. If  $A > B$  holds, then exists a directed sequence  $P_0, \dots, P_k$ , such that the selected atom of  $P_0$  is  $A_1$ , the selected atom of  $P_k$  is  $B_1$ ,  $A^{inp} = A_1^{inp}$ ,  $B^{inp} = B_1^{inp}$ . If  $B > A$  holds, then exists a directed sequence  $Q_0, \dots, Q_m$ , such that the selected atom of  $Q_0$  is  $B_2$ , the selected atom of  $Q_m$  is  $A_2$ ,  $A^{inp} = A_2^{inp}$ ,  $B^{inp} = B_2^{inp}$ . By repetitive application of Corollary 2 an infinite branch is build and the contradiction to the finiteness of the LD-tree is obtained.
    - (c) Transitivity. If  $A > B$  holds, then exists a directed sequence  $P_0, \dots, P_k$ , such that the selected atom of  $P_0$  is  $A_1$ , the selected atom of  $P_k$  is  $B_1$ ,  $A^{inp} = A_1^{inp}$ ,  $B^{inp} = B_1^{inp}$ . If  $B > C$  holds, then exists a directed sequence  $Q_0, \dots, Q_m$ , such that the selected atom of  $Q_0$  is  $B_2$ , the selected atom of  $Q_m$  is  $C_2$ ,  $B^{inp} = B_2^{inp}$ ,  $C^{inp} = C_2^{inp}$ . By applying the Corollary 2 new directed sequence is build, such that the selected atom of its first element is  $A_1$  and the selected atom of its last element is  $C_2$ . By definition of  $>$  holds that  $A > C$ .

2.  $>$  is output-independent. Assume that there are two atoms  $A$  and  $B$ , such that  $A^{inp} = B^{inp}$ , but  $A > B$ . Then exists a directed sequence  $P_0, \dots, P_k$ , such that the selected atom of  $P_0$  is  $A_1$ , the selected atom of  $P_k$  is  $B_1$ ,  $A_1^{inp} = A^{inp}$  and  $B_1^{inp} = B^{inp}$ . However,  $B_1^{inp} = A^{inp}$ . Thus, by repetitive application of Corollary 2 an infinite branch is build and the contradiction to the finiteness of the LD-tree is obtained.
3.  $>$  is well-founded. Assume that there is an infinitely decreasing chain  $A_1 > A_2 > \dots$ . This means that there is an infinite directed sequence in the tree (concatenation of infinitely many finite ones), contradicting the finiteness of the tree.
4.  $P$  is term-acceptable w.r.t.  $>$  and  $I$ . Let  $A \leftarrow B_1, \dots, B_n$ . Let  $\theta$  be a substitution, such that  $(A\theta)^{inp}, B_1\theta, \dots, B_{i-1}\theta$  are ground and  $I \models B_1\theta \wedge \dots \wedge B_{i-1}\theta$ . The goal  $\leftarrow A\theta$  is a well-moded goal, however, it is not necessary simply moded. Thus, we define a new goal  $A'$  such that it will coincide with  $A\theta$  on its input positions, and its output positions will be occupied by a linear set of variables.

More formally, let  $\theta_1$  be  $\theta$  restricted to  $\text{Var}(A)^{inp}$ . Then  $(A\theta_1)^{inp} = (A\theta)^{inp}$ , and thus,  $(A\theta_1)^{inp}$  is ground, while  $(A\theta_1)^{out} = A^{out}$ , and thus,  $(A\theta_1)^{out}$  is a linear sequence of variables. Summing up,  $\leftarrow A\theta_1$  is well-moded and simply moded goal. Thus, it terminates w.r.t.  $P$  and its derivations has been considered while defining  $>$ .

By definition of  $\theta_1$  exists some substitution  $\sigma$ , such that  $\theta = \theta_1\sigma$ . Thus,  $B_1\theta, \dots, B_{i-1}\theta = (B_1\theta_1, \dots, B_{i-1}\theta_1)\sigma$ . Since  $I$  is a least Herbrand model  $\sigma$  is a correct answer substitution of  $B_1\theta_1, \dots, B_{i-1}\theta_1$  and, since  $P$  and  $\leftarrow B_1\theta_1, \dots, B_{i-1}\theta_1$  are well-moded (the later as the LD-resolvent of the well-moded clause and the well-moded goal),  $\sigma$  is a computed answer substitution as well [1]. Thus, the next goal to be considered in the derivation is  $\leftarrow B_i\theta_1\sigma, \dots, B_n\theta_1\sigma$ . This is a directed descendant of  $A\theta_1$ , thus, by definition of  $>$ ,  $A\theta_1 > B_i\theta_1\sigma$ . By definition of  $\sigma$ ,  $B_i\theta_1\sigma = B_i\theta$ , and by the definition of  $\theta_1$ ,  $(A\theta_1)^{inp} = (A\theta)^{inp}$ . Thus,  $A\theta > B_i\theta$ .

We have introduced so far two different notions of term-acceptability: notion of term-acceptability w.r.t. a set of queries and the notion of term-acceptability (w.r.t. an order and a model). We study the relationship between these two notions for well-moded and simply-moded programs and goals.

Observe, that the well-modedness and simply modedness are not sufficient to reduce the term-acceptability w.r.t. a set to the term-acceptability.

*Example 19.*

$$\begin{array}{l} q \\ p \leftarrow p \end{array}$$

This program is not term-acceptable, but it is term-acceptable w.r.t. a set  $\{q\}$ .

In order to eliminate this kind of examples we assume also in Theorem 5, that for every clause in  $P$  there is a goal in  $\text{Call}(P, S)$ , that can be unified with its head.

Once more we precede the theorem with a small lemma.

**Lemma 11.** *Let  $P$  be a well-moded and simply moded program, and  $S$  be a set of well-moded and simply moded goals. Then, given an output-independent  $\succ$ , for any clause goal  $A \in \text{Call}(P, S)$  and any  $A' \leftarrow B_1, \dots, B_n$ , s.t.  $\text{mgu}(A, A') = \theta$  exists, holds that  $A =_{\succ} A'\theta$ .*

*Proof.* If  $G \in S$  and  $P$  are well-moded then  $A$  is well-moded as well [1]. Analogously,  $A$  is simply moded [2]. Thus, the input positions of  $A$  are ground and the output positions of  $A$  are occupied by distinct variables. Therefore,  $\theta$  cannot affect the input positions of  $A$ , and  $A^{\text{inp}} = (A\theta)^{\text{inp}}$ , i.e.,  $A =_{\succ} A\theta$ . Since  $A\theta$  is identical to  $A'\theta$ ,  $A =_{\succ} A'\theta$  holds.

**Theorem 5.** *Let  $S$  be a set of well-moded and simply moded goals,  $P$  be a well-moded and simply moded program, such that for every clause in  $P$  there is a goal in  $\text{Call}(P, S)$ , that can be unified with its head. Then, given an output-independent  $\succ$ , if  $P$  is term-acceptable w.r.t.  $S$  and  $\succ$ , then  $P$  is term-acceptable w.r.t. some well-founded order  $\succ$  and least Herbrand model of  $P$ .*

*Proof.* We define  $\succ$  in the following way:

$$t_1 \succ t_2 \text{ if } \begin{cases} \text{rel}(t_1) \sqsupset \text{rel}(t_2) \\ \text{rel}(t_1) \simeq \text{rel}(t_2) \text{ and } t_1 > t_2 \end{cases}$$

The properties of  $\succ$  follow from the corresponding properties of  $>$  and  $\sqsupset$ .

Let  $A' \leftarrow B_1, \dots, B_n$  be a clause. We have to prove that for any substitution  $\gamma$ , such that  $(A'\gamma)^{\text{inp}}$  and  $(B_1, \dots, B_{i-1})\gamma$  are ground and  $I \models B_1\gamma \wedge \dots \wedge B_{i-1}\gamma$  holds that  $A'\gamma \succ B_i\gamma$ .

Then, by the property of  $P$  stated above there is some  $A \in \text{Call}(P, S)$ , unifiable with  $A'$ . Let  $\theta$  be the most general unifier of  $A$  and  $A'$ . By Lemma 11  $A =_{\succ} A'\theta$ . On the other hand,  $(A'\theta)^{\text{inp}} = (A'\gamma)^{\text{inp}}$ . On the other hand, the output argument positions of  $A'\theta$  are occupied by a sequence of distinct variables (simply modedness of  $P$  and  $A$  - we can always assume that fresh variants of the clauses are considered [2]). Thus, there is some  $\sigma$ , such that  $\gamma = \theta\sigma$ .

$I \models B_1\gamma \wedge \dots \wedge B_{i-1}\gamma$ . Since  $I$  is the least Herbrand model,  $\gamma$  is a correct answer substitution for  $\leftarrow (B_1, \dots, B_{i-1})$ . Since  $P$  and any  $G \in S$  are well-moded  $\gamma$  is also computed answer substitution. Thus, for the goal  $\leftarrow (B_1, \dots, B_{i-1})\theta$  the computed answer substitution will be  $\sigma$ .

- $\text{rel}(B_i) \simeq \text{rel}(A')$ . Then, by the definition of term-acceptability w.r.t. a set,  $A > B_i\theta\sigma$ . However,  $A =_{\succ} A'\gamma$  and  $B_i\theta\sigma$  is  $B_i\gamma$ . Thus, by claiming  $A'\gamma > B_i\gamma$ , and  $A'\gamma \succ B_i\gamma$ .
- $\text{rel}(B_i) \not\simeq \text{rel}(A')$ . This means that  $\text{rel}(A') \sqsupset \text{rel}(B_i)$ , i.e.,  $A'\gamma \succ B_i\gamma$ .

To conclude, we briefly discuss why it is difficult to extend the notions of term-acceptability to the non well-moded case, using a notion of boundedness, as it was done for standard acceptability [3]. In acceptability based on level mappings, boundedness ensures that the level mapping of a (non-ground) goal can

only increase up to some finite bound when the goal becomes more instantiated. Observe that every ground goal is trivially bounded.

One particular possible generalisation of boundedness to term-orderings, which is useful for maintaining most of our results, is:

An atom  $A$  is *bounded* with respect to an ordering  $<$ , if there exists an atom  $C$  such that for all ground instances  $A\theta$  of  $A$ :  $A\theta < C$ , and  $\{B \in B_P^E \mid B < C\}$  is finite.

Such a definition imposes constraints which are very similar to the ones imposed by standard boundedness in the context of level mappings. However, one thing we loose is that it is no longer generalisation of groundness. Consider an atom  $p(a)$  and assume that our language contains a functor  $f/1$  and a constant  $b$ . Then one particular well-founded ordering is

$$p(a) > \dots > p(f(f(b))) > p(f(b)) > p(b).$$

So,  $p(a)$  is not bounded with respect to this ordering.

Because of such complications, we felt that the rigidity-based results of the previous section are the preferred generalisations to general term orders.

## 8 Towards automation of the approach

In this section we present an approach leading towards automatic verification of the term-acceptability condition. The basic idea for the approach is inspired on the “constraint based” termination analysis proposed in [10]. We start off from the conditions imposed by term-acceptability, and systematically reduce these conditions to more explicit constraints on the objects of our search: the order  $>$  and the interargument relations,  $R_p$ , or model  $I$ .

The approach presented below has been applied successfully to a number of examples that appear in the literature on termination, such as different versions of *permute* [5, 16, 10], *dis-con* [7], *transitive closure* [16], *add-mult* [18], *combine*, *reverse*, *odd-even*, *at\_least\_double* and *normalisation* [10], *quicksort* program [19, 1], *derivative* [13], *distributive law* [12], *boolean ring* [14], *flatten* [4].

In the remainder of the paper, we explain the approach using some of these examples.

We start by showing how the analysis of Example 10, presented before, can be performed systematically. We stress the main steps of an algorithm.

*Example 20.*  $>$  should be rigid on  $\text{Call}(P, S)$ . To enforce the rigidity,  $>$  should ignore all argument positions in atoms in  $\text{Call}(P, S)$  that might be occupied by free variables, i.e., the second argument position of *permute* and the first and the third argument positions of *delete*. Moreover, since the first argument of *permute* and the second argument of *delete* are general nil-terminated lists, the first argument of  $./2$  should be ignored as well.

The  $>$ -decreases imposed in the term-acceptability w.r.t. a set  $S$  are:

$$\text{delete}(X, [H|T], [H|T1])\theta > \text{delete}(X, T, T1)\theta$$



$$\begin{aligned} \text{delete}(El, L, L_1)\theta \text{ satisfies } R_{\text{delete}} \text{ implies} \\ \text{permute}(L, [El|T])\theta > \text{permute}(L_1, T)\theta \end{aligned}$$

Each of these conditions we simplify by replacing the predicate argument positions that should be ignored by some arbitrary term  $t$ . The following conditions are obtained:

$$\text{delete}(t, [H|T]\theta, t) > \text{delete}(t, T\theta, t) \quad (1)$$

$$\begin{aligned} \text{delete}(El, L, L_1)\theta \text{ satisfies } R_{\text{delete}} \text{ implies} \\ \text{permute}(L\theta, t) > \text{permute}(L_1\theta, t) \end{aligned} \quad (2)$$

Observe that this only partially deals with the requirements that the rigidity conditions expressed above impose: rigidity on functor arguments (the first argument of  $. / 2$  should be invariant w.r.t. the order) is not expressed. We keep track of such constraints implicitly, and only verify them at a later stage when additional constraints on the order are derived.

For each of the conditions (1) and (2), we have two options on how to enforce it:

Option 1): The decrease required in the condition can be achieved by imposing some property on  $>$ , which is consistent with the constraints that were already imposed on  $>$  before.

In our example, condition (1) is satisfied by imposing the subterm property for the second argument of  $. / 2$  and monotonicity on the second argument of  $\text{delete}$ . The second argument of  $. / 2$  does not belong to a set of functor argument positions that should be ignored. Then,  $[t_1|t_2] > t_2$  holds for any terms  $t_1$  and  $t_2$ , and by the monotonicity of  $>$  in the second argument of  $\text{delete}$  (1) holds.

In general we can select from a bunch of term-order properties, or even specific term-orders, that were proposed in the literature.

Option 2): The required decrease is imposed as a constraint on the interargument relation(s)  $R$  of the preceding atoms.

In the  $\text{permute}$  example, the decrease  $\text{permute}(L\theta, t) > \text{permute}(L_1\theta, t)$  cannot directly be achieved by imposing some constraint on  $>$ . Thus, we impose that the underlying decrease  $L\theta > L_1\theta$  should hold for the intermediate body atoms ( $\text{delete}(El, L, L_1)\theta$ ) that satisfy the interargument relation  $R_{\text{delete}}$ .

Thus, in the example, the constraint is that  $R_{\text{delete}}$  should be such that for all  $\text{delete}(t_1, t_2, t_3)$  that satisfy  $R_{\text{delete}}$ :  $t_2 > t_3$ . As we have observed, the interargument relation is valid if it forms a model for its predicate. Thus, one way to constructively verify that a valid interargument relation  $R_{\text{delete}}$  exists, such that the property  $t_2 > t_3$  holds for  $\text{delete}(t_1, t_2, t_3)$  atoms is to simply impose that  $M = \{\text{delete}(t_1, t_2, t_3) \mid t_2 > t_3\}$  itself is a model for the  $\text{delete}$  clauses in the program.

So our new constraint on  $R_{\text{delete}}$  is that it should include  $M$ . Practically we can enforce this by imposing that  $T_P(M) \subseteq M$  should hold. That is, the following constraints are obtained:

$$\begin{aligned} [t_1|t_2] > t_2 \\ t_2 > t_3 \text{ implies } [t|t_2] > [t|t_3] \end{aligned}$$

These are again fed into our Option 1) step, imposing a monotonicity property on the second argument of  $\cdot/2$  for  $> \cdot$ . At this point the proof is complete.  $\square$

Recall, that we do not need to construct actually the order, but only to prove that there is some, that meets all the requirements posed.

The previous example does not illustrate the approach in full generality. It might happen that more than one intermediate goal preceded the recursive atom in the body of the clause. In this case we refer to the whole conjunction as to “one” subgoal. Formally, given a sequence of intermediate body atoms  $B_1, \dots, B_n$  a (generalised) clause  $B_1, \dots, B_n \leftarrow B_1, \dots, B_n$  is constructed and one step of unfolding is performed on each atom in its body, producing a generalised program  $P'$ .

*Example 21.* The following is the version of the *permute* program that appeared in [16].

$$\begin{array}{ll} \text{perm}([], []). & \text{ap}_1([], L, L). \\ \text{perm}(L, [H|T]) \leftarrow & \text{ap}_1([H|L1], L2, [H|L3]) \leftarrow \\ & \text{ap}_2(V, [H|U], L), \quad \text{ap}_1(L1, L2, L3). \\ \text{ap}_1(V, U, W), & \text{ap}_2([], L, L). \\ \text{perm}(W, T). & \text{ap}_2([H|L1], L2, [H|L3]) \leftarrow \\ & \text{ap}_2(L1, L2, L3). \end{array}$$

This example is analysed, based on Theorem 3 for the well-moded case. We would like to prove termination of the goals  $\text{perm}(t_1, t_2)$ , where  $t_1$  is a ground list and  $t_2$  a free variable.

Assume the modes

$$\text{perm}(in, out), \text{ap}_1(in, in, out), \text{ap}_2(out, out, in)$$

The term-acceptability imposes, among the others, the following  $>$ -decrease:  $I \models \text{ap}_2(V, [H|U], L)\theta \wedge \text{ap}_1(V, U, W)\theta$  implies  $\text{perm}(L)\theta > \text{perm}(W)\theta$ . Note that the underlying decrease  $L\theta > W\theta$  cannot be achieved by reasoning on  $\text{ap}_1/3$  or  $\text{ap}_2/3$  alone. Therefore, we construct a following program  $P'$ :

$$\begin{array}{l} \text{ap}_2([], [t_1|t_2], [t_1|t_2]), \text{ap}_1([], t_2, t_2). \\ \text{ap}_2([t_6|t_1], [t_5|t_2], [t_6|t_3]), \text{ap}_1([t_6|t_1], t_2, [t_6|t_4]) \leftarrow \\ \text{ap}_2(t_1, [t_5|t_2], t_3), \text{ap}_1(t_1, t_2, t_4). \end{array}$$

Now, we need to verify that  $M = \{\text{ap}_2(a_1, a_2, a_3), \text{ap}_1(b_1, b_2, b_3) \mid a_3 > b_3\}$  satisfies  $T_{P'}(M) \subseteq M$ . Using the 2 clauses, this is reduced to “ $[t_1|t_2] > t_2$ ” and “ $t_3 > t_4$  implies  $[t_6|t_3] > [t_6|t_4]$ ”, imposing monotonicity and subterm properties on  $>$ . The proof is completed analogously to the previous example.  $\square$

As a last example, we return to the motivating Example 1, on computing higher derivatives of polynomial functions in one variable.

*Example 22.* We are interested in proving termination of the queries that belong to  $S = \{d(t_1, t_2) \mid t_1 \text{ is a repeated derivative of a function in a variable } u \text{ and } t_2 \text{ is a free variable}\}$ . So  $S$  consists of atoms of the form  $d(\text{der}(u), X)$  or  $d(\text{der}(u * u + u), Y)$  or  $d(\text{der}(\text{der}(u + u)), Z)$ , etc. Observe, that  $\text{Call}(P, S)$  coincides with  $S$ .

We start by analysing the requirements that imposes the rigidity of  $>$  on  $\text{Call}(P, S)$ . First, the second argument position of  $d$  should be ignored, since it might be occupied by a free variable. Second, the first argument position of  $d$  is occupied by a ground term. Thus, rigidity does not pose any restrictions on functors argument positions.

Then, we construct the  $>$ -decreases that follow from the rigid term-acceptability. The arguments that should be ignored are replaced by a term  $t$ .

$$d(\text{der}(X + Y)\theta, t) > d(\text{der}(X)\theta, t) \quad (3)$$

$$d(\text{der}(X), DX)\theta \text{ satisfies } R_d$$

$$d(\text{der}(X + Y)\theta, t) > d(\text{der}(Y)\theta, t) \quad (4)$$

$$d(\text{der}(X * Y)\theta, t) > d(\text{der}(X)\theta, t) \quad (5)$$

$$d(\text{der}(X), DX)\theta \text{ satisfies } R_d$$

$$d(\text{der}(X * Y)\theta, t) > d(\text{der}(Y)\theta, t) \quad (6)$$

$$d(\text{der}(\text{der}(X))\theta, t) > d(\text{der}(X)\theta, t) \quad (7)$$

$$d(\text{der}(X), DX)\theta \text{ satisfies } R_d$$

$$d(\text{der}(\text{der}(X))\theta, t) > d(\text{der}(DX)\theta, t) \quad (8)$$

Conditions (3)-(7) impose monotonicity and subset properties to hold on the first argument of  $d$ . In order to satisfy condition (8), it is sufficient to prove that for any  $(t_1, t_2) \in R_d$  holds that  $t_1 > t_2$ . That is if  $M = \{d(t_1, t_2) \mid t_1 > t_2\}$  then  $T_P(M) \subseteq M$ . This may be reduced to the following conditions:

$$\text{der}(t) > 1 \quad (9)$$

$$t_1 \in R_{\text{number}} \text{ implies } \text{der}(t_1) > 0 \quad (10)$$

$$\text{der}(t_1) > t_2 \ \& \ \text{der}(t_3) > t_4 \text{ implies } \text{der}(t_1 + t_3) > t_2 + t_4 \quad (11)$$

$$\text{der}(t_1) > t_2 \ \& \ \text{der}(t_3) > t_4 \text{ implies } \text{der}(t_1 * t_3) > t_1 * t_4 + t_2 * t_3 \quad (12)$$

$$\text{der}(t_1) > t_2 \ \& \ \text{der}(t_2) > t_3 \text{ implies } \text{der}(\text{der}(t_1)) > t_3 \quad (13)$$

Condition (13) follows from monotonicity and transitivity of  $>$ . However, (10)-(12) are not satisfied by general properties of  $>$  and we choose, to specify the order. The order that meets these conditions is the recursive path ordering [11] with  $\text{der}$  having the highest priority.  $\square$

This example demonstrates the main steps of our algorithm. First, given a program  $P$  and a set  $S$  of goals, *compute the set of calls*  $\text{Call}(P, S)$  (for instance through the abstract interpretation of [15]). Second, *enforce the rigidity of  $>$  on*  $\text{Call}(P, S)$ , i.e., ignore all predicate or functor argument positions that might be occupied by free variables in  $\text{Call}(P, S)$ . Third, repeatedly *construct  $>$ -decreases*, such that rigid term-acceptability condition will hold and check if those can be verified by some of the predefined orders.

## 9 Conclusion

We have presented a non-transformational approach to termination analysis of logic programs, based on general term-orderings. The problem of termination was studied by a number of authors (see [7] for the survey). More recent work on this topic can be found in [17, 9, 10].

Our approach gets its power from integrating the traditional notion of acceptability [3] with the wide class of term-orderings that have been studied in the context of the term-rewriting systems. In theory, such an integration is unnecessary: acceptability (based on level mappings only) is already equivalent to LD-termination. In practice, the required level mappings may sometimes be very complex (such as for Example 1 or *distributive law* [12], *boolean ring* [14] or *flattening of a binary tree* [4]), and automatic systems for proving termination are unable to generate them. In such cases, generating an appropriate term-ordering, replacing the level mapping, may often be much easier, especially since we can reuse the impressive machinery on term-orders developed for term-rewrite systems. In some other cases, such as *turn* [6], simple level mappings do exist (in the case of *turn*: a norm counting the number of 0s before the first occurrence of 1 in the list is sufficient), but most systems based on level mappings will not even find this level mapping, because they only consider mappings based on term-size or list-length norms. Again, our approach is able to deal with such cases.

Unlike transformational approaches, that establish the termination results for logic programs by the reasoning on termination of term-rewriting systems, we apply the term-orderings directly to the logic programs, thus, avoiding transformations. This could both be regarded as an advantage and as a drawback of our approach. It may be considered as a drawback, because reasoning on successful instances of intermediate body-atoms introduces an additional complication in our approach, for which there is no counterpart in transformational methods (except for in the transformation step itself). On the other hand, we consider it as an advantage, because it is precisely this reasoning on intermediate body atoms that gives more insight in the property of *logic program termination* (as opposed to *term-rewrite system termination*).

So, in a sense our approach provides the best of both worlds: a means to incorporate into ‘direct’ approaches the generality of general term-orderings.

We consider as a future work a full implementation of the approach. Although we already tested very many examples manually, an implementation will allow us to conduct a much more extensive experimentation, comparing the technique also in terms of efficiency with other systems. Since we apply a demand-driven approach, systematically reducing required conditions to more simple constraints on the ordering and the model, we expect that the method can lead to very efficient verification.

## 10 Acknowledgement

Alexander Serebrenik is supported by GOA: “ $LP^+$ : a second generation logic programming language”. Danny De Schreye is a senior research associate of FWO Flanders, Belgium.

## References

1. K. R. Apt. *From Logic Programming to Prolog*. Prentice-Hall International Series in Computer Science. Prentice Hall, 1997.
2. K. R. Apt and S. Etalle. On the unification free Prolog programs. In A. M. Borzyszkowski and S. Sokolowski, editors, *18th International Symposium on Mathematical Foundations of Computer Science*, pages 1–19. Springer Verlag, 1993. Lecture Notes in Computer Science, volume 711.
3. K. R. Apt and D. Pedreschi. Studies in Pure Prolog: Termination. In J. W. Lloyd, editor, *Proceedings Esprit Symposium on Computational Logic*, pages 150–176. Springer Verlag, 1990.
4. T. Arts. *Automatically proving termination and innermost normalisation of term rewriting systems*. PhD thesis, Faculteit Wiskunde & Informatica, Universiteit Utrecht, 1997.
5. T. Arts and H. Zantema. Termination of logic programs using semantic unification. In M. Proietti, editor, *Logic Programming Synthesis and Transformation, 5th International Workshop. LOPSTR'95*, pages 219–233. Springer Verlag, 1995. Lecture Notes in Computer Science, volume 1048.
6. A. Bossi, N. Cocco, and M. Fabris. Norms on terms and their use in proving universal termination of a logic program. *Theoretical Computer Science*, 124(2):297–328, February 1994.
7. D. De Schreye and S. Decorte. Termination of logic programs: The never-ending story. *The Journal of Logic Programming*, 19/20:199–260, May/July 1994.
8. D. De Schreye, K. Verschaetse, and M. Bruynooghe. A framework for analyzing the termination of definite logic programs with respect to call patterns. In I. Staff, editor, *Fifth Generation Computer Systems '92: Proceedings of the International Conference on Fifth Generation Computer Systems.*, pages 481–488. IOS Press, 1992.
9. S. Decorte and D. De Schreye. Termination analysis: some practical properties of the norm and level mapping space. In J. Jaffar, editor, *Proceedings of the 1998 Joint International Conference and Symposium on Logic Programming*, pages 235–249. MIT Press, June 1998.
10. S. Decorte, D. De Schreye, and H. Vandecasteele. Constraint-based termination analysis of logic programs. *ACM Transactions on Programming Languages and Systems (TOPLAS)*, 21(6):1137–1195, November 1999.
11. N. Dershowitz. Termination. In C. Kirchner, editor, *Proceedings of the First International Conference on Rewriting Techniques and Applications*, pages 180–224. Springer Verlag, 1985. Lecture Notes in Computer Science, volume 202.
12. N. Dershowitz and C. Hoot. Topics in termination. In C. Kirchner, editor, *Rewriting Techniques and Applications, 5th International Conference*, pages 198–212. Springer Verlag, 1993. Lecture Notes in Computer Science, volume 690.
13. N. Dershowitz and Z. Manna. Proving termination with multiset orderings. *Communications of the ACM (CACM)*, 22(8):465–476, August 1979.

14. J. Hsiang. Rewrite method for theorem proving in first order theory with equality. *Journal of Symbolic Computation*, 8:133–151, 1987.
15. G. Janssens and M. Bruynooghe. Deriving descriptions of possible values of program variables by means of abstract interpretation. *J. Logic Programming*, 13(2&3):205–258, July 1992.
16. M. Krishna Rao, D. Kapur, and R. Shyamasundar. Transformational methodology for proving termination of logic programs. *The Journal of Logic Programming*, 34:1–41, 1998.
17. N. Lindenstrauss and Y. Sagiv. Automatic termination analysis of logic programs. In L. Naish, editor, *Proceedings of the Fourteenth International Conference on Logic Programming*, pages 63–77. MIT Press, July 1997.
18. L. Plümer. *Termination Proofs for Logic Programs*. Lecture Notes in Artificial Intelligence, volume 446. Springer Verlag, 1990.
19. L. Sterling and E. Shapiro. *The Art of Prolog*. The MIT Press, 1994.
20. K. Verschaetse. *Static termination analysis for definite Horn clause programs*. PhD thesis, Departement Computerweyenschappen, K.U.Leuven, 1992.